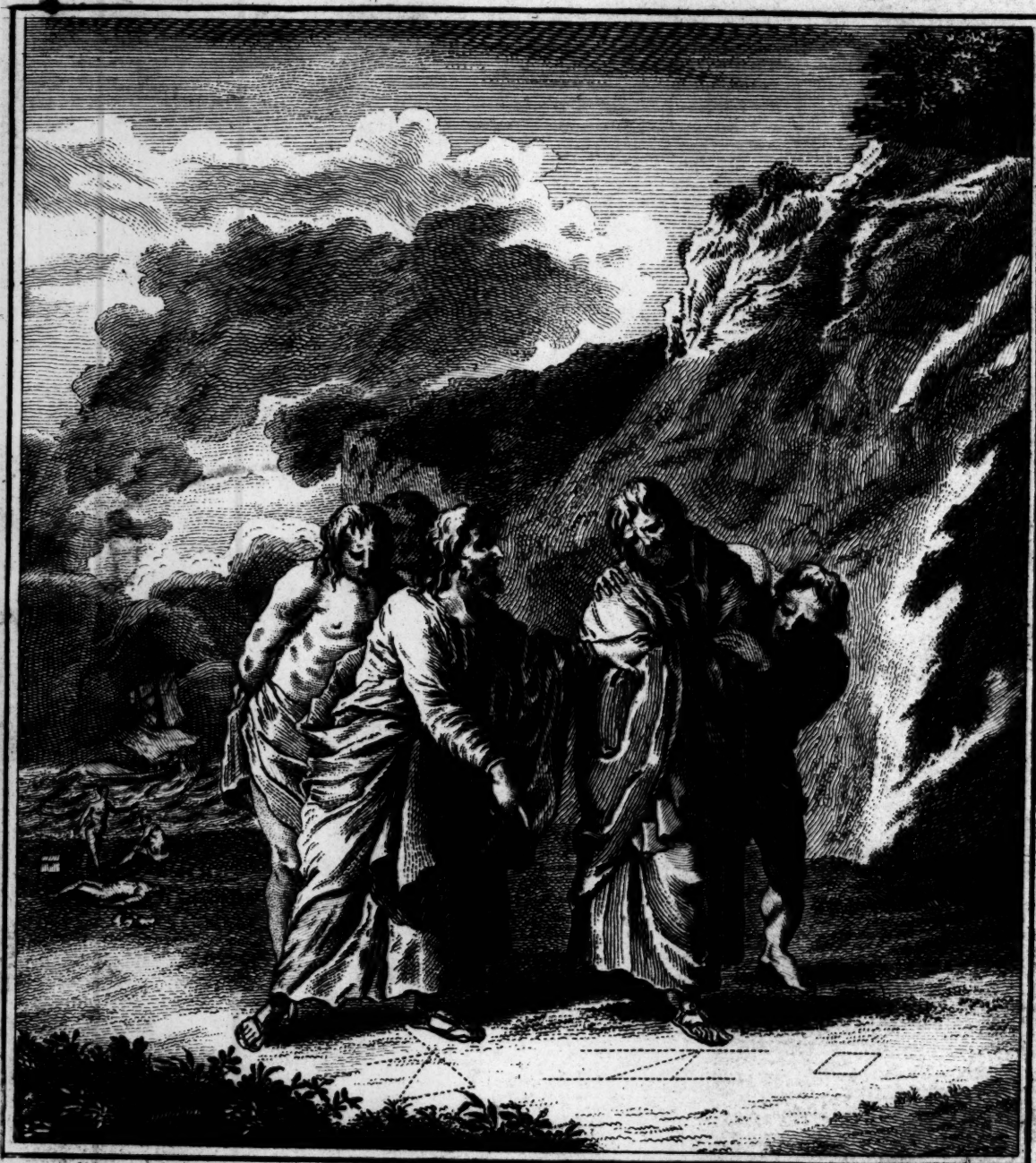
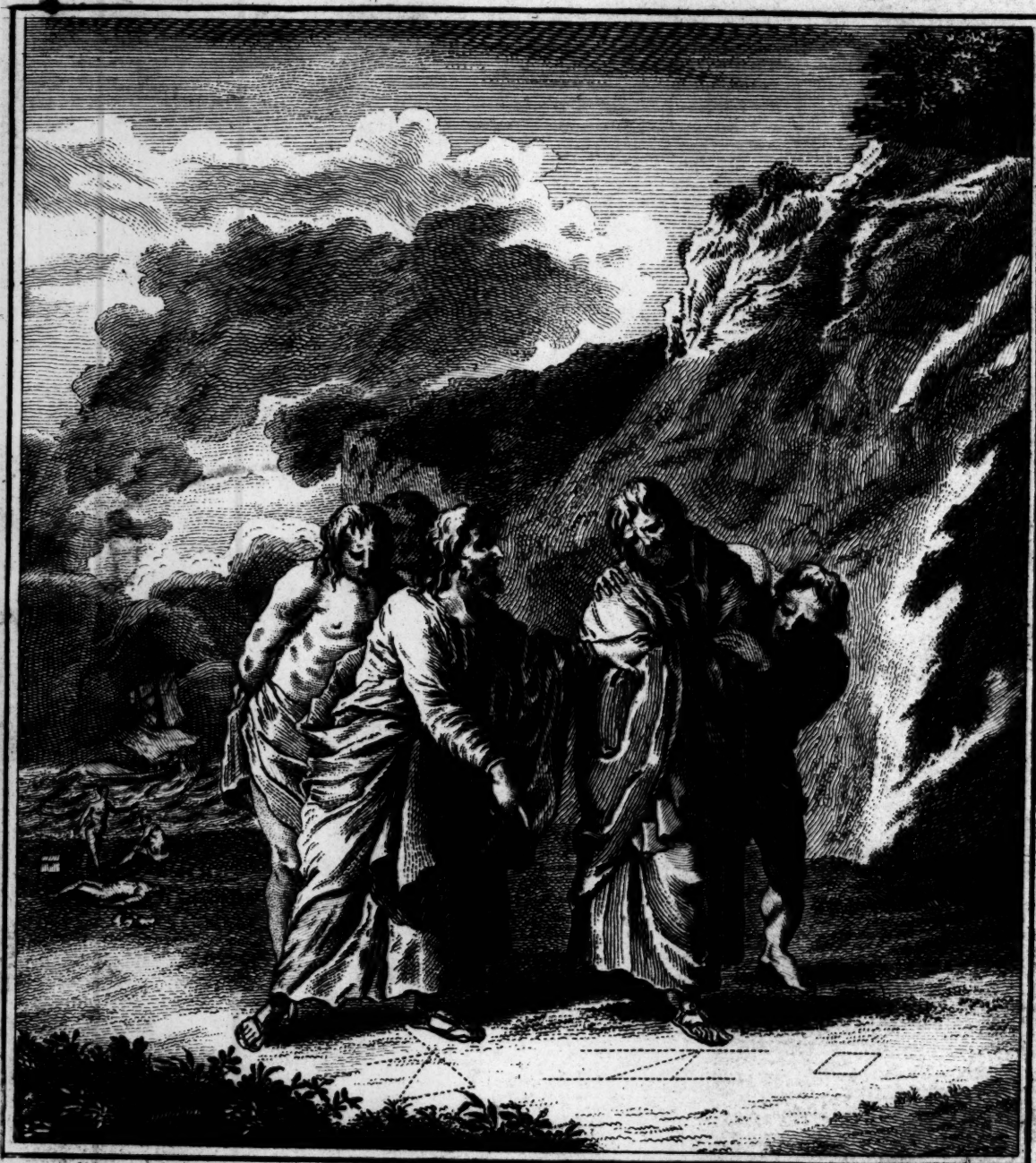




Aristippus Philosophus Socraticus, naufragio cum ejectus ad
Rhodiensium litus animadvertisset Geometrica schemata descripta,
exclamavisse ad comites ita dicitur,
 Bene speremus, Hominum enim vestigia video.
Vitruv. Architect. lib. 6. Praef.



H.P. Sculp.

*Aristippus Philosophus Socraticus, naufragio cum eiectus ad
Rhodiensium litus animadvertisset Geometrica schemata descripta,
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Bene speremus, Hominum enim vestigia video.

Vitruv. Architect. lib. 6. Praef.

FIRST VOLUME
OF THE
INSTRUCTIONS
GIVEN IN THE
DRAWING SCHOOL
ESTABLISHED BY THE
DUBLIN-SOCIETY,

Pursuant to their RESOLUTION of the Fourth
of FEBRUARY, 1768;

To enable Youth to become PROFICIENTS in the different
Branches of that Art, and to pursue with Success, GEOGRA-
PHICAL, NAUTICAL, MECHANICAL, COMMERCIAL, and
MILITARY STUDIES.

Under the Direction of JOSEPH FENN, heretofore Professor of
PHILOSOPHY in the University of NANTS. K

*Quid munus Reipublicæ majus aut melius afferre possumus, quam si Ju-
ventutem bene Erudiamus?* CICERO.

D U B L I N :

Printed by ALEX. M'CULLOH, in Henry-street, M,DCC,LXIX.



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H.P. Sculpsit.

A U S P I C I I S
FREDERICI HARVEY, EPISCOPI DERRENSIS SUPREMÆ CURIÆ, &c.
 PROMOVENTE SOCIETATE DUBLINENSI.
 F A V E N T I B U S
JOSEPHO HENRY, ROGER PALMER ET GULIELMO DEANE,
 ARMIGERIS, OMNIGENÆ ERUDITIONIS MÆCENATIBUS.

Josephus Fenn olim in Academia Nanatensi Philosophiæ Professor, puræ et mixtæ
 Matheos Elementa digessit et publicavit, in usum Scholæ ad propagandas Ar-
 tes in Hibernia fundatæ.

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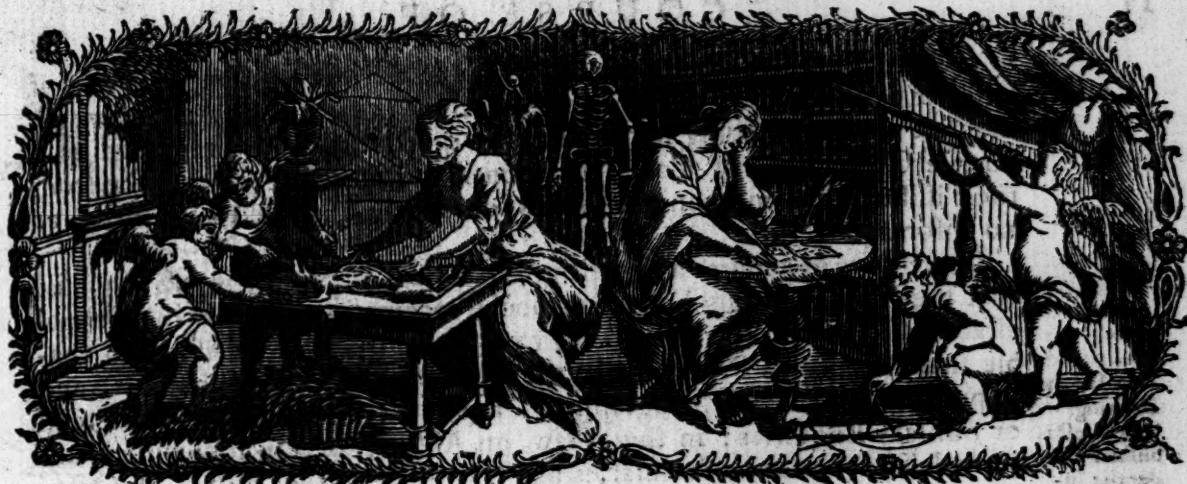
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PLAN of the INSTRUCTIONS given in the DRAWING-SCHOOL established by the DUBLIN SOCIETY, to enable YOUTH to become Proficients in the different Branches of that Art, and to pursue with Success geographical, nautical, mechanical, commercial or military Inquiries.

L'Oisiveté & l' Ignorance sont les deux Sources empoisonnées de tous les Désordres, & les plus grands Fléaux de la Société.

THE Education of Youth is considered in all Countries as the Object which interests most immediately the Happiness of Families, as well as that of the State. To this End, the ablest Hands are employed in forming Plans of Instruction, the best calculated for the various Professions of Life, and Societies are formed, composed of Men distinguished, as well by their Birth and Rank, as by their Experience and Knowledge, under whose Inspection, and by whose Care they are carried into Execution, by Persons of acknowledged Abilities in their different Departments: And thus the Education of Youth is conducted, from their earliest Years, in a Manner the best suited to engage their Minds in the Love of useful Knowledge, to improve their Understandings, to form their Taste and ripen their Judgments, to fix in them an Habit of Thinking with Steadiness and Attention, to promote their Address and Penetration, and to raise their Ambition to excel in their respective Provinces.

However necessary such Regulations may appear to every reasonable Person, however wished for by every Parent who feels the Loss of a proper Education in his own Practice; nevertheless they had not been even thought of in this Country, where that Extent of Knowledge, requisite

Wise Regulations relative to the Education of Youth, in England, Scotland, and other Parts of Europe.

Fatal Consequences resulting from the Neglect of this Object

to prepare Youth to appear with Dignity in the various Employments of Life, or to enable them to bring to Perfection the different Arts for which they are designed, being not attended to; Education was regarded as a puerile Object, and of Course abandoned to illiterate Persons, who from their illiberal and mechanic Methods of teaching gave Youth little or no Information.

How far the Drawing-School established under the Inspection of the Dublin-Society put on a proper Footing, has supplied this Defect,

To remove so general and well grounded a Complaint, it was proposed that the Youth of this Kingdom should receive in the Drawing-School established by the DUBLIN-SOCIETY, the Instructions necessary to enable them to become Proficients in the different Branches of that Art, and to pursue with Success, geographical, nautical, mechanical, commercial or military Enquiries: in this View, an Abstract of the following Plans were delivered to their Secretaries and Treasurer in the Month of October, 1764, to be laid before the Society; and to prevent an Undertaking of National Utility, to be defeated through the Suggestions of Design or Ignorance, the Plans were printed; which being received by the Public with general Approbation, the DUBLIN-SOCIETY, pursuant to the Report of their Committee appointed to examine into the Merit of the Plans, and the Character of the Proposer, resolved, the 4th of February, 1768, that they should be carried into Execution by the Author, under their immediate Inspection.

The PLANS are as follow.

I.

PLAN of a Course of pure Mathematicks, absolutely necessary for the right understanding any Branches of practical Mathematicks in their Application to geographical, nautical, mechanical, commercial, and military Enquiries.

II.

PLAN of the physical and moral System of the World, including the Instructions relative to young Noblemen and Gentlemen of Fortune.

III.

PLAN of the military Art, including the Instructions relative to Engineers, Gentlemen of the Artillery, and, in general, to all Land-Officers.

IV.

PLAN of the merchantile Arts, or the Instructions relative to those who are intended for Trade.

PLAN

M A T H E M A T I C K S.

V

V.

PLAN of the naval Art, including the Instructions relative to Ship-Builders, Sea-Officers, and to all those concerned in the Business of the Sea.

VI.

PLAN of a School of Mechanic Arts, where all Artists, such as Architects, Painters, Sculptors, Engravers, Clock-makers, &c. receive the Instructions in Geometry, Perspective, Statics, Dynamicks, Physics, &c. which suit their respective Professions, and may contribute to improve their Taste and their Talents.

The Youth of this Kingdom destitute of the most important Means of Instruction.

Those PLANS have convinced the Noblemen and Gentlemen of Fortune of this Kingdom, that their Children, and in general, the Youth of this Country, were destitute of the most important Means of Instruction, and would ever be destitute of them, until they had resolved that Men of Genius and Education should be encouraged to appear as Teachers.

PLAN of a Course of pure Mathematicks, absolutely necessary for the right understanding any Branches of practical Mathematicks in their Application to geographical, nautical, mechanical, commercial, and military Inquiries.

Vix quicquam in universa Matheſi ita difficile aut arduum occurrere poſſe, quo non inoffenſo Pede per hanc Methodum penetrare liceat.

I.

PURE Mathematicks comprehend Arithmetick, and Geometry. Practical Mathematicks, their Application to particular Objects, as the Laws of Equilibrium, and Motion of solid and fluid Bodies, the Motion of the heavenly Bodies, &c. they extend to all Branches of human Knowledge, and strengthening our intellectual Powers, by forming in the Mind an Habit of Thinking closely, and Reasoning accurately, serve to bring to Perfection, with an entire Certitude, all Arts which Man can acquire by his Reason alone. It is therefore of the highest Importance, that the Youth * of this Country should be methodically brought acquainted with a Course of pure Mathematicks, to serve as an Introduction to such Branches of Knowledge as are requisite to qualify them for their future Stations in Life. The Noblemen and Gentlemen of Fortune, therefore, have unanimously resolved, that such a Course should be given on the most approved Plan, in the DRAWING SCHOOL established under their Inspection, by a Person, who, on account of the Readiness and Knowledge he has acquired in these Matters, during the many Years that he has made them his principal Occupation, is qualified for making the Entry to those abstruse Sciences, accessible to the meanest Capacity.

Utility of the Mathematicks

* The proper Age to commence this Course is 14.

II.

Method of
teaching Ma-
thematics.

The Synthe-
tick Method
should not
extend fur-
ther than the
simple Ele-
ments.

The Anali-
tick Method
is the Key of
all mathema-
tical Discove-
ries.

As to the Method of teaching Mathematicks, the synthetic Method being necessary to discover the principal Properties of geometrical Figures, which cannot be rightly deduced but from their Formation, and suiting Beginners, who, little accustomed to what demands a serious Attention, stand in Need of having their Imagination helped by sensible Objects, such as Figures, and by a certain Detail in the Demonstrations, is followed in the Elements (a). But as this Method, when applied to any other Research, attains its Point, but after many Windings and perplexing Circuits, viz. by multiplying Figures, by describing a vast many Lines and Arches, whose Position and Angles are carefully to be observed, and by drawing from these Operations a great Number of incidental Propositions which are so many Accessaries to the Subject; and very few having Courage enough, or even are capable of so earnest an Application as is necessary to follow the Thread of such complicated Demonstrations: afterwards a Method more easy and less fatiguing to the Attention is pursued. This Method is the analitic Art, the ingenious Artifice of reducing Problems to the most simple and easiest Calculations that the Question proposed can admit of; it is the universal Key of Mathematicks, and has opened the Door to a great Number of Persons, to whom it would be ever shut, without its Help; by its Means, Art supplies Genius, and Genius, aided by Art so useful, has had Successes that it would never have obtained by its own Force alone; it is by it that the Theory of curve Lines have been unfolded, and have been distributed in different Orders, Classes, Genders, and Species, which as in an Arsenal, where Arms are properly arranged, puts us in a State of chusing readily those which serve in the Resolution of a Problem proposed, either in Mathematicks, Astronomy, Opticks, &c. It is it which has conducted the great Sir *Isaac Newton* to the wonderful Discoveries he has made, and enabled the Men of Genius, who have come after him, to improve them. The Method of Fluxions, both direct and inverse, is only an Extention of it, the first be-

(a) It is for these Reasons that in all the public mathematical Schools established in England, Scotland, &c. the Masters commence their Courses by the Elements of Geometry; we shall only instance that of Edinburgh, where a hundred young Gentlemen attend from the first of November to the first of August, and are divided into five Classes, in each of which the Master employs a full Hour every Day. In the first or lowest Class, he teaches the first six Books of Euclid's Elements, plain Trigonometry, practical Geometry, the Elements of Fortification, and an Introduction to Algebra. The second Class studies Algebra, the 11th and 12th Books of Euclid, spherical Trigonometry, conic Sections, and the general Principles of Astronomy. The third Class goes on in Astronomy and Perspective, read a Part of Sir *Isaac Newton's Principia*, and have a Course of Experiments for illustrating them, performed and explained to them: the Master afterwards reads and demonstrates the Elements of Fluxions. Those in the fourth Class read a System of Fluxions, the Doctrine of Chances, and the rest of *Newton's Principia*, with the Improvements they have received from the united Efforts of the first Mathematicians of Europe.

ing the Art of finding Magnitudes infinitely small, which are the Elements of finite Magnitudes; the second the Art of finding again, by the Means of Magnitudes infinitely small, the finite Quantities to which they belong; the first as it were resolves a Quantity, the last restores it to its first State; but what one resolves, the other does not always reinstate, and it is only by analitic Artifices that it has been brought to any Degree of Perfection, and perhaps, in Time, will be rendered universal, and at the same Time more simple. What cannot we expect, in this Respect, from the united and constant Application of the first Mathematicians in *Europe*, who, not content to make use of this sublime Art, in all their Discoveries, have perfected the Art itself, and continue so to do.

This Method has also the Advantage of Clearness and Evidence, and the Brevity that accompanies it every where does not require too strong an Attention. A few Years moderate Study suffices to raise a Person, of some Talents, above these Geniuses who were the Admiration of Antiquity; and we have seen a young Man of Sixteen, publish a Work, (*'Traite' des Courbes à double Courbure par Clairaut*) that *Archimedes* would have wished to have composed at the End of his Days. The Teacher of Mathematicks, therefore, should be acquainted with the different Pieces upon the analitic Art, dispersed in the Works of the most eminent Mathematicians, make a judicious Choice of the most general and essential Methods, and lead his Pupils, as it were, by the Hand, in the intricate Roads of the Labyrinth of Calculation; that by this Means Beginners, exempted from that close Attention of Mind, which would give them a Distaste for a Science they are desirous to attain, and methodically brought acquainted with all its preliminary Principles, might be enabled in a short Time, not only to understand the Writings of the most eminent Mathematicians, but, reflecting on their Method of Proceeding, to make Discoveries honourable to themselves and useful to the Public.

Has the Advantage of Clearness, Evidence, and Brevity.

III.

Arithmetick comprehends the Art of Numbering and Algebra, consequently is distinguished into particular and universal Arithmetick, because the Demonstrations which are made by Algebra are general, and nothing can be proved by Numbers but by Induction. The Nature and Formation of Numbers are clearly stated, from whence the Manner of performing the principal Operations, as Addition, Subtraction, Multiplication and Division are deduced. The Explication of the Signs and Symbols used in Algebra follow, and the Method of reducing, adding, subtracting, multiplying, dividing, algebraic Quantities simple and compound. This prepares the Way for the Theory of vulgar, algebraical, and decimal Fractions, where the Nature, Value, Man-

How Arithmetick numeral and Specious is treated.

The Art of
solving Equa-
tions.

The Nature
and Laws of
Chance.

Manner of comparing them, and their Operations, are carefully unfolded. The Composition and Resolution of Quantities comes after, including the Method of raising Quantities to any Power, extracting of Roots, the Manner of performing upon the Roots of imperfect Powers, radical or incommensurable Quantities, the various Operations of which they are susceptible. The Composition and Resolution of Quantities being finished, the Doctrine of Equations presents itself next, where their Genesis, the Nature and Number of their Roots, the different Reductions and Transformations that are in Use, the Manner of solving them, and the Rules imagined for this Purpose, such as Transposition, Multiplication, Division, Substitution, and the Extraction of their Roots, are accurately treated. After having considered Quantities in themselves, it remains to examine their Relations; this Doctrine comprehends arithmetical and geometrical Ratios, Proportions and Progressions: The Theory of Series follow, where their Formation, Methods for discovering their Convergency, or Divergency, the Operations of which they are susceptible, their Reversion, Summation, their Use in the Investigation of the Roots of Equations, Construction of Logarithms, &c. are taught. In fine, the Art of Combinations, and its Application for determining the Degrees of Probability in civil, moral and political Enquiries are disclosed. *Ars cujus Usus et Necessitas ita universale est, ut sine illa, nec Sapientia Philosophi, nec Historici Exactitudo, nec Medici Dexteritas, aut Politici Prudentia, consistere queat. Omnis enim horum Labor in conjectando, et omnis Conjectura in Trutinandis Causarum Complexionibus aut Combinationibus versatur.*

IV.

Division of
Geometry
into Element-
ary, Trans-
cendental
and Su-
blime.

GEOMETRY is divided into ELEMENTARY, TRANSCENDENTAL and SUBLIME.

To open to Youth an accurate and easy Method for acquiring a Knowledge of the Elements of Geometry, all the Propositions in *Euclid* (a) in the Order they are found in the best Editions, are retained with

(a) "Perspicuity in the Method and Form of Reasoning, is the peculiar Characteristic of Euclid's Elements, not as interpolated by Campanus and Clavius, anatomised by Herigone and Barrow, or depraved by Tacquet and Deschales, but of the Original, handed down to us by Antiquity. His Demonstrations being conducted with the most express Design of reducing the Principles assumed to the fewest Number, and most evident that might be, and in a Method the most natural, as it is the most conducive towards a just and complete Comprehension of the Subject, by beginning with such Particulars as are most easily conceived, and flow most readily from the Principles laid down; thence by gradually proceeding to such as are more obscure, and require a longer Chain of Argument, and have thence been regarded in all Ages, as the most perfect in their Kind." Such is the Judgment of the ROYAL SOCIETY, who have express'd at the same Time their Dislike to the new modelled Elements that at present every where abound; and to the illiberal and mechanic Methods of teaching those most perfect Arts; which is to be hoped, will never be countenanced in the Public Schools in England and Scotland, &c.

all possible Attention, as also the Form, Connection and Accuracy of his Demonstrations. The essential Parts of his Propositions being set forth with all the Clearness imaginable, the Sense of his Reasoning are explained and placed in so advantageous a Light, that the Eye the least attentive may perceive them. To render these Elements still more easy, the different Operations and Arguments essential to a good Demonstration, are distinguished in several separate Articles; and as Beginners, in order to make a Progress in the Study of Mathematicks, should apply themselves chiefly to discover the Connection and Relation of the different Propositions, to form a just Idea of the Number and Qualities of the Arguments, which serve to establish a new Truth; in fine, to discover all the intrinsical Parts of a Demonstration, which it being impossible for them to do without knowing what enters into the Composition of a Theorem and Problem, First, The Preparation and Demonstration are distinguished from each other. Secondly, The Proposition being set down, what is supposed in this Proposition is made known under the Title of Hypothesis, and what is affirmed, under that of Thesis. Thirdly, All the Operations necessary to make known Truths, serve as a Proof to an unknown one, are ranged in separate Articles. Fourthly, The Foundation of each Proposition relative to the Figure, which forms the Minor of the Argument, are made known by Citations, and a marginal Citation recalls the Truths already demonstrated, which is the Major: In one Word, nothing is omitted which may fix the Attention of Beginners, make them perceive the Chain, and teach them to follow the Thread of geometrical Reasoning.

Methodical Order in which the Elements of Euclid are digested.

V.

Transcendental Geometry presupposes the algebraic Calculation; it commences by the Solution of the Problems of the second Degree by Means of the Right-line and Circle: This Theory produces important and curious Remarks upon the positive and negative Roots, upon the Position of the Lines which express them, upon the different Solutions that a Problem is susceptible of; from thence they pass to the general Principles of the Application of Algebra to curve Lines, which consist, First, In explaining how the Relation between the Ordinates and Abcisses of a Curve is represented by an Equation. Secondly, How by solving this Equation we discover the Course of the Curve, its different Branches, and its Asymptots. Thirdly, The Manner of finding by the direct Method of Fluxions, the Tangents, the Points of Maxima, and Minima. Fourthly, How the Areas of Curves are found by the inverse Method of Fluxions.

Transcendental Geometry.

In what it consists.

The Conic Sections follow; the best Method of treating them is to consider them as Lines of the second Order, to divide them into their Species. When the most simple Equations of the Parabola,

Best Method of treating Conic Sections.

Ellipse, and Hyperbola are found, then it is easily shewn that these Curves are generated in the Cone. The Conic Sections are terminated by the Solution of the Problems of the third and fourth Degree, by the Means of these Curves.

The different Orders of Curves.

The Conic Sections being finished, they pass to Curves of a superior Order, beginning by the Theory of multiple Points, of Points of Inflection, Points of contrary Inflection, of Serpement, &c. These Theories are founded partly upon the simple algebraic Calculation, and partly on the direct Method of Fluxions. Then they are brought acquainted with the Theory of the Evolute and Causiques by Reflection and Refraction. They afterwards enter into a Detail of the Curves of different Orders, assigning their Classes, Species, and principal Properties, treating more amply of the best known, as the Folium, the Conchoid, the Cissoid, &c.

The mechanic Curves follow the geometrical ones, beginning by the exponential Curves, which are a mean Species between the geometrical Curves and the mechanical ones; afterwards having laid down the general Principles of the Construction of mechanic Curves, by the Means of their fluxional Equations, and the Quadrature of Curves, they enter into the Detail of the best known, as the Spiral, the Quadratrice, the Cycloid, the Trochoid, &c.

VI.

Sublime Geometry.

Sublime Geometry comprehends the inverse Method of Fluxions, and its Application to the Quadrature, and Rectification of Curves, the cubing of Solids, &c.

Its Division.

Fluxional Quantities, involve one or more variable Quantities; the natural Division therefore of the inverse Method of Fluxions is into the Method of finding the Fluents of fluxionary Quantities, containing one variable Quantity, or involving two or more variable Quantities; the Rule for finding the Fluents of fluxional Quantities of the most simple Form, is laid down, then applied to different Cases, which are more composed, and the Difficulties which some Times occur, and which embarrass Beginners, are solved.

What the first Part comprehends.

These Researches prepare the Way for finding the Fluents of fluxional Binomials, and Trinomials, rational Fractions, and such fluxional Quantities as can be reduced to the Form of rational Fractions; from thence they pass to the Method of finding the Fluents of such fluxional Quantities which suppose the Rectification of the Ellipse and Hyperbola, as well as the fluxional Quantities, whose Fluents depend on the Quadrature of the Curves of the third Order; in fine, the Researches which Mr. Newton has given in his Quadrature of Curves, relative to the Quadrature of Curves whose Equations are composed of three or four Terms;

and this first Part is terminated by the Methods of finding the Fluents of fluxional, logarithmetical, and exponential Quantities, and those which are affected with many Signs of Integration, and the various Methods of Approximation, for the Solution of Problems, which can be reduced to the Quadrature of Curves.

The second Part of the inverse Method of Fluxions, which treats of fluxional Quantities, including two or more variable Quantities, commences by shewing how to find the Fluents of such fluxional Quantities as require no previous Preparation; the Methods for knowing and distinguishing these Quantities or Equations; afterwards they pass to the Methods of finding the Fluents of fluxional Quantities, which have need of being prepared by some particular Operation, and as this Operation consists most commonly in separating the indeterminate Quantities, after being taught how to construct differential Equations, in which the indeterminate Quantities are separated, they enter into the Detail of the different Methods for separating the variable Quantities in a proposed Equation, either by Multiplication, Division, or Transformations, being shewed their Application, first to homogeneous Equations, and after being taught how to construct these Equations in all Cases, the Manner of reducing Equations to their Form is then explained. How the Method of indeterminate Co-efficients can be employed for finding the Fluents of fluxional Equations, including a certain Number of variable Quantities, and how by this Method, the Fluent can be determined by certain Conditions given of a fluxional Equation. Fluxional Quantities of different Orders follow; it is shewn, first, that fluxional Equations of the third Order, have three Fluents of the second Order, but the last Fluent of a fluxionary Equation of any Order is simple; then the various Methods imagined by the most eminent Mathematicians for finding these Fluents, supposing the Fluxion of any one variable Quantity constant, are explained, and the Whole, in fine, terminated by the Application of this Doctrine to the Quadrature and Rectification of Curves, Cubing of Solids, &c.

What the second Part comprehends.

VII.

Such is the Plan of a Course of pure Mathematicks traced by *Newton*, improved by *Cotes*, *Bernoulli*, *Euler*, *Clairaut*, *D'Alembert*, *M'Laurin*, *Simpson*, *Fontain*, * &c. which serves as a Basis to the Instructions requisite to qualify Youth to appear with Dignity in the different Employments of Life, or to enable them in Time, to bring to Perfection the various Arts for which they are intended.

* *Quadratura curvarum, harmonia mensurarum, &c.*

PLAN of the System of the Physical and Moral World, including the Instructions relative to young Noblemen and Gentlemen of Fortune.

PLAN of the System of the Physical World.

————— *Nubem pellente matthesi,
Claustra patent cæli, rerumque immobilis ordo:
Jam superum penetrare domos, atque ardua cæli
Scandere, sublimis genii concessit acumen.*

I.

Utility of
the Study
of the Sy-
stem of the
World.

Is a Pre-
servative
against the
Passions.

Leads to
Virtue.

STUDY in general is necessary to Mankind, and essentially contri-
butes to the Happiness of those who have experienced that active
Curiosity which induceth them to penetrate the Wonders of Nature.
It is, besides, a Preservative against the Disorders of the Passions; a
kind of Study therefore which elevates the Mind, which applies it
closely, consequently, which furnishes the most assured, arms against
the Dangers we speak of, merits particular Distinction. "It is not
" sufficient, says *Seneca*, to know what we owe to our Country, to our
" Family, to our Friends, and to ourselves, if we have not Strength of
" Mind to perform those Duties, it is not sufficient to establish Precepts,
" we must remove Impediments, *ut ad præcepta quæ damus possit animus*
" *ire, solvendus est.* (Epist. 95.) Nothing answers better this Purpose
than the Application to the Study of the System of the World; the
Wonders which are discovered captivate the Mind, and occupy it in a
noble Manner; they elevate the Imagination, improve the Understand-
ing, and satiate the Heart: The greatest Philosophers of Antiquity
have been of this Opinion. *Pythagoras* was accustomed to say, that
Men should have but two Studies, that of Nature, to enlighten their
Understandings, and of Virtue to regulate their Hearts; in effect to be-
come virtuous, not through Weakness but by Principle, we must be
able to reflect and think closely; we must by dint of Study be delivered
from Prejudices which makes us err in our Judgments, and which are
so many Impediments to the Progress of our Reason, and the Improve-
ment of our Mind. *Plato* held the Study of Nature in the highest
Esteem; he even goes so far as to say, that Eyes were given to Man to
contemplate the Heavens: To which alludes the following Passage of
Ovid.

*Finxit in effigiem moderantum cuncta deorum,
Pronaque cum spectant animalia cetera terram,
Os homini sublime dedit, cælumque tueri
Jussit, et erectos ad sidera tollere vultus.*

II.

The Poets who have illustrated *Greece* and *Italy*, and whose Works are now sure of Immortality, were perfectly acquainted with the Heavens, and this Knowledge has been the Source of many Beauties in their Works: *Homer, Hesiod, Aratus*, among the *Greeks*: *Horace, Virgil, Ovid, Lucretius, Manilius, Lucan, Claudian*, among the *Latins*; make use of it in several Places, and have expressed a singular Admiration for this Science.

Is celebrated
by the
Poets.

Ovid after having announced in his *Fasti*, that he proposes celebrating the Principles on which the Division of the Roman Year is founded, enters on his Subject by the following pompous Elogium of the first Discoverers of the System of the World.

*Felices animos, quibus hæc cognoscere primis,
Inque domos superas scandere cura fuit,
Credibile est illos pariter vitiiisque locisque,
Altius humanis exeruisse caput.
Non venus aut vinum sublimia pectora fregit,
Officiumve fori militiæque labor,
Nec levis ambitio perfusaque gloria fuco,
Magnarumve fames sollicitavit opum.
Admovere oculis distantia sidera nostris,
Ætheraque ingenio supposuere suo.
Sic petitur cælum.*

Claudian in the following Verses, celebrates *Archimedes* on his Invention of a Sphere admirably contrived to represent the celestial Motions.

*Jupiter in parvo cum cerneret æthera vitro,
Risit, et ad superos talia dicta dedit:
Hucine mortalis progressa potentia curæ!
Jam meus in fragili luditur orbe labor.
Jura poli, rerumque fidem legesque deorum
Ecce Syracusius transtulit Arte senex;
Inclusus Variis famulatur spiritus astris,
Et vivum certis motibus urget opus;
Percurrit proprium mentitus signifer, annum,
Et simulata novo Cynthia mense redit:
Jamque suum volvens audax industria mundum
Gaudet, et humana sidera mente regit.*

Virgil seems desirous of renouncing all other Study, to contemplate the Wonders of Nature.

*Me vero primum dulces ante omnia musæ,
Quarum sacra fero ingenti percussus amore,
Accipiant, cœlique vias et sydera monstrent
Defectus solis varios, lunæque labores,
Unde tremor terris, qua vi maria alta tumescant
Objicibus ruptis, rursusque in seipsa residunt,
Quid tantum oceano properent se tingere soles
Hyberni, vel quæ tardis mora noctibus obstet
Felix qui potuit rerum cognoscere causas.*

GEOR. II. 475.

La Fontaine imitates the Regrets of *Virgil* in a masterly Manner, where he says,

*Quand pourront les neuf sœurs loin des cours et des villes,
M'occuper tout entier, et m'apprendre des cieux
Les divers mouvements inconnus à nos yeux,
Les noms et les vertues de ces clartés errantes.*

Songe dun habitant du Mogol.

Voltaire, the first Poet of our Age, has testified in many Parts of his Works, his Taste for Astronomy, and his Esteem for Astronomers, whom he has celebrated in the finest Poetry. What he says of *Newton* is worthy of Attention.

*Confidens du Tres Haut, Substances eternelles,
Qui parez de vos feux, qui couvrez des vos ailes,
Le trone ou votre maitre est assis parmi vous :
Parles ! du grand NEWTON n'etiez vous point jaloux.*

To which we can only oppose what *Pope* has said on the same Subject :

Nature and Nature's Laws lay hid in Night ;
God said, Let *Newton* be, and all was Light.

The great Geniuses of every Species have been surprized at the Indifference which Men shew for the Spectacle of Nature. *Tasso* puts Reflections in the Mouth of *Rinaldo*, which merit to be recited for the Instruction of those to whom the same Reproach may be applied ; it is at the Time when marching before Day towards *Mount Olivet*, he contemplates the Beauty of the Firmament.

Con gli occhi alzati contemplando intorno,
 Quinci notturne e quindi matutine
 Bellezze, incorruttibili e divine;
 Fra se stesso pensava, o quanto belle
 Luci, il tempio celeste in se raguna!
 Ha il suo gran carro il de, l'aurata stelle
 Spiega la notte, e l'argentata Luna;
 Ma non è chi vagheggi o questa, o quelle;
 E miriam noi torbida luce e bruna,
 Ch'un girar d'occhi, un balenar di riso
 Scopre in breve confin di fragil viso.

JERUS. Cant. xviii. St. 12, 13.

III.

The Knowledge of the System of the World has delivered us from Effects the Apprehensions which Ignorance occasions; can we recal without which the Ignorance of the System of the World has produced. Compassion, the Stupidity of those People, who believed that by making a great Noise when the Moon was eclipsed, this Goddess received Relief from her Sufferances, or that Eclipses were produced by Inchantments (a)?

Cum frustra resonant Æra auxiliaria Lunæ. Met. iv. 333.
Cantus et e Curru Lunam deducere tentant,
Et faceret si non Æra repulsa sonent. Tib. El. 8.

The Knowledge of the System of the World has dissipated the Errors of The Knowledge of the System of the World has dissipated the Errors of Astrology, by whose foolish Predictions Mankind had been so long abused. The Adventure of 1186, should have covered with Shame the Astrologers of Europe; they were all, Christians, Jews and Arabians, united to announce, seven Years before, by Letters published throughout Europe, a Conjunction of all the Planets, which would be attended with such terrible Ravages, that a general Dissolution of Nature was much to be dreaded, so that nothing less than the End of the World was expected: this Year notwithstanding passed as others. But a hundred Lies, each as well attested, would not be sufficient to wain ignorant and credulous Men from the Prejudices of their Infancy. It was necessary that a Spirit of Philosophy, and Research, should spread itself among Mankind, open their Understandings, unveil the Limits of Nature, and accustom them not to be terrified without Examination, and without Proof.

IV.

The Comets, as it is well known, were one of the great Objects of Terror which the Knowledge of the System of the World has, in fine,

(a) Seneca, Hipolit. 787. Tacit. Ann. Plutarch in Pericle, et de defectu Oraculorum.

removed. It is not without Concern we find such strange Prejudices in the finest Poem of the last Age, whereby they are transmitted to the latest Posterity.

*Qual colle cbiome sanguinose borende,
Splender cometa suol per l'aria adusta,
Che i regni muta, ei fieri morti adduce,
Ai purperei tiranni infauusta luce.* JERUS. Lib. 7. St. 52.

The Charms of Poetry are actually employed in a Manner more philosophical and useful, witness the following fine Passage.

*Cometes que l'on craint à legal du tonnerre,
Cessez d'epouvanter les peuples de la terre;
Dans une Ellipse immense achevez votre cours;
Remontez, descendez pres de l'astre des jours;
Lancez vos feux, volez, et revenant sans cesse,
Des mondes epuisez ranimez la vielleffe.*

Thus the profound Study of the System of the World has dissipated absurd Prejudices, and re-established human Reason in its inalienable Rights.

V.

The Knowledge of the System of the World useful in Geography and Navigation, and consequently of the greatest Importance to these Kingdoms.

To the Knowledge of the System of the World, are owing the Improvements in Cosmography, Geography, and Navigation; the Observation of the Height of the Pole, taught Men that the Earth was round, the Eclipses of the Moon taught how to determine the Longitudes of the different Countries of the World, or their mutual Distances from East to West. The Discovery of the Satellites of *Jupiter*, has contributed more effectually to improve geographical or marine Charts, than ten thousand Years Navigation; and when their Theory will be better known, the Method of Longitudes will be still more exact and more easy. The Extent of the *Mediterranean* was almost unknown in 1600, and To-Day, is as exactly determined as that of *England* or *Ireland*. By it the new World was discovered. *Christopher Columbus* had a more intimate Knowledge of the Sphere, than any Man of his Time, since it gave him that Certainty, and inspired him with that Confidence with which he directed his Course towards the West, certain to rejoin by the East the Continent of *Asia*, or to find a new one. And nothing seems to be wished for, to render Navigation more perfect and secure, but a Method for finding with Ease, the Longitude at Sea, which is now obtained by the Means of the Moon: And if the Navigators of this Kingdom were initiated in Astronomy, by able Teachers, as is practised

in other Parts of *Europe*, their Estimation would approach within twenty Miles of the Truth, whilst in ordinary Voyages, the Uncertainty amounts to more than three hundred Leagues, by which the Lives and Fortunes of Thousands are endangered. The Utility therefore of the Marine to those Kingdoms, where Empire, Power, Commerce, even Peace and War, are decided at Sea, proves that of the Knowledge of the System of the World.

VI.

The actual State of the Laws, and of the ecclesiastical Administration, is essentially connected with the System of the World; St. *Augustine* recommended the Study of it particularly for this Reason; St. *Hyppolite* applied himself to it, as also many Fathers of the Church, notwithstanding our Kalendar was in such a State of Imperfection, that the *Jews* and *Turks* were astonished at our Ignorance. *Nicholas V*, *Leon X*, &c. had formed a Design of re-establishing Order in the Kalendar, but there were at that Time no Philosophers, whose Reputation merited sufficient Confidence. *Gregory* the XIIIth, governed at a Time when the Sciences began to be cultivated, and he alone had the Honour of this Reformation.

The Reformation of the Kalendar depended on it.

VII.

Agriculture borrowed formerly from the Motions of the celestial Bodies, its Rules and its Indications; *Job*, *Hesiod*, *Varro*, *Eudoxus*, *Aratus*, *Ovid*, *Pliny*, *Columella*, *Manilius*, furnish a thousand Proofs of it. The *Pleyades*, *Arcturus*, *Orion*, *Syrius*, gave to *Greece* and *Egypt* the Signal of the different Works; the rising of *Syrius* announced to the *Greeks* the Harvest; to the *Egyptians* the overflowing of the *Nile*. The Kalendar answers this Purpose exactly.

Is useful in Agriculture

VIII.

Ancient Chronology deduces from the Knowledge and Calculation of Eclipses, the most fixed Points which can be found, and in remote Times we find but Obscurity. The *Chinese* Chronology is entirely founded upon Eclipses, and we would have no Uncertainty in the ancient History of Nations as to the Dates, if there were always Philosophers. (See the Art of verifying Dates.)

Is the Foundation of Chronology

IX.

It is from the System of the World we borrow the Division of Time, and the Art of regulating Clocks and Watches; and it may be said, that the Order and Multitude of our Affairs, our Duties, our Amusements, our Taste, for Exactness and Precision, our Habitudes have rendered this Measure of Time almost indispensable, and has placed it in the Number of the Necessaries of Life; if instead of Clocks and Watches, Meridians and solar Dials are traced, it is an Advantage that the Knowledge of the System of the World has procured us, Dial-

Furnishes the Means of measuring Time.

ling being the Application of spherical Trigonometry; a Projection of the Sphere upon a Plane, or a Section of a Cone, according to the Forms given to a Dial.

X.

Is useful in
Physick.

The Knowledge of the Changes of the Air, Winds, Rain, dry Weather, Motions of the Thermometer, Barometer, have certainly an essential and immediate Relation with the Health of the human Body; the Knowledge of the System of the World will be of sensible Utility, when, by repeated Observations, the physical Influences of the Sun and Moon upon the Atmosphere, and the Revolutions which result will be discovered. *Galen* advises the Sick not to call to their Assistance Physicians, who are not acquainted with the Motions of the celestial Bodies, because Remedies given at unseasonable Times are useless or hurtful, and the ablest Physicians of our Days are convinced, that the Attractions which elevate the Waters of the Ocean twice a Day, influence the State of the Atmosphere, and that the Crisis and Paroxysms of Disorders correspond with the Situation of the Moon in respect of the Equator, Sygies, and Apfides. See *Mead, Hofman, &c.*

XI.

Cultivated
in all Ages
by all the
civilized Na-
tions of the
World.

Those Advantages which result from the Knowledge of the System of the World, has caused it to be cultivated and held in singular Esteem by all the civilized People of the Earth. The ancient Kings of *Persia*, and the Priests of *Egypt*, were always chosen amongst the most expert in this Science. The Kings of *Lacedemon* had always Philosophers in their Council. *Alexander* was always accompanied by them in his military Expeditions, and *Aristotle* gave him strict Charge to do nothing without their Advice. It is well known how much *Ptolemeus* the second King of *Egypt*, encouraged this Science; in his Time flourished *Hyparchus*, *Calimachus*, *Apollonius*, *Aratus*, *Bion*, *Theocrites*, *Conon*. *Julius Caesar* was very curious in making Experiments and Observations, as it appears by the Discourse which *Lucan* makes him hold with *Achore* Priest of *Egypt*, at the Feast of *Cleopatra*.

Media inter prelia semper
Stellarum cœlique plagis superisque vacavi,
Nec meus Eudoxi vincetur fastibus annus. PHAR.

Has been
the favorite
Study of
great
Princes.

The Emperor *Tiberius* applied himself to the Study of the System of the World, as *Suetonius* relates; the Emperor *Claudius* foresaw there would be an Eclipse the Day of his Anniversary, and fearing it might occasion Commotions at *Rome*, he ordered an Advertisement to be published, in which he explains the Circumstances, and the Causes of this Phenomenon. It was cultivated particularly by the Emperors *Adrian*

and *Severus*, by *Charlemagne*, by *Leon V*, Emperor of *Constantinople*, by *Alphonso X*, King of *Castile*, by *Frederick II*, Emperor of the * *West*, by *Calife Almamon*, the Prince *Ulubeigh*, and many other Monarchs of *Asia*.

Among the Heroes who also cultivated it, are reckoned *Mabomet II*, Conqueror of the *Greek Empire*; the Emperor *Charles V*, and *Lewis XIV*. In fine, the Establishments of different Philosophical Societies in *England*, *Scotland*, *France*, *Italy*, *Germany*, *Poland*, *Sweden*, *Russia*, &c. have given the Monarchs, Nobility, and Gentry of those Countries, a Taste for the more refined Pleasures attending the Study of the Sciences, and particularly of the System of the World, an Example worthy to be imitated by those of this Kingdom.

XII.

Besides those renowned Societies which have all contributed to the Progress of every Branch of human Knowledge, and particularly of the System of the World, there has been established in the different Parts of *Europe* public Schools, conducted by Men of superior Talents and Abilities, who make it their Business to guide and instruct the young Nobility and Gentry in this noble Science, and furnish those who discover singular Dispositions with every Means of Improvement.

An illustrious *Englishman*, *Henry Saville*, founded in the University of *Oxford* two Schools, which have been of vast Utility to *England*; the Masters have been Men all eminent in this Science, *John Bainbridge* in 1619, *John Greaves* in 1643, *Seth Ward*, *Christopher Wren*, *Edward Bernard* in 1673, *David Gregory* in 1691, *Briggs*, *Wallis*, and *J. Caswell* in 1708, *Keill* in 1712, *Hornsby*, &c.

The Schools established at *Cambridge*, among whose Masters were *Barrow*, *Newton*, *Cotes*, *Wiston*, *Smyth*, and *Long*, all celebrated Astronomers.

The School of *Gresham* at *Bishops-Gate* in *London*, which has essentially contributed to the Progress of Astronomy; among the Masters of this School were Doctor *Hook*, and other eminent Men.

The Royal mathematical School at *Christ's-Hospital*, where *Hodgson*, *Robertson*, &c. have bred up a great Number of expert Navigators and Astronomers.

The Schools of *Edinburgh*, *Glasgow*, and *Aberdeen*, are known all over *Europe*; the Nobility, and Gentlemen of Fortune of *Scotland*, superintending them, and taking every Method of encouraging both Masters and Students to Assiduity and Attention, to go through their respective Tasks with Alacrity and Spirit; the Names of *Gregory*, *M^cLaurin*, *Stuart*, *Simpson*, &c. the famous Masters, will never be forgotten.

Publick Schools established in the different Parts of Europe for instructing young Noblemen and Gentle men of Fortune in what regards the System of the World.

Foundation of Henry Saville.

Foundations of Lownds and-Lucas. College of Gresham.

Mathematical School of Christ's Hospital.

Mathematical Schools in Scotland.

* He ordered the Works of Ptolemy to be translated into Latin, and publickly to be taught at Naples.

The Royal
College.

The Royal School of *France*, founded by *Francis I.*, has essentially contributed to the Progress of the Knowledge of the System of the World. *Orance, Fine, Stadius, Morin, Gassendi, de la Hire, de Lisle*, who were successively Masters of it, have been celebrated Astronomers, &c.

XIII.

Observato-
ries and
Schools of
Experimen-
tal Philoso-
phy.

Experiments and Observations are the Foundation of all real Knowledge, those which serve as a Basis to the Discoveries relative to the System of the World, are made and learned in Experimental Schools and Observatories: The first Observatory of any Celebrity, was built by *William V.* Landgrave of *Hesse*, where he collected all the Instruments, Machines, Models, &c. which were known in his Time, and put it under the Direction of *Rothman* and *Byrgius*, the first an Astronomer, the second an expert Instrument-Maker: The Duke of *Broglie*, General of the *French* Army, having rendered himself Master of *Cassel* in 1760, took a Copy of the Observations and Experiments made in this Observatory, and deposited it in the Library of the Academy.

Of Cassel.

Of Urani-
bourg.

Frederick I. King of *Denmark*, being informed of the singular Merit of *Ticho Brabe*, granted him the Island of *Venusia*, opposite *Copenhagen*, and built for him the Castle of *Uranibourgh*, furnished it with the largest, and the most perfect Instruments, and gave Pensions to a Number of Observers, Calculators, and Experiment-Makers, to assist him, which enabled him in the Space of 16 Years, to lay the Foundation of the System of the World, in a Manner more stable, than was ever before effected. The most eminent Men took Pleasure in visiting this incomparable Philosopher: The King of *Scotland* going to espouse the Princess *Anne*, Sister of the King of *Denmark*, passed into the Island of *Venusia* with all his Court, and was so charmed at the Operations and Success of *Tycho*, that he composed his Elogium in *Latin* Poetry: So much Merit raised him Enemies, and the Death of King *Frederick II.* furnished them the Means of succeeding in their Machinations. A Minister called *Walchendorp*, (whose Name should be devoted to the Execration of the Learned of all Ages) deprived him of his Island of *Venusia*, and forbid him to continue at *Copenhagen* his Experiments and Observations.

XIV.

Of Dantzick

The first Observatory of the last Age, was that of *Hevelius*, established at *Dantzick*; it is described in his great Work, intitled, *Machina Celestis*.

Of Copen-
hagen.

The Astronomical Tower of *Copenhagen* was finished in 1656, built by *Christian IV.* at the Solicitation of *Longomontanus*.

Of Pekin.

There has been an Experimental School and Observatory at *Pekin* these 400 Years, built on the Walls of the City: Father *Verbieft* being made President of the Tribunal of Mathematicks in 1669, obtained of the Emperor *Cam-by*, that all the *European* Instruments, Machines,

Models, &c. should be added to those with which it was already furnished. (See the Description of *China* by *Dubald*.) There has been made there a vast Collection of useful Experiments and Observations, a Copy of which is deposited in the *French Academy*.

XV.

The Royal Observatory of *England* was built by *Charles II.* under the Direction of Sir *J. Moore*, four Miles from *London*, to the Eastward upon a high Hill: It will be for ever famous by the immortal Labours of *Flamsteed*, *Halley*, and *Bradley*; *Flamsteed* was put in Possession of this Observatory in 1676, where, during the Space of 33 Years, he made a prodigious Number of Observations contained in his History of the Heavens: *Halley* succeeded him, and was, without Doubt, the greatest Astronomer *England* produced; at the Age of Twenty he went to the Island of *St. Helen*, to form a Catalogue of the Southern Stars, which he published in 1679; then he went to *Dantzick* to confer with *Hewellius*, he travelled also through *Italy* and *France* for his Improvement; in 1683 he published his Theory of the Variation of the Magnetic Needle; in 1686 he superintended the Impression of the *Principia Mathematica Philosophiæ Naturalis*, which its immortal Author could not resolve with himself to publish. The same Year he published his History of the Trade Winds; in 1698 he received the Command of a Vessel to traverse the Atalantic Ocean, and visit the *English* Settlements, in order to discover whether the Variation of the Magnetic Needle, found by Experiment, agreed with his Theory, and to attempt new Discoveries; he advanced as far as 52 Degrees South Latitude, where the Ice impeded his further Progress; he visited the Coast of *Brazil*, the *Canaries*, the Islands of *Cape Verde*, *Barbadoes*, &c. and found every where the Variation of the Compass conformable to his Theory; in 1701 he was commissioned to traverse the *English* Channel, to observe the Tides, and to take a Survey of the Coasts; in 1708 he visited the Ports of *Trieste* and *Bocari* in the Gulph of *Venice*, and repaired the first, accompanied by the chief Engineer of the Emperor; he published in 1705 the Return of the Comets of which he was the first Discoverer; and we have seen in 1759 the Accomplishment of his Prediction; in 1713 he was made Secretary of the Royal Society; he examined the different Methods for finding the Longitude at Sea, and proved that those which depend on the Observations of the Moon were the only practicable ones, and as those Methods required accurate Tables of this Planet, which did not differ from Observation more than two Minutes, he set about rectifying them, having discovered that to obtain this Point it was sufficient to determine, every Day during 18 Years, the Place of the Moon by Observation, and to know how much the Tables differed from it, the Errors every Period afterwards being the same, and returning in the same Order: It was

The Royal Observatory and Experimental School at Greenwich rendered famous by the Labours of *Flamsteed*, *Halley* and *Bradley*.

in 1722 that this courageous Astronomer, in the 65th Year of his Age, undertook this immense Work, and after having completed it, and published the Success of his Labours for foretelling accurately the Moon's Place, and deducing the Longitude at Sea; we lost this great Man the 25th of *January* 1742. *Bradley* succeeded him, who enriched Astronomy with his Discoveries and accurate Observations. He departed this Life the 13th of *July* 1762, in the 70th Year of his Age. *M. Maskeline*, his Successor, continues his Observations with the most active Zeal and happy Dispositions.

Other Observatories and Experimental Schools in England.

The Royal Observatory not being sufficient for all those who pursue the Study of natural Philosophy, there has been formed several Observatories in *London* and the different Parts of *England*, for Example, the Observatory of *Sherburn* near *Oxford*, where the Lord *Maclesfield*, late President of the Royal Society, *M. Hornsby*, &c. have made Experiments and Observations for many Years.

Those of Edinburgh, &c.

The Experimental School and Observatory of *Edinburgh*, built by the Subscription of the Nobility and Gentry of that Kingdom, has been rendered famous by *M^r Laurin*. The Royal Academy of Sciences deputed in 1747 the King's Astronomer, *Le Monier*, to observe there an annular Eclipse of the Sun.

XVI.

The Royal Observatory of Paris.

The Royal Observatory of *Paris*, the most sumptuous Monument that ever was consecrated to Astronomy, was built under the Direction of the great *Colbert*, immortal Protector of the Arts and Sciences. It is near 200 Feet in Front, 140 from North to South, and 100 in Height, the Vaults are near eighty Feet deep; there are also several others in *Paris*, and in other Parts of *France*, as that of *M. Lemonier* at the *Capuchines* of *St. Honore*, that of *M. Delisle* at the *Hotel de Cluny*, that of *M. La Caille* at the College of *Masarin*, that of the Palace of *Luxembourg*, that of *M. de Fouchy* in *Rue des Postes*, and that of *M. Pingre* at *St. Genevieve*; the Observatory of *Marseilles* which *F. Pezenas* has rendered famous, that of *Lyons* where *F. Beraud* made Experiments and Observations for a long Time, that of *Rowen* and *Toulouse* from which *M. Bowin* and *M. Dulange*, *M. d'Auguier* send annually to the Academy a great Number of useful and curious Experiments and Observations; that of *Strasbourg* where *M. Brakenaffer* has made some.

Other Observatories and Experimental Schools in France.

XVII.

Of Nuremberg in 1678.

The Senate of the Republic of *Nuremberg*, erected an Observatory in 1678, and put it under the Direction of *Geo. Christopher Eimmart*. *Phil. Wurzelban* built another in 1692, described in his Book *Uranies Noricæ Basis*. The Administrators of the University of *Leyden*, established in 1690, an Experimental School and Observatory. *Frederick I.* King of *Prussia*, having founded in 1700, an Academy of Sciences at

Of Leiden in 1690.

Berlin, built an Experimental School, with an Observatory. The present King of *Prussia*, added a superb Edifice, where the Academy actually holds its Assemblies. The Institution of *Bologn*, a famous Academy, established in 1709, by the Count of *Marfigli*, with the Permission of *Clement XI.* has a fine Experimental School and Observatory, which *Manfredi* and *Zanotti* have rendered famous. There are four Experimental Schools, with Observatories, at *Rome*; that of *Blanchini*, that of the Convent of *Ara Caeli*, that of the Convent of *Minerva*, and that of *Trinite du Mont*. There is also one at *Genoa*, founded by the Marquis of *Salvagi*; one at *Florence*, which *Ximenes* has rendered famous; one at *Milan*, erected in the College of *Brera*, in 1713. The Superiors of the University of *Altorf*, in the Territory of *Nuremberg*, erected an Experimental School, and an Observatory, and furnished it with all the necessary Implements. In 1714, the Landgrave of *Hesse*, *Charles I.* Heir of the States and Talents of the celebrated Landgrave we have already spoke of, built a new Experimental School and Observatory, and put it under the Direction of *Zumbach*. In 1722, the King of *Portugal*, *John V.* erected an Experimental School and Observatory, in his Palace at *Lisbon*; there is also one in the College of *St. Antony*. The Experimental School and Observatory at *Petersbourg*, is one of the most magnificent in *Europe*, it is situated in the Middle of the superb Edifice of the Imperial Academy of *Petersbourg*, it is composed of three Flights of Halls, adapted for making Experiments and Observations, and is 150 Feet high. In 1726, the Magistrates of the Republic of *Utrecht*, built an Experimental School, and an Observatory, in which the famous *Muschembroek* made his Experiments and Observations. In 1739, the King of *Sweden* erected one at *Upsal*, and put it under the Direction of *Wargentin*. In 1740, the Prince of *Hesse Darmstadt*, erected another at *Gieffen*, near *Marborough*. There are two Experimental Schools and Observatories, at *Vienna*, where *F. Hell*, and *F. Liganig*, distinguish themselves actually. There is one at *Tyrnaw* in *Hungary*; one in *Poland*, at *Wilna*, &c. &c.

Of Berlin
in 1700.Of Italy
in 1709
and 1713.Of Altorf
in 1714.Of Lisbon
in 1722.Of Petersbourg
in 1726.
Of Utrecht
in 1726.Of Upsal
in 1739.

Of Vienna.

Of Wilna.

Such are the renowned Establishments to which we are indebted for our Knowledge of the System of the World, and the Improvements it receives every Day; but there are a great many Branches, which require such long Operations, and so great a Space of Time, that Posterity will always have new Observations and Discoveries to make. *Multum egerunt qui ante nos fuerunt, sed non peregerunt, multum adhuc restat Operis multumque restabit; nec ulli nato post mille Sæcula præcludetur Occasus aliquid adhuc adjiciendi.* (SENEC. Epist. 64.)

XVIII.

Those great Examples of all the civilized Nations of the World, have at length brought the Noblemen and Gentlemen of this Country,

to a true Sense of the Importance of procuring to their Children, those Means of Instruction, which may prevent their regretting in a more advanced Age, the mis-spent Time of their Youth; which is the only Period of Life in which they can apply themselves with Success, to the Study of Nature: In this happy Age, when the Mind begins to think, and the Heart has no Passions violent enough to trouble it. Shortly, the Passions and Pleasures of their Age will engross their Time, and when the Fire of Youth is abated, and they have paid to the Tumult of the World the Tribute of their Age and Rank, Ambition will gain the Ascendant. And though in a more advanced Age, which will not however be more ripe, they should apply themselves to the Study of the Sciences, their Minds having lost that Flexibility which they had in their youthful Days, it is only by the Dint of Study, they can attain what they might acquire before with the greatest Ease.

Publick
School
establish'd in
the City of
Dublin for
instructing
Youth in
evry Branch
of pure and
mixt Mathe-
matics pur-
suant to the
Resolution
of the No-
blemen and
Gentlemen
of Fortune
of the King-
dom of Ire-
land the 4th
of February
1768.

To improve therefore the Dawn of their Reason, to secure them from Ignorance, so common among People of Condition, which exposes them daily to be scandalously imposed upon, to accustom them early to the Habit of thinking and acting on rational Principles, a SCHOOL has been established on the most approved PLAN, where, after having spent some Time in learning ELEMENTARY MATHEMATICKS, they are initiated in the Misteries of SUBLIME GEOMETRY, and of the INFINITESIMAL CALCULATION; from those abstract Truths, they are led to the Discovery of the Phenomena of Nature, they are taught how to discern their Causes, and measure their Effects; from thence they are conducted as far as the Heavens, those immense Globes which roll over our Heads with so much Majesty, Variety and Harmony, letting themselves be approached; they are taught how to observe their Motions, and investigate the Laws according to which this material World, and all Things in it, are so wisely framed, maintained and preserved.

To relax their Minds after those Speculations, they are brought back to Earth, where, free from all Spirit of System and Research of Causes, they are taught how to contemplate the Wonders of Nature in detail. But as it presents an immense Field, whose whole Extent the greatest Genius cannot compass, and the Inquiries the most valuable, and the only worthy of a true Citizen are those by which the Good of Society is promoted, they are confined particularly to the Study of what may contribute to the Perfection of useful Arts, such as AGRICULTURE and COMMERCE, that thus initiated in the true Principles of the different Branches of Knowledge suitable to their Rank, having completed their Studies in this School, far from being obliged to forget what they have learned, as hitherto has been the Case, they may be enabled to pursue with Success, such Inquiries as are best adapted to their Genius.

Progress of the Discoveries relative to the System of the World.

I.

THE first Views which Philosophers had of the System of the World, were no better than those of the Vulgar, being the immediate Suggestions of Sense; but they corrected them; thus the first System supposed the Earth to be an extended Plane, and the Center round which the Heavenly Bodies revolved.

First views of Philosophers of the System of the World

II.

The *Babylonians* from examining the Appearances of Sence were the first who discovered the Earth to be round, and the Sun to be the Center of the Universe (a) in these Points they were followed by *Pythagoras* and his School.

Discoveries of the Babylonians, and of Pythagoras.

III.

The true System of the World being discovered, it may appear surprising that the Notion of the Earth's being the Center of the Celestial Motions should generally prevail: for tho' on a superficial Survey it seems to be recommended by its Simplicity, and to square exactly with the Appearances of Sence, yet on Examination it is found entirely insufficient to explain the Phenomena, and to account for the Heavenly Motions: This constrained *Ptolemy* and his followers to incumber and embarrass the Heavens with a Number of Circles and Epicycles equally arduous to be conceived and employed, for nothing so difficult as to substitute Error in the room of Truth.

Efforts that have been made to maintain the Earth to be at rest in the System of Ptolemy.

Probably the Influence of *Aristotle's* Authority, whose Writings in *Ptolemy's* Time were held in the highest Esteem, and considered as the Standard of Truth, lead this Philosopher into Error: But why did not *Aristotle* declare in favour of the true System, which he knew, since he endeavoured to overthrow it; this Reflection is sufficiently mortifying to the Pride of the Human Understanding, whatever was the Cause, thus much is certain, that the Ptolomaic System generally prevailed to the Time of *Copernicus*.

IV.

This great Man revived the ancient System of the *Babylonians*, and of *Pythagoras* which he confirmed by so many Arguments and Discoveries that Error could no longer maintain its Ground against the Evidence of Demonstration; thus the Sun was reinstated by *Copernicus* in the Center of the World, or to speak more exactly, in the Center of our Planetary System.

Copernicus revives the ancient System of Pythagoras.

(a) *NEWTON* in his Book *DE SYSTEMATE MUNDI* attributes this Opinion to *Numa Pomilius*, and says, (Page 1.) it was to represent the Sun in the Center of the Celestial Orbits that *Numa* caused a round Temple to be built in honour of *Vesta*, the Goddess of Fire in the Middle of which a perpetual Fire was preserved.

V.

System of Ticho Brabe The Copernican System easily accounts for all the Celestial Phenomena, and tho' Observation and Argument are equally favourable to it, yet *Ticho Brabe* an eminent Philosopher of that Age refused his assent to the Evidence of these Discoveries, whether deluded by an ill-formed Experiment, (b) or carried away by the Vanity of making a new System, he composed one which steers a middle Course between those of *Ptolomy* and *Copernicus*; he supposed the Earth to be at rest and the other Planets which move round the Sun, to revolve with him round the Earth, in the Space of 24 Hours; thus retaining the most exceptionable Part of *Ptolomy's* System, viz. the inconceivable Rapidity with which the *primum Mobile* is supposed to revolve, from whence we may learn into what dangerous Errors the misapplication of Genius may lead us.

The Discoveries relative to the System of the World, improved by Tycho.

Tho' *Tycho* erred in the Manner he made the Celestial Bodies move, yet he contributed very much to the Progress of the Discoveries relative to the System of the World, by the Accuracy and long Series of his Observations. He determined the Position of a vast Number of Stars to a Degree of exactness unknown before; he discovered the Refraction of the Atmosphere, by which the Celestial Phenomena are so much influenced; he was the first who proved from the Parallax of the Comets, that they ascend above the Moon; he was the first who observed what is called the *Moon's variation*; and in fine, it is from his Observations on the Motions of the Planets, that *Kepler* who resided with him, near Prague, during the last Years of his Life, deduced his admirable Theory of the Motions of the Heavenly Bodies.

VI.

How much remained to be discovered after Copernicus.

Copernicus undoubtedly rendered important Services to Human Reason by re-establishing the true System of the World: It was already a great point gained that Human Vanity condescended to place the Earth in the Number of the simple Planets; but much still remained to be discovered: neither the Forms of the Planetary Orbits, nor the Laws by which their Motions are regulated, were known; for these important Discoveries we are indebted to *Kepler*.

(b) It was objected to *Copernicus*, that the Motion of the Earth would produce Effects which did not take Place; that, for Example, if the Earth moved, a Stone dropp'd from the Top of a Tower, ought not to fall at the Foot of it, because the Earth moved during the Time of the Stone's descent, that notwithstanding it falls at the Foot of the Tower. *COPERNICUS* replied, that the Situation of the Earth with respect to Bodies that fall on its Surface was the same as that of a Ship in Motion, with respect to Bodies that are made to fall in it; he asserted, that a Stone let fall from the Top of the Mast of a Vessel in Motion, would fall at the Foot of it. This Experiment which is now incontestible was then ill-made, and was the Cause or the Pretext which made *Ticho* refuse his assent to the Discoveries of *Copernicus*.

This eminent Philosopher found out, that the Notion which generally prevailed before his time, that the Planets revolved in circular Orbits, was erroneous; and he discovered, by the means of Ticho's Observations, that the Planets move in Ellipses, the Sun residing in one of the Foci: and that they move over the different Parts of their Orbit, with different Velocities, so that the Area described by a Planet, that is, the Space included between the straight lines drawn from the Sun to any two Places of the Planet, is always proportional to the time which the Planet employs to pass from one to the other.

Discoveries
of Kepler.
the ellipticity
of the orbits.
the proportionality
of the areas and
the times.

Some years afterwards, comparing the Times of the Revolutions of the different Planets about the Sun, with their different Distances from him, he found that the Planets which are placed the farthest from the Sun to move slowest, and examining whether this Proportion was that of their Distances, he discovered after many Trials, in the Year 1618, that the Times of their Revolutions were as the Square Roots of the Cubes of their mean Distances from the Sun.

Relation
which subsists
between
the periodic
times and
the distances.

VII.

Kepler not only discovered these two Laws, which retain his Name, and which regulate the Motions of all the Planets, and the Curve they describe, but had also some Notion of the Force which makes them describe this Curve; in the Preface to his Commentaries on the Planet Mars, we discover the first Hints of the attractive Power; he even goes so far as to say, that the Flux and Reflux of the Sea, arises from the gravity of the Waters towards the Moon: but he did not deduce from this Principle what might be expected from his Genius and indefatigable Industry. For in his Epitome of Astronomy(c) he proposes a physical Account of the planetary Motions from quite different Principles; and in this same Book of the Planet Mars, he supposes in the Planets a friendly and a hostile Hemisphere, that the Sun attracts the one and repels the other, the friendly Hemisphere being turned to the Sun in the Planets descent to its Perihelium, and the Hostile in its Recess.

VIII.

The Attraction of the Celestial Bodies was suggested much more clearly by M. Hook, in his Treatise on the Motion of the Earth, printed in the Year 1674, twelve Years before the Principia appeared. *These are his Words,* Page 27, "I shall explain hereafter a System of the World, different in many Particulars from any yet known, answering in all Things to the common Rules of Mechanical Motions. This depends on the three following Suppositions."

(c) See Gregory, Book 1, Page 69.

Singular anecdote concerning attraction.

1st That all celestial Bodies, whatever, have an Attraction, or gravitating Power towards their own Centers, whereby they attract, not only their own Parts and keep them from flying from them, as we may observe the Earth to do, but that they do also attract all the other celestial Bodies that are within the Sphere of their Activity; and consequently not only the Sun and the Moon have an Influence upon the Body and Motion of the Earth, and the Earth on the Sun and Moon, but also, that Mercury, Venus, Mars, Jupiter and Saturn, by their attractive Powers, have a considerable Influence upon the Motion of the Earth, as in the same Manner the corresponding attractive Power of the Earth hath a considerable influence upon the Motion of the Planets."

2d That all Bodies whatever that are put into a direct and simple Motion, will so continue to move forward in a straight Line, till they are by some other effectual Power deflected and turned into a Motion, describing a Circle, an Ellipse, or some other more compounded Curve Line."

3d That these attractive Powers are so much the more powerful in operating, by how much the nearer the Body wrought upon is to their own Center."

"These several Degrees I have not yet experimentally verified, but it is a Notion which if fully prosecuted as it ought to be, will mightily assist the Astronomer to reduce all the celestial Motions to a certain Rule, which I doubt will never be done true without it. He that understands the Nature of the circular Pendulum and circular Motion, will easily understand the whole Ground of this Principle, and know where to find Directions in Nature for the true stating thereof. This I only hint at present to such as have a Capacity and Opportunity of prosecuting this Enquiry, &c."

We are not to imagine, that this Hint thrown out casually by *Hook*, detracts from the Glory of *Newton*, who even took Care to make Mention of it in his Book *de Systemate mundi* (d). the Example of *Hook* and *Kepler* makes us perceive the wide Difference between having a Notion of the Truth, and being able to establish it by irrefragable Demonstration; it also shews us how little the greatest Sagacity can penetrate into the Laws and Constitution of Nature, without the Aid and Direction of Geometry.

Strange notions of Kepler.

Kepler, who made such important Discoveries, whilst he followed this unerring Guide, affords us a convincing Proof of the Errors into which the brightest Genius may be seduced, by indulging the pleasing Vanity of inventing Systems; who could believe, for Instance, that such a Man could

adopt the wild Fancies and whimsical Reveries of the Pythagoreans, concerning Numbers: yet he thought that the Number and Interval of the primary Planets bore some Relation to the five regular Solids of Elementary Geometry (e), imagining that a Cube inscribed in the Sphere of Saturn would touch the Orb of Jupiter with its six Planes, and that the other four regular Solids, in like Manner, fitted the Intervals that are betwixt the Spheres of the other Planets: afterwards on discovering that this Hypothesis did not square with the Distances of the Planets, he fancied that the celestial Motions are performed in Proportions corresponding with those, according to which a Cord is divided in order to produce the Tones which compose the Octave in Music (f);

Kepler having sent to Ticho a Copy of the Work, in which he attempted to establish those Reveries. Ticho recommended to him, in his Answer (g), to relinquish all Speculations deduced from first Principles, all reasoning a Priori, and rather study to establish his Researches on the sure and firm Ground of Observation. Wife counsel of Ticho to Kepler.

The great *Hughens* himself (h) believed that the fourth Satellite of Saturn, which retains his Name, making up with our Moon and the four Satellites of Jupiter six secondary Planets, the Number of the Planets was complete, and it was labour lost to attempt to discover any more, because the principle Planets are also six in Number, and the Number Six is a *perfect* Number, as being equal to the Sum of its aliquot Parts, 1, 2 and 3. Whimsical notion of Hughens.

XI.

It was by never deviating from the most profound Geometry, that *Newton* discovered the Proportion in which Gravity acts, and that in his Hands the Principle of which *Kepler* and *Hook* had only some faint Notion, became the Source of the most admirable and unhop'd for Discoveries.

One of the Causes which prevented *Kepler* from applying the Principles of Attraction to explain the Phænomena of Nature with Success, was his Ignorance of the true Laws of Motion. *Newton* had the Advantage over *Kepler* of profiting of the Laws of Motion, established by *Hughens*, which he has carried to so great a Height in his Mathematical Principles of Natural Philosophy. Advantages of *Newton* over *Kepler* in his time, the theory of motion was better understood.

XII.

The Mathematical Principles of Natural Philosophy consist of three Books, besides the Definitions, the Laws of Motion and their Corollaries; the first Book is composed of fourteen Sections, the second contains nine, Analysis of the principles

(e) *Mysterium Cosmographicum.*

(f) *Mytletium Cosmographicum.*

(g) *Uti suspensis speculationibus a priori descenditibus animum potius ad observationes quas simul offerebat considerandas adjicerem* (it is *Kepler* who speaks) *notæ in secundæ editionem mysterii cosmographici.*

and the third, the Application of the two first to the Explication of the Phænomena of the System of the World.

XIII.

Definitions. The Principia commence with eight Definitions; *Newton* shews in the two first how the *Quantity of Matter* and the *Quantity of Motion* should be measured; he defines in the third, the *Vis inertiae*, or resisting Force, which all Matter is endued with; he explains in the fourth what is to be understood by *active Force*; he defines in the fifth the *centripetal Force*, and lays down in the sixth, seventh and eighth the Manner of measuring its *absolute Quantity*, its *motrix Quantity*, and its *accelerative Quantity*; afterwards he establishes the three following Laws of Motion.

XIV.

Laws of motion. 1st. That a Body always perseveres of itself, in its State of Rest, or of uniform Motion in a straight Line.
2d. That the change of Motion, is proportional to the Force impressed, and is produced in the straight Line in which that Force acts.
3d. That Action and Reaction are always equal with opposite Directions.

XV.

First book, *Newton* having explained those Laws, and deduced from them several Corollaries, commences his first Book with eleven Lemmas, which compose the first Section, he unfolds in those eleven Lemmas his Method of *Prime and ultimate Ratios*; this Method is the Foundation of infinitesimal Geometry, and by its Assistance, this Geometry is rendered as certain as that of the Ancients.

the other 13 The thirteen other Sections of the first Book of the Principia, are employed in demonstrating general Propositions on the Motion of Bodies, Abstracting from the Species of these Bodies and of the Medium in which they move.

It is in this first Book that *Newton* unfolds all his Theory of the gravitation of the celestial Bodies, but does not confine himself to examine the Questions relative to it; he has rendered his Solutions general, and has given a great Number of Applications of those Solutions.

XVI.

Second book In the second Book, *Newton* treats of the Motion of Bodies in resisting Mediums.

it treats of This second Book which contains a very profound Theory of Fluids, and of the Motion of Bodies which are immersed in them, seems to have been intended to overthrow the System of Vortices, though it is only in the Scholium of the last Proposition, that *Newton* openly attacks *Descartes*, and proves that the celestial Motions are not produced by Vortices.

XVII.

In fine, the third Book of the Principia treats of the System of the World; Third book. it treats of the system of the world. in this Book, *Newton* applies the Propositions of the two first: in this Application we shall endeavour to follow *Newton*, and point out the Connection of his Principles, and shew how naturally they unravel the Mechanism of the Universe.

XVIII.

The Term, Attraction, I employ in the Sense in which *Newton* has defined it, understanding by it nothing more than that Force, by which Bodies tend towards a Center, without pretending to assign the Cause of this Tendency. What is meant by the word attraction.

Principal Phenomena of the System of the World.

I.

THE Knowledge of the Disposition and Motions of the Celestial Bodies must precede a just Enquiry into their Causes. It will not therefore appear unnecessary to prepare our Readers by a succinct description of our planetary System for our Account of the manner *Newton* demonstrates the powers which govern the Celestial Motions and produce their mutual Influences. This Description must necessarily comprize some Truths, discovered by that illustrious Philosopher, the Manner he attained them will be described in the Sequel.

II.

The celestial Bodies that compose our planetary System, are divided into *Primary Planets*, that is, those which revolve round the Sun, as their *Center* and *Secondary Planets*, otherwise, called *Satellites*, which revolve round their respective Primaries as Centers: There are six Primary Planets whose Names and Characters are as follows,

First division of the celestial bodies of our planetary system into principal and secondary planets.

- ♂ *Mercury,*
- ♀ *Venus,*
- ♂ *The Earth,*
- ♂ *Mars,*
- ♂ *Jupiter,*
- ♂ *Saturn.*

Names and characters of the principal planets.

In enumerating the Primary Planets, we follow the Order of their Distances from the Sun, commencing with those which are nearest to him.

The Earth, Jupiter, and Saturn, are the only Planets which have been discovered to be attended by Secondaries: The Earth has only one Satellite, namely, the Moon; Jupiter has four, and Saturn five, exclusive of his Ring, so that our Planetary System is composed of eighteen celestial Bodies, including the Sun and the Ring of Saturn.

Which are the planets that have satellites, enumeration of the celestial bodies of our planetary system.

III.

The Primary Planets are divided into *superior* and *inferior Planets*, the *inferior Planets* are those which are nearer the Sun than the Earth is; these

Second division of the planets into superior and inferior.

which are the inferior planets and what is their arrangement. are *Mercury* and *Venus*; the Orbit (a) of *Venus* includes that of *Mercury* and also the Sun, and the Orbit of the Earth is exterior to those of *Mercury* and of *Venus*, and incloses them and the Sun also.

This order is discovered, by *Venus* and *Mercury* sometimes appearing to be interposed between the Sun and us, which could never happen unless these Planets revolved nearer the Sun than the Earth, and it is very perceivable that *Venus* recedes farther from the Sun than *Mercury* does, and consequently its Orbit includes that of *Mercury*.

which are the superior planets and what is their arrangement. The superior Planets are those which are more distant from the Sun than the Earth is, these are three in Number, *Mars*, *Jupiter* and *Saturn*; we know that the Orbits of these Planets inclose the Orbit of the Earth, because the Earth is sometimes interposed between them and the Sun.

The Orbit of *Mars* incloses that of the Earth, the Orbit of *Jupiter* that of *Mars*, and the Orbit of *Saturn* that of *Jupiter*; so that of the three superior Planets *Saturn* is the remotest from the Earth, and *Mars* is the nearest.

how it has been discovered. This Arrangement is discovered by those Planets which are nearer the Earth (b) sometimes coming between the Eye and the Remoter, and intercepting them from our View.

IV.

The planets are opaque bodies. All the Planets are opaque Bodies; this appears of *Venus* and *Mercury*, because when they pass between us and the Sun, they resemble black Spots traversing his Body, and assume all those various Appearances which are called Phases, that is, the Quantity of their Illumination depends on their Position in respect to the Sun and us.

The planets are spherical. For the same Reason, since *Mars* has *Phases* we infer his Opacity, and the same Conclusion is extended to *Jupiter* and *Saturn*, because their Satellites do not appear illuminated while their Primaries are between them, and the Sun which proves that that Hemisphere of those Planets which is turned from the Sun is opaque: Lastly, we know that the Planets are spherical Bodies, because, whatever be their Position, in respect of us, their Surface always appears to be terminated by a Curve.

We conclude that the Earth is spherical, because in Eclipses her Shadow, always appears to be bounded by a Curve, and when a Ship sails out of sight, it gradually disappears, first the Hulk, next the Sails, and lastly the Mast, sinking to the Eye and vanishing, and moreover, if the Earth was an extended Plane, Navigation would have discovered its Limits and Boundaries the contrary of which is proved by many Voyagers, such as Drake, Forbish, and Lord Anson, who have sailed round the World.

(a) Orbit is the Curve which a Planet describes in revolving round the Body which serves it as a Center.

(b) Wolf's Elements of Astronomy.

V.

All that we know therefore concerning the primary Planets, proves that they are opaque, solid and spherical Bodies.

The Sun appears to be a Body of a Nature entirely different from the Planets; we know not whether the Parts of which it is composed be solid or fluid; all that we can discover is, that those Parts emit light & heat, and burn when condensed and assembled in sufficient Quantity; hence we may probably conclude, that the Sun is a Globe of Fire resembling terrestrial Fire, since the Effects produced by this and the solar Rays, are exactly the same.

The planets appear to be all of the same nature,

It is probable that the Sun is a globe of fire.

VI.

All the celestial Bodies complete their Revolutions round the Sun in Ellipses (c), more or less excentric, the Sun residing in the common Focus of all their Orbits; hence the Planets in their Revolutions sometimes approach nearer, and sometimes recede farther from the Sun; a right Line passing through the Sun and terminating in the two Points of the Orbit of a Planet, which are nearest and remotest from the Sun, is called the *Line of the Absides*, the Point of the Orbit which is nearest the Sun is called the *Perihelium*, and the Point of the Orbit which is remotest from the Sun is called the *Aphelium*.

In what curve the celestial bodies revolve about the sun, what is the line of the absides the aphelium and perihelium.

The primary Planets in their Revolutions round the Sun, carry also their Satellites, which at the same Time revolve round them as their Centers.

In what direction the planets revolve.

All these Revolutions are performed in a direction from West to East (d). There appear from Time to Time Stars that move in all Directions, and with astonishing Rapidity, when they are sufficiently near to be visible, these are called Comets.

Of the comets.

We have not yet collected Observations sufficient to determine their Number, all that we know concerning them, and 'tis but lately that the Discovery has been made; is that they are Planets revolving round the Sun like the other Bodies of our System, and that they describe Ellipses so very excentric as to be visible only while they are moving over a very small Part of their Orbit.

The comets are planets.

VII.

All the Planets in their Revolutions round the Sun, observe the two Laws of Kepler.

The planets and comets observe the laws of Kepler

Observations evince, that the Comets observe the first of these Laws, namely, that which makes the celestial Bodies (e) describe equal Areas in equal

(c) A Species of Curve, which is the same with what is commonly called an Oval, the foci are the points in which Gardeners fix their pegs in order to trace this curve of which they make a frequent use.

(d) The Spectator is supposed to be placed on the Earth.

(e) By the Word Area, in general is understood a Surface, here it signifies the Space included between two Lines drawn from the Center to two Points where the Planet is found;

qual Times; and in the sequel it will be shewn, that all the Observations that have hitherto been made, concerning their Motions, render it highly probable that they are regulated by the second Law, that is, that their periodic (f) Times are in the sesquiplicate ratio of their mean Distances.

VIII.

Proofs of the
motion of
the earth

Admitting these two Laws of *Kepler*, confirmed by all astronomical Observations, from them we may derive several convincing Proofs of the Motion of the Earth, a Point which had been so long contested; for supposing the Earth to be the Center of the Celestial Motions, these two Laws are not observed; the Planets do not describe Areas proportional to the Times around the Earth, and the periodic Times of the Sun and the Moon, for instance, round this Planet, are not as the Square Roots of the Cubes of their mean Distances from the Earth; for the periodic Time of the Sun around the Earth, being nearly thirteen Times greater than that of the Moon, its Distance from the Earth would be, according to *Kepler's Rule*, between five and six Times greater than that of the Moon, but Observations demonstrate, that this Distance is about four-hundred Times greater, therefore, admitting the Laws of *Kepler*, the Earth is not the Center of the celestial Revolutions.

The centripetal Force(g) which *Newton* has demonstrated to be the Cause of the Revolutions of the Planets renders the Curve they describe around their Center concave (h) towards it, since this Force is exerted in drawing them off from the tangent (i); now the Orbits of Mercury and Venus, in some Parts, are convex to the Earth; of consequence, the inferior Planets do not revolve round the Earth.

The same may easily be proved of the superior Planets; for these are those Areas are proportional to the Times, that is, they are greater or less, as the Times in which they are described are longer or shorter.

(f) Periodical Time is the Time that a Planet employs in completing its Revolution in its Orbit. An Example, of Sesquiplicate Ratio will render it more intelligible than a Definition; Suppose then the mean Distance of Mercury from the Sun, to be 4, that of Venus 9, the periodical Time of Mercury 40 Days, and let the periodical Time of Venus be required, cubing the two first Numbers 4 and 9, there will result 64 and 729; afterwards extracting the Square-Roots of these two Numbers, there will be found 8 for that of the first, and 27 for that of the second, and by the Rule of three you will have $8 : 27 :: 40 : 135$. That is the Square-Root of the Cube of the mean Distance of Mercury from the Sun, is to the Square Root of the Cube of the mean Distance of Venus from the Sun, as the periodic Time of Mercury round the Sun is to the periodic Time sought of Venus round the Sun, which is found to be 135, according to the Suppositions which have been made, and this is what is called Sesquiplicate Ratio.

(g) The Word CENTRIPETAL FORCE carries its Definition along with it, for it signifies no more than that Force which makes a Body tend to a Center.

(h) The two Sides of the Crystal of a Watch may serve to explain those Words CONCAVE and CONVEX; the Side exterior to the Watch is CONVEX, and that which is on the Side of the Dial-plate is CONCAVE.

(i) A Tangent is a right Line which touches a Curve, without cutting it.

sometimes observed to be *direct* (k), sometimes *stationary*, and afterwards *retrograde*; all those Irregularities are only apparent and would vanish if the Earth was the Center around which the heavenly Bodies revolved, for none of these Appearances would be observed by a Spectator placed in the Sun, since they result only from the Motion of the Earth in its Orbit combined with the Motion of those planets in their respective Orbits; from hence we may see the Reason why the Sun and the Moon are the only heavenly Bodies that appear always direct; for as the Sun describes no Orbit, its Motion cannot be combined with that of the Earth, and as the Earth is the Center of the Moon's Motion, to us she should always appear direct; as would all the Planets to a Spectator placed in the Sun.

When *Copernicus* first proposed his System, an Objection was raised against it, taken from the Planet Venus by some who alledged, that if that Planet revolved round the Sun she should appear to have Phases as the Moon, to which *Copernicus* answered, if your Eyes were sufficiently acute you would actually observe such Phases, and that perhaps in Time some Art may be discovered so to improve and enlarge the visual Powers, as to render those Phases perceivable: This Prediction of Copernicus was first verified by *Galileo*, and every Discovery that has been made since on the Motion of the heavenly Bodies has confirmed it.

IX.

The Planes (l) of the Orbits of all the Planets intersect in right Lines passing through the center of the Sun, so that a Spectator placed in the Center of the Sun would be in the Planes of all those Orbits.

The Right Line, which is the common Section of the Plane of each Orbit, with the Plane of the Ecliptic, that is, the Plane in which the Earth moves, is called the *Line* of the nodes of that Orbit, and the extreme Points of this Section, are called the *Nodes* of that Orbit.

The Quantities of the Inclination of the Planes of the different Orbits with the Plane of the Ecliptic, are as follows, the Plane of the Orbit of Saturn is inclined to the Plane of the Ecliptic in an Angle of $2^d \frac{1}{2}$, that of Jupiter $1^d \frac{1}{2}$, that of Mars in an angle somewhat less than 2^d , that of Venus somewhat more than $3^d \frac{1}{2}$, and that of Mercury about 7^d .

X.

The Orbits of the primary Planets being Ellipses, having the Sun in one of their Foci, all these Orbits are consequently excentric, and are more or less so, according to the Distance between their Centers and the Point where the Sun is placed.

(k) A Planet is said to be *DIRECT* when it appears to move according to the Order of the Signs, that is, from Aries to Taurus, from Taurus to Gemini, &c. which is also said to move in consequentia, it is stationary when it appears to correspond for some Time to the same Points of the Heavens, and in fine it is *RETROGRADE* when it appears to move contrary to the Order of the Signs, which is also said to move in Antecedentia, that is, from Gemini to Taurus, from Taurus to Aries, &c.

(l) The plane of the Orbit of a Planet is the surface on which it is supposed to move.

excentricity
of the pla
nets in semi
diameters
of the earth

The excentricity of all those Orbits have been measured, and have been found as follows, in decimal Parts of the semidiameter of the Earth's orbit, supposed to be divided into 100,000 Parts,

That of Saturn,	54207 Parts.
That of Jupiter,	25058
That of Mars,	14115
That of the Earth,	4692
That of Venus,	500
And in fine, that of Mercury,	8149 Parts.

excentricity
of the pla
nets in semi
diameters of
their great
Orbit

The excentricity of the Planets measured in decimal Parts of the semidiameter of their Orbits, supposed to be divided into 100,000 Parts, are as follows,

That of Saturn,	5683 Parts.
That of Jupiter,	4822
That of Mars,	9263
That of the Earth,	5700
That of Venus,	694
That of Mercury,	21000 Parts.

Whence it appears that the Excentricity of Mercury is almost insensible.

XI.

Proportion
of the dia
meters of
the planets.

The Planets are of different Magnitudes; of the Earth alone we know the absolute Diameter, because this Planet is the only one whose Circumference admits of actual Mensuration, but the relative Magnitudes of the Diameters of the other Planets have been discovered, and the Diameter of the Sun being taken for a common Measure, and supposed to be divided into 1000 Parts:

That of Saturn is	137
That of Jupiter	181
That of Mars	6
That of the Earth	7
That of Venus	12
That of Mercury	4

Hence we see that Mercury is the least of all the Planets, for Spheres are as the Cubes of their Diameters.

XII.

Distances of
the planets
from the sun

The Planets are placed at different Distances from the Sun, taking the Distance of the Earth from the Sun for a common Measure, and supposing it divided into 100,000 Parts, the mean Distances of the Planets are as follows,

That of Mercury is	38710
That of Venus	7233
That of the Earth	10000
That of Mars	152369
That of Jupiter	520110
In fine, that of Saturn	953800

The mean Distances of the Sun and the Planets from the Earth, have also been computed in Semidiameters of the Earth; the mean Distances of the Sun, Mercury and Venus from the Earth are nearly equal, and amount to 22000 Semidiameters of the Earth, that of Mars is 33500, that of Jupiter 115000, and that of Saturn 210000. Distances of the planets from the earth

XIII.

The Times of the Revolutions of the Planets round the Sun, are less in Proportion of their Proximity, thus Mercury the nearest revolves in 87 Days, Venus next in Order revolves in 224, the Earth in 365, Mars in 686, Jupiter in 4332, and Saturn the remotest from the Sun in 10759, the whole in round Numbers. Periodic times of the planets round the sun

XIV.

The Planets, besides their Motion of Translation round the Sun, have another Motion of Rotation round their Axis, called their *Diurnal Revolution*. Rotation of the planets

We only know the diurnal Revolution of the Sun and of four Planets, namely of the Earth, Mars, Jupiter and Venus; this Revolution has been discovered by Means of the Spots observed on their Discs, (m) and which successively appear and vanish; Mars, Jupiter and Venus having Spots on their Surface, by the regular Return and successive Disappearance of the same Spots it has been found, that these Planets turn round their Axes, and in what Time they compleat their Rotation; thus it has been observed, that Mars makes his Rotation in 23^h. 20^m. and Jupiter in 9^h. 56^m. Means employed to discover it In what place this rotation has been perceived

Astronomers are not agreed about the Time in which Venus revolves round its Axis; most suppose the Time of rotation to be about 23^h. But Sign. Bianchini who observed the Motions of this Planet with particular Attention, thinks she employs 24 Days in turning round; but as he was compelled to remove his Instruments during the Time he was observing, an House having intercepted Venus from his View; and as he lost an Hour in this Operation, 'tis probable that the Spot he was observing during this Interval changed its Appearance; however this be his authority in Astronomical Matters deserves we should suspend our Judgment till more accurate Observations have decided the Point. Incertitude with regard to the time of the rotation of Venus

M. de la Hire observed with a Telescope 16 Feet long, Mountains in Venus higher than those of the Moon.

The extraordinary brightness of Mercury arising from his proximity to the Sun, prevents our discovering by Observation its Rotation; and Saturn is too remote to have his Spots observed. The rotation of Mercury and of Saturn cannot be discovered by observation and why

In the Year 1715 *Cassini* observed with a Telescope 118 Feet long; three Belts in Saturn resembling those observed in Jupiter, but probably these Observations could not be pursued with accuracy sufficient to conclude the Rotation of Saturn about its Axis.

(m) By the Disk of a Planet is understood that Part of its surface which is visible to us.

but analogy
authorises
to conclude
that those
planets re-
volve round
their Axes.

As Mercury and Saturn are subject to the same Laws that direct the Courses of the other Planets, and as far as has been discovered appear to be Bodies of the same Nature, Analogy authorizes us to conclude that they also revolve, round their Axes; and perhaps future Astronomers may be able to observe this Motion, and to determine its Period.

XV.

There appear from Time to Time Spots upon the Sun, which have served to discover that it has a rotatory Motion about its Axis.

How the ro-
tation of the
Sun about
its axis has
been disco-
vered

It was long after the Discovery of those Spots, before Astronomers could observe any, sufficiently durable and permanent, to enable them to determine the Time of his Revolution. Keill in the 5th Lecture of his Astronomy, relates, that some Spots have been observed to pass from the Western Limb of the Sun to the Eastern Margent in 13 Days and half, and after 13 Days and half to re-appear in the Western Verge of his Disk, from whence he infers that the Sun revolves round its Axis in the Space of about 27 Days from West to East, that is in the same Direction of the Planets; by means of those Spots it has been discovered, that the Axis round which the Sun revolves, is inclined to the plane of the Ecliptic in an Angle of 7d.

Faquier, in his Commentary on *Newton*, has made some Reflections on these Spots that deserve to be remarked; as no Observations prove the Times of their Occultation to be equal, but on the contrary, all the Observations he could collect, prove them to be unequal; and, that the Time during which they are concealed, has been always longer than that, during which they have been visible, from hence he concluded (as also *Wolf* Art. 411. of his Astronomy) that those Spots are not inherent to the Sun, but removed from his Surface to some distance.

The Solar Spots were first discovered in *Germany*, in the Year 1611, by *John Fabricius*, (n) who from thence concluded, the diurnal Revolution of the Sun. They were afterwards observed by *Scheiner*, (o) who published the Result of his Observations. The same Discovery was made by *Galileo* in *Italy*.

Scheiner observed more than fifty Spots on the Surface of the Sun; this may serve to account for a Phenomenon, related by many Historians, that the Sun, sometimes for the Space of a whole Year, has appeared very Pale, as this Effect would naturally follow from a Number of Spots sufficiently large and permanent, to obscure a considerable Portion of his Surface.

(n) *Wolf*. *Elementa Astronomiæ* Cap. 1.

(o) *Scheiner* having informed his Superior that he had discovered Spots in the Sun, he gravely replied, "that is impossible, I have read Aristotle two or three-times over, and have found not the least mention of it."

It is no longer doubted that the Earth turns round her Axis in 23h 56m which compose our astronomical Day; from this Rotation arise the changes of Day and Night, which all the Climates of the Earth enjoy.

XVI.

This Motion of the Celestial Bodies about their Centers alters their Figures, for it is known that Bodies revolving in Circles, acquire a Force which is so much the greater, the Time of their Revolution being the same as the Circle which they describe is greater. This Force is called *Centrifugal Force*; that is, the Force which *repels them from the Center*; wherefore, from their diurnal Rotation, the Parts of the Planets acquire a Centrifugal Force, so much greater as they are nearer the Equators of these Planets: (since the Equator is the greatest Circle of the Sphere,) and so much less as they are nearer the Poles (p); supposing therefore the Heavenly Bodies in their State of Rest, to have been perfect Spheres, their Rotation about their Axes must have elevated their equatorial and depressed their polar Regions, and of Consequence changed their spherical Figures into that of Oblate Spheroids, flat towards the Poles.

the effects of the rotatory motion of the planets consist in raising their equators. what is the centripetal force.

The Theory thus leads us to conclude, that all the Planets, in Consequence of their Rotation, should be flat towards the Poles, but this is only sensible in Jupiter and the Earth. In the Sequel it will appear, that the Proportion of the Axes (q), in the Sun, is assignable from Theory, but is too inconsiderable to be observed.

which are the planets in which the elevation of the equator is perceived.

The Measures of Degrees of the Meridian, taken at the Polar Circle in France, and at the Equator, fix the Proportion of the Axes of the Earth to be as 173 to 174. By the Help of Telescopes the oblate Figure of Jupiter has been perceived. And the Disproportion of his Diameters is much greater than that of the Earth, because this Planet is a great deal bigger, and revolves with greater Rapidity about its Axis than the Earth; the Proportion of the Axes of Jupiter is esteemed to be as 13 to 14.

XVII.

As the Spots of Venus, Mars and Jupiter are variable, and frequently change their Appearance, it is probable that these Planets, like our Earth, are surrounded by dense Atmospheres, the Alterations in which, produce these Phenomena in respect of the Sun, as his Spots are not inherent on his Disk, and as they frequently appear and disappear, it is manifest that he is surrounded by a gross Atmosphere, contiguous to his Body, in which these Spots are successively generated and dissolved.

observation proves that the Earth, Mars, Jupiter, Venus and the Sun are surrounded by atmospheres.

(p) The Poles are the Points about which the Body revolves, and the Equator, the Circle equi distant from those Points dividing the Sphere into two equal Parts.

(q) Axis or Diameter, in general, is a Line which passes through the Center. and is terminated at the Circumference, In the present Case, the Axes are two Lines which pass through the Center, one of which is terminated at the Poles, and the other at the Equator.

XVIII.

What has hitherto been set forth was known before the Time of *Newton*, but no one thought before him, that it was possible to discover the Quantities of Matter in the Planets, their Densities, and the different Weights of one and the same Body successively transferred to the Surfaces of the different Planets. How *Newton* attained to those astonishing Discoveries will be explained in the Sequel; at present it suffices to say, that he found out that the Masses of the Sun, Jupiter, Saturn, and the Earth, that is the Quantities of Matter those Bodies contain, are to one another, as $1 : 1087,3623$ & 105182 , supposing (r) the Parallax of the Sun to be $10'' 3'''$; that their Densities are as 100, 94, 67, and 400; & that the Weights of the same Body, placed successively on the Surfaces of the Sun, Jupiter, Saturn, and the Earth, would be as 10000, 943, 529, and 435; in determining those Proportions, *Newton* has supposed the Semidiameters of the Sun, Jupiter, Saturn, and the Earth, to be as 10000, 997, 791, and 109. it will be shewn hereafter why neither the Density, nor the Quantity of Matter of Mercury, Venus, and Mars, or the Weights of Bodies at their respective Surfaces, are known.

XIX.

It follows from all those Proportions that Saturn is nearly 500 Times less than the Sun, and contains 3000 Times less Matter, that Jupiter is 1000 Times less than the Sun, and contains 1033 Times less Matter. Compared with the Sun the Earth is only as a Point, being 100,000 Times less; and in fine, that the Sun is 116 Times greater, than all the Planets together.

XX.

Comparing the Planets with one another, we find that Mercury and Mars are the only Planets less than the Earth; that Jupiter is not only the biggest of all the Planets, but is bigger than all the Planets together, and that this Planet is two thousand Times bigger than the Earth.

XXI.

The Earth besides her annual and diurnal Motion, has also a third Motion, by which her Axis recedes from its Parallelism, (f) & after a certain Time is directed to different Points of the Heavens, from this Motion arises what is called *the Procession of the Equinoxes* that is, the Regression of the equinoctial Points, or those Points in which the terrestrial Equator cuts the Ecliptic. The equinoctial Points move contrary to the Order of the Signs, and their Motion is so very slow, that they do not compleat a Revolution in less than 25920 Years, they recede a Degree in 72 Years, and the annual Quantity is about, $50''$.

(r) The parallax of the Sun, is the Angle, under which the Semidiameter of the Earth is seen from the Sun, and in general the parallax of any celestial Body, with respect to the Earth, is the Angle under which the Semidiameter of the Earth would be seen from that Body.

(f) A line is said to be parallel when it always preserves the same position with respect to a Point supposed fixed.

Newton found, as will appear in the Sequel, the Cause of this Motion in the Attraction of the Sun and Moon on the Elevation of the equatorial Parts of the Earth.

The Precession of the Equinoxes has caused a Distinction of the Year into the *tropical* and *sydereal*. The tropical Year is the Interval of Time elapsed between two successive vernal or autumnal Equinoxes, in two annual Revolutions of the Earth. This Year is somewhat shorter than the *sydereal* Year, or the Time intervening the Earth's Departure from any Point of her Orbit, and her Return to the same.

Tropical year.
Sydereal year.

XXII.

It remains to describe the secondary Planets, which exclusive of the Ring of Saturn, are 10 in Number; namely, the 5 Satellites of Saturn, the 4 of Jupiter, and the Moon, the only Satellite attending the Earth.

The secondary planets.

Observation proves that these Satellites in revolving round their Primaries, observe the Laws of *Kepler*.

They observe the laws of *Kepler*.

The Satellites of Jupiter have been but lately discovered: The Discovery before the Invention of Telescopes was impossible. *Gallileo* discovered the four Satellites of Jupiter, which in Honour of his Patron, he termed the *Medicean* Stars. These are of the greatest Utility in Geography and Astronomy.

Discovery of the satellites of Jupiter.

Hugens was the first who discovered one of Saturn's Satellites; it still retains his Name, and is the fourth. Afterwards *Cassini* discovered the four others.

And of those of Saturn.

XXIII.

Taking the Semidiameter of Jupiter as a common Measure, his 4 Satellites revolve at the following Distances; the first at the Distance of 5 Semidiameters, the second of 9, the third of 14, and the fourth of 25, neglecting Fractions. These Determinations have been deduced by *Cassini* from his Observations of their Eclipses.

Distances of the moons of Jupiter from this planet.

Their periodic Times round Jupiter are so much the longer as they are remoter from this Planet. The first revolves in 42 Hours, the second in 85, the third in 171, and the fourth in 400, neglecting the Minutes.

Their periodic times about Jupiter.

The diurnal Rotations, Diameters, Bulks, Masses, Densities, and attractive Forces of these Satellites, have not as yet been discovered; and the best Telescopes represent them so vastly small, that there is no Hopes of ever attaining Certainty in these points; the same is the Case with regard to the Satellites of Saturn: These are placed still further beyond the reach of our Researches.

XXIV.

Taking the Diameter of Saturn's Ring for a common Measure, the Distances of the Satellites of Saturn commencing with the innermost, are in the following Proportions.

Distances of the moons of Saturn from Saturn.

& their periodic times round this planet.

The first is expressed by 1, the second by 2, the third by 3, the fourth by 8, and the fifth by 24, neglecting Fractions; and their periodic Times, according to *Cassini*, are 45^h , 65^h , 109^h , 382^h , and 1903^h respectively.

The Moons of Saturn, all revolve in the Plane of the Equator of that Planet, except the fifth, which recedes from it about 15 or 16 Degrees.

Conjectures of *Hughens* concerning a sixth satellite of Saturn.

Several Philosophers, and among them *Hughens*, have suspected, that if Telescopes were once brought to perfection, a sixth Satellite of Saturn between the fourth and fifth would be discovered, the Distance between those two Satellites being two great in Proportion to that which separates the others; but there would then occur this other Difficulty, that this Satellite, which would be the fifth, notwithstanding must be less than any of the four interior Moons, since with our most perfect Telescopes it cannot be perceived.

The Orbits of the Satellites of Jupiter, and of Saturn, are nearly concentric to those Planets.

Observation of *Maraldi* concerning the satellites of Jupiter.

Maraldi has observed Spots on the Moons of Jupiter, but no Consequences could as yet be derived from this Observation, which if properly pursued and accurately repeated, might conduct us to the Knowledge of several interesting Particulars respecting the Motions of the Satellites.

XXV.

Of the ring of Saturn. It does not adhere to the body of this planet. Its distance from the body of the planet. Its diameter. Its breadth. Its thickness. It is an opaque body subject to phases.

Saturn, exclusive of his five Moons, is also surrounded by a Ring, no where adhering to his Body; for through the Interval which separates his Body from the Ring, we can view the fixed Stars: The Diameter of this Ring is to the Diameter of Saturn as 9 to 4, according to *Hughens*, that is more than the Double of the Diameter of Saturn; the Distance of the Body of Saturn from his Ring, is nearly equal to his Semidiameter; so that the Breadth of the Ring is nearly equal to the Distance between its interior Limb and the Globe of Saturn. Its Thickness is very inconsiderable, for when it turns its Edge to the Eye, it is no longer visible, but only appears as a black Line extended across the Globe of Saturn. Thus this Ring undergoes Phases according to the Position of Saturn in his Orbit, which proves it to be an opaque Body; and which like the other Bodies that compose our planetary System, shines only by reflecting the Light it receives from the Sun.

We cannot discover whether the Ring of Saturn has any Motion of Rotation, as no Changes in its Aspect are observed to authorise us to conclude this Rotation.

The Plane of this Ring always forms with the Plane of the Ecliptic an Angle of $23^\circ \frac{1}{2}$, hence its Axis remains always parallel to itself in its Revolution round the Sun.

Of the discovery of this ring. Opinion concerning it.

The Discovery of the Ring of Saturn, the only Phenomenon of the Kind observed in the Heavens is due to *Hughens*. Before his Time, Astronomers observed Phases in Saturn, for they confounded Saturn with his Ring; but those Phases were so different from those of the other Planets as to be utterly inex-

plicable. In *Hevelius* may be seen the Names he gives to those Appearances of Saturn, and how far (t) he was from assigning the true Cause.

fore Hu-
ghens.

Hughens comparing the different Appearances of Saturn, found they were produced by a Ring surrounding his Body; and this Supposition is so conformable with all Telescopic Discoveries, as to be now generally received.

Gregory describing the Notion of *Halley*, that the terrestrial Globe is only an Assemblage of Shells concentric to an internal Nucleus, proposes a Conjecture concerning this Ring, that it is formed of several concentric Shells detached from the Body of that Planet, whose Diameter was formerly equal to the Sum of its actual Diameter, and the Breadth of the Ring.

Notion of
Gregory con-
cerning this
ring.

Another Conjecture has also been proposed, that the Ring of Saturn is only an Assemblage of Moons, which from the immense Distance appear to be contiguous; but those Conjectures are not grounded on any Observation.

The satel-
lites of Jupi-
ter and Sa-
turn are
spherical bo-
dies.

By the Shadows of the Satellites of Jupiter and Saturn projected on their Primaries, it has been discovered, that they are spherical Bodies,

XXVI.

The Earth has only one Satellite, namely the Moon; but her Proximity has enabled us to push our Enquiries concerning this Satellite much further than about the others.

Of the moon

The Moon performs its Revolution round the Earth in an Ellipse, the Earth being placed in one of the Foci; The Form and Position of this Ellipse is continually changing; these Variations are caused by the Action of the Sun, as will appear in the Sequel.

What curve
it describes
round the
earth.

The Moon in her Revolution round the Earth observes the first of the two Laws of *Kepler*, and recedes from it only by the Action of the Sun upon her; she compleats her Revolution round the Earth from West to East in 27 d. 7 h. 43 m. which is called its *periodical Month*.

Its periodic
month.

The Disc of the Moon is sometimes totally, and at other times partially, illuminated by the Sun. The illuminated Part is greater, or less, according to its Position with respect to the Sun and the Earth; these are called her *Phases*. She assumes all those various Phases during the Time of her *synodic* Revolution, or the Interval between two successive Conjunctions with the Sun. This *synodic* Month of the Moon consists of 29 Days $\frac{1}{2}$ nearly.

Her phases.
Her synodic
month.

The Phases of the Moon prove that she is an opaque Body, shining only by reflecting the Light of the Sun.

The moon
is an opaque
and spheri-
cal body.

We know that the Moon is a spherical Body, because she always appears to be bounded by a Curve.

The Earth enlightens the Moon during her Nights, as the Moon does the Earth during ours; and it is by the reflected Light of the Earth that we see the Moon, when she is not illustrated by the Sun.

The earth
enlightens
the moon
during her
nights.

(t) *Hevelius* in opusculo de Saturni Nativa facie distinguishes the different Aspects of Saturn by the Names of Monasphericum, Trisphericum, Spherico-ansatum, elliptico-ansatum, sphericocuspdatum, and subdivides them again into other Phases.

Proportion
of this illu-
mination.

As the Surface of the Earth is about 14 times greater than that of the Moon, the Earth seen from the Moon would appear 14 times brighter, and reflect 14 times more rays to the Moon, than the Moon does to us, supposing both equally capable of reflecting Light.

Inclination
of the orbit
of the moon

The Plane of the lunar Orbit forms with the Plane of the Ecliptic, an Angle of about 5° .

The great Axis of the Ellipse which the Moon describes round the Earth, is called *the Line of the Apfides* (u) *of the Moon*.

The Moon accompanies the Earth in her annual Revolution round the Sun.

Time of the
revolution
of the line
of the ap-
fides.

If the Orbit of the Moon had no other Motion but that by which it is carried round the Sun along with the Earth, the Axis of this Orbit would always remain parallel to itself; and Moon being in her *Apogee*, and in her *Perigee*, would be always at the same Distances from the Earth, and would always correspond to the same Points of the Heavens; but the Line of the Apfides of the Moon revolves with an angular Motion round the Earth, according to the Order of the Signs; and the Apogee and Perigee of the Moon do not return to the same Points in less than 9 Years, which is the Time of the Revolution of the Line of the Apfides of the Moon.

Revolution
of the nodes
of the moon

The Orbit of the Moon intersects the Orbit of the Earth in two Points, which are called her *Nodes*; these Points are not always the same, but change perpetually by a retrogressive Motion that is contrary to the Order of the Signs, and this Motion is such, that in the space of 19 Years the Nodes perform a whole Revolution, after which they return to the same Points of the Orbit of the Earth, or of the Ecliptic.

Time of its
revolution.

Excentrici-
ty of the
moon.

The Excentricity of the Orbit of the Moon changes also continually; this Excentricity sometimes increases, sometimes diminishes, so that the Difference of the greatest and least Excentricity exceeds half the least.

Its motion
round its
axis.

It will be explained in the Sequel how *Newton* discovered the Cause of all those Inequalities of the Moon.

In what
time it is
performed.

The only uniform Motion that the Moon has, is its Motion of Rotation about her Axis; this Motion is performed exactly in the same Time as its Revolution about the Earth, hence its Days consist of 27 of our Days, $7^{\text{h}} 43^{\text{m}}$.

This equality of the lunar Day and the periodic Month makes the Moon always present to us nearly the same Disc.

Libration of
the moon.

The uniform Motion of the Moon about its Axis, combined with the Inequality of its Motion round the Earth, produces the apparent Oscillation of the Moon about her Axis, sometimes Eastward, and at other times Westward, and this is what is called *her Libration*; by this Motion she presents

(u) The Line of the Apfides of the Moon is the Line which passes through the Apogee and Perigee; apogee is the Point of the Orbit the Remotest from the Earth, and the Perigee is the Point of the Orbit the nearest to the Earth; and in general, the Apfides of any Orbit are the Points the Remotest from, and nearest to, the central Point.

to us sometimes Parts which were concealed, and conceals others that were visible.

This Libration of the Moon arises from her Motion in an Elliptic Orbit, its cause. for if she revolved in a circular Orbit, having the Earth for its Center, and turned about her Axis in the Time of her periodic Motion round the Earth, she would in all Positions turn the same Disc exactly towards the Earth.

We are ignorant of the Form of the Surface of the Moon, which is on the other Side of her Disc with Respect to us. Some Philosophers have even attempted to explain its Libration, by assigning a conical Figure to that Part of its Surface, which is concealed from us, and who deny her Rotation round her Axis.

The Surface of the Moon is full of Eminences and Cavities, for which reason she reflects on every Side the Light of the Sun, for if her Surface was even and polished like a Mirror, she would only reflect to us the Image of the Sun.

The mean Distance of the Moon from the Earth is nearly $60 \frac{1}{2}$ Semi-diameters of the Earth.

The Diameter of the Moon is to the Diameter of the Earth, as 100 to 365, its Mass is to the Mass of the Earth, as 1 to 39, 788 and its Density is to the Density of the Earth, as 11 to 9.

And lastly, a Body which would weigh 3 Pounds at the Surface of the Earth, transferred to the Surface of the Moon would weigh one Pound.

All these Proportions are known in the Moon and not in the other Satellites, because this Planet supplies a peculiar Element, namely her Action on the Sea, which *Newton* knew how to measure and to employ for determining her Mass, the Method he pursued in this Enquiry will be unfolded in the Sequel.

Theory of the Primary Planets.

I.

In accounting for the celestial Motions, the first Phenomenon that occurs to be explained is the perpetual Circulation of the Planets round the Center of their Revolutions.

By the first Law of Nature every Body in Motion perseveres in that rectilinear Course in which it commenced, therefore that a Planet may be deflected from the straight Line it tends to describe incessantly, it is Necessary that a Force different from that which makes it tend to describe this straight Line should incessantly Act on it in order to bend its Course into a Curve, in the same Manner as when a Stone is whirled round in a Sling. The Sling incessantly restrains the Stone from flying off in the Direction of the Tangent to the Circle it describes.

To explain this Phenomenon, the Ancients invented their solid Orbs and *Descartes* Vortices, but both one and the other of those Explications

Distance of
the moon
from the
earth.
Its diameter
Its mass.
Its density.

What bodies
weigh on
its surface.

How the
ancient phi-
losophers
and Descar-

tes explain
the circula-
tion of the
planets in
their orbits.

It is a cen-
tripetal
force which
hinders the
planets
from flying
off by the
tangent.

And the pro-
jectile force
hinders
them from
falling to
the center

Of the cen-
trifugal
force of the
planets.

were mere Hypotheses devoid of Proof, and though *Descartes* Explanation was more Philosophical, it was no less Fictitious and Imaginary.

II.

Newton begins with proving in the first Proposition (a), that the Areas described by a Body revolving round an immoveable Center to which it is continually urged, are proportional to the Times, and reciprocally in the Second, that if a Body revolving round a Center describes about it Areas proportional to the Times, that Body is actuated by a Force directed to that Center. Since therefore according to *Kepler's* Discoveries, the Planets describe round the Sun Areas proportional to the Times, they are actuated by a centripetal Force, urging them towards the Sun, and retaining them in their Orbits.

Newton has also shewn (Cor. 1. Prop. 2.) that if the Force acting on a Body, urges it to different Points, it would accelerate or retard the Description of the Areas, which would consequently be no longer proportional to the Times: Therefore if the Areas be proportional to the Times, the revolving Body is not only actuated by a centripetal Force, directed to the central Body, but this Force makes it tend to one and the same Point.

III.

As the Revolutions of the Planets in their Orbits prove the Existence of a centripetal Force drawing them from the Tangent, so by their not descending in a straight Line towards the Center of their Revolution, we may conclude that they are acted upon by another Force different from the Centripetal. *Newton* has examined (b) in what Time each Planet would descend from its present Distance to the Sun if they were actuated by no other Force but the Sun's Action, & he has found (P. 36) that the different Planets would employ in their Descent, the Half of the periodic Time of the Revolution round the Sun of a Body placed at Half their present Distances, and consequently these Times would be to their periodic Times, as 1 to $4\sqrt{2}$. Thus, *Venus* for Example would take about 40 Days to descend to the Sun, for $40 : 224 :: 1 : 4\sqrt{2}$ nearly; *Jupiter* would employ two Years and a Month in his Descent, and the Earth and the Moon sixty-six Days and nineteen Hours, &c. since then the Planets do not descend to the Sun, some Force must necessarily counteract the Force which make them tend to the Sun, and this Force is called the *Projectile Force*.

IV.

The Effort exerted by the Planets in Consequence of this Force to recede from the Center of their Motion, is what is called their *Centrifugal Force*, hence in the Planets, the centrifugal Force is that Part of the projectile Force, which removes them directly from the Center of their Revolution.

(a) When the Propositions are quoted without quoting the Book, they are the Propositions of the first Book.

(b) *De systemate mundi*, page 31. edition 1731.

V.

The projectile Force has the same Direction in all the Planets, for they all revolve round the Sun from West to East.

Supposing the Medium in which the Planets move to be void of all Resistance, the Conservation of the projectile Motion in the Planets, is accounted for from the Inertia of Matter, and the first Law of Motion, but its Physical Cause, and the Reason of its Direction are as yet unknown.

VI.

After having proved that the Planets are retained in their Orbits by a Force directed to the Sun, *Newton* demonstrates (Prop. 4.) that the centripetal Forces of Bodies revolving in Circles are to one another as the Squares of the Arcs of those Circles described in equal Times, divided by their Rays, from whence he deduces (cor. 6.) that if the periodic Times of Bodies revolving in Circles be in the sesquiplicate Ratio of their Rays, the centripetal Force which urges them to the Center of those Circles, is in the inverse Ratio of the Squares of those same Rays, that is of the Distance of those Bodies from the Center: But by the second Law of *Kepler*, which all the Planets observe, their periodic Times are in the sesquiplicate Ratio of their Distances from their Center; consequently, the Force which urges the Planets towards the Sun, decreases as the Square of their Distance from the Sun increases, supposing them to revolve in Circles concentric to the Sun.

Newton discovers the force urging the planets to the Sun to be in the inverse ratio of the square of their distances from the ratio of their periodic times and distances. First in the supposition of their orbits being circular.

VII.

The first and most natural Notion that we form concerning the Orbits of the Planets, is that they perform their Revolutions in concentric Circles; but the Difference in their apparent Diameters, and more accuracy in the Observations, have long since made known that their Orbits cannot be concentric to the Sun; their Courses therefore, before *Kepler's* Time, were explained by excentric Circles, which answered pretty well to the Observations on the Motions of the Sun and the Planets, except Mercury and Mars.

Before *Kepler's* time it was thought that the planets revolved about the Sun in excentric circles. But *Kepler* has shewn that they revolve in ellipses.

From considering the Course of this last Planet, *Kepler* suspected that the Orbits of the Planets might possibly be Ellipses, having the Sun placed in one of the Foci, and this Curve agrees so exactly with all the Phenomena, that it is now universally acknowledged by Astronomers, that the Planets revolve round the Sun in elliptic Orbits, having the Sun in one of the Foci.

VIII.

Assuming this Discovery, *Newton* examines what is the Law of centripetal Force, required to make the Planets describe an Ellipse, and he found (Prop. 11.) that this Force must follow the inverse Ratio of the Planet's Distance from the Focus of this Ellipse. But having found before (cor. 6. Prop. 4.) that if the periodic Times of Bodies revolving in Circles be in the sesquiplicate Ratio of their Rays, the centripetal Forces would be in the inverse Ratio of those same Distances; he had no more to do to invincibly

prove that the centripetal Force which directs the celestial Bodies in their Courses, follows the inverse Ratio of the Square of the Distances; but to examine if the periodic Times follow the same Proportion in Ellipses as in Circles.

Newton demonstrates that in ellipses the periodic times are in the same proportion as in circles. Consequently the centripetal force which retains the planets in their orbits decreases as the square of the distance.

The centripetal force being in this proportion the planets can only describe conic sections the Sun being placed in one of the foci.

Manner of determining the orbit of a planet supposing the law of centripetal force to be given.

But *Newton* demonstrates (Prop. 15.) that the periodic Times in Ellipses are in the sesquiplicate Ratio of their great Axes; that is, that those Times are in the same Proportion in Ellipses, and Circles whose Diameters are equal to the great Axes of those Ellipses.

This Curve which the Planets describe in their Revolution is endued with this Property, that if small Arcs described in equal Times be taken, the Space bounded by the Line drawn from one of the Extremities of this Arc, and by the Tangent drawn from the other Extremity increases in the same Ratio as the Square of the Distance from the Focus decreases; from whence it follows, that the attractive Power which is proportional to this Space, follows also this same Proportion.

IX.

Newton, not content with examining the Law that makes the Planets describe Ellipses; he enquired further whether in consequence of this Law: Bodies might not describe other Curves, and he found (Cor. 1. Prop. 13.) that this Law would only make them describe a conic Section, the Center of the Force being placed in the Focus, let the projectile Force be what it would.

Other Laws, by which Bodies might describe conic Sections, would make them describe them about Points different from the Focus. *Newton* found, for example, (Prop. 10.) that if the Force be as the Distance from the Center, it will make the Body describe a conic Section, whose Center would be the Center of Forces, thus *Newton* has discovered not only the Law which the centripetal Force observes in our planetary System, but he has also shewn that no other Law could subsist in our World in its present State.

X.

Newton afterwards examines (Prop. 17.) the Curve a Body would describe with a centripetal Force decreasing in the inverse Ratio of the Square of the Distance, supposing the Body let go from a given Point, with a Direction and Velocity assumed at Pleasure.

To solve this Problem, he sets out with the Remark he had made, (Prop. 16.) that the Velocities of Bodies describing conic Sections, are in each Point of those Curves, as the Square-Roots of the principal Parameters, divided by the Perpendiculars, let fall from the Foci on the Tangents to those Points.

This Proposition is not only very interesting, considered merely as a geometrical Problem, but also of great use in Astronomy; for finding by Observation the Velocity and Direction of a Planet in any Part of its Orbit, by the Assistance of this Proposition, the Remainder of its Orbit is found out, and the Determination of the Orbits of Comets, may in a great Measure be deduced from this Proposition.

XI.

It is easy to conceive that in consequence of other Laws of centripetal Force different from that of the Square of the Distances Bodies would describe other Curves, that there are some Laws by which notwithstanding the projectile Force, they would descend to the Sun, and others by which notwithstanding the centripetal Force, they would recede in infinitum in the Heavenly Spaces; others would make them describe Spirals, &c. and *Newton* in the 42d Proposition, investigates what are the Curves described in all Sorts of Hypothesis of centripetal Forces.

XII.

It evidently appears from all that has been said that the perpetual Circulation of the Planets in their Orbits depends on the Proportion between the centripetal and the projectile Force, and those who ask why the Planets arriving at their Perihelia, reascend to their Aphelia, are ignorant of this Proportion; for in the higher Apfis the centripetal Force exceeds the Centrifugal Force, since in descending the Body approaches the Centre, and in the lower Apfis on the Contrary, the centrifugal Force surpasses in its turn the centripetal Force, since in reascending the Body recedes from the Centre: A certain Combination between the centripetal Force and the centrifugal Force was therefore requisit, that they might alternately prevail and cause the Body to descend to the lower, and reascend to the higher Apfis perpetually.

Another Objection was alledged with regard to the Continuation of the Heavenly Motions, derived from the Resistance they should undergo in the Medium in which they move. This Objection *Newton* has answered in (Prop. 10. B. 3.) where he shews that the Resistance of Mediums diminishes in the Ratio of their Weight and their Density; but he proved in the Scholium of (Proposition 22. B. 2.) that at the Height of two hundred Miles above the Surface of the Earth, the Air is more rarified than at the Surface, in the Ratio of 30 to 0,0000000000003998 or nearly as 7500000000000 to 1, from whence he concludes (Prop. 10. B. 3.) supposing the Resistance of the Medium in which Jupiter moves to be of this Density, this Planet describing five of its Semidiameters in 30 days, would from the Resistance of this Medium, in 1000000 years scarcely lose 1000000th Part of its Motion; from hence we see that the Medium in which the Planets move may be so rare and subtle, that its Resistance may be regarded as Void; and the Proportionality constantly observed, between the Areas and the Times, is a convincing Proof that this Resistance is actually insensible.

XIII.

As we have shewn that the Proportionality of the Times and of the Areas which the Planets describe around the Sun, proves that they tend to the Sun as to their Centre, and that the Ratio subsisting between their periodic Times and their Distances, shews that this Force decreases in the inverse

Ratio of the Square of the Distances. If the Planets which perform their Revolutions round the Sun be furrounded by others which revolve round them, and observing the same Proportions in their Revolutions, we may conclude that these Satellites are urged by a centripetal Force directed to their Primaries, and that this Force decreases as that of the Sun in the duplicate Ratio of the Distance.

We can discover only three Planets attended with Satellites, Jupiter, the Earth, and Saturn; we know that the Satellites of those three Planets describe around them Areas proportional to the Times, and consequently are urged by a Force tending to those Planets.

The comparison of the periodic times and distances of the satellites of Saturn and Jupiter, proves that the centripetal force of those planets is also in the inverse ratio of the square of the distances. How Newton discovered that the attractive force of the Earth follows the same proportion.

XIV.

Jupiter and Saturn having each several Satellites whose periodic Times and Distances are known, it is easy to discover whether the Times of their Revolution about their Planet, are to their Distance in the Proportion discovered by *Kepler*; and Observations evince that the Satellites of Jupiter and Saturn observe also this second Law of *Kepler* in revolving round their Primaries, and of consequence the centripetal Force of Jupiter and of Saturn decrease in the Ratio of the Square of the Distances of Bodies from the Centre of those Planets.

XV.

As the Earth is attended only by one Satellite, namely the Moon, it appears at first View difficult to determine the Proportion in which the Force acts that makes the Moon revolve in her Orbit round the Earth, as in this Case we have no Term of Comparison.

Newton has found the Means of supplying this Defect; his Method is as follows: All Bodies which fall on the Surface of the Earth, describe according to the Progression discovered by *Gallileo*, Spaces which are as the Squares of the Times of their Descent. We know the mean Distance of the Moon from the Earth which in round Numbers is about 60 Semidiameters of the Earth; and all Bodies near the Surface of the Earth are considered as equidistant from the Centre; therefore if the same Force produces the Descent of heavy Bodies, and the Revolution of the Moon in her Orbit; and if this Force decreases in the Ratio of the Square of the Distance, its Action on Bodies near the Surface of the Earth should be 3600 Times greater than what it exerts on the Moon, since the Moon is 60 Times remoter from the Centre of the Earth; we know the Moon's Orbit, because we know at present the Measure of the Earth, we know that the Moon describes this Orbit in 27 Days, 7 Hours, 43 Minutes, hence we know the Arc she describes in one Minute; now by (Cor. 9 Prop. 4.) the Arc described in a given Time by a Body revolving uniformly in a Circle with a given centripetal Force, is a mean Proportional between the Diameter of this Circle and the right Line described in the Body's descent during that Time.

It is true that the Moon does not revolve round the Earth in an exact Circle, but we may suppose it such in the present Case without any sensible Error, and in this Hypothesis, the Line expressing the Quantity of the Moon's descent in one Minute, produced by the centripetal Force, is found to be nearly 15 Feet.

But the Moon according to the Progression discovered by *Gallileo*, at her present Distance would describe a Space 3600 Times less in a Second than in a Minute, and Bodies near the Surface of the Earth describe, according to the Experiments of Pendulums, for which we are indebted to *Hughens*, about 15 feet in a Second, that is, 3600 Times more Space than the Moon describes in the same Time; therefore the Force causing their Descent acts 3600 Times more powerfully on them than it does on the Moon; but this is exactly the inverse Proportion of the Squares of their Distances.

By this Example we see the Advantage of knowing the Measure of the Earth; for in order to compare the Verse Sine which expresses the Quantity of the Moon's descent towards the Earth, with the cotemporary Space described by Bodies falling by the Force of Gravity near the Earth, we must know the absolute Distance of the Moon from the Earth, reduced into Feet, as also the Length of the Pendulum vibrating Seconds; for in this Case it is not sufficient to know the Ratio of Quantities, but their absolute Magnitudes.

The measure of the Earth was necessary for making this discovery.

XVI.

Jupiter, Saturn, and our Earth therefore attract Bodies, in the same Proportion that the Sun attracts those Planets, and Induction authorises us to conclude that Gravity follows the same Proportion in Mars, Venus, and Mercury; for by all that we can discover of these three Planets, they appear to be Bodies of the same Nature with the Earth, Jupiter, and Saturn; from whence we may conclude, with the highest Probability, that they are endued with the attractive Force, and that this Force decreases as the Square of the Distances.

Induction authorises us to conclude, that attraction follows the same proportion in the planets which have no satellites.

XVII.

It being proved by Observation and Induction that all the Planets are endued with the attractive Power decreasing as the Square of the Distances; and by the second Law of Motion, Action is always equal to Reaction, we should conclude with *Newton*, (Prop. 5. B. 3.) that all the Planets gravitate to one another, and that as the Sun attracts the Planets, he is reciprocally attracted by them; for as the Earth, Jupiter, and Saturn act on their Satellites in the inverse Ratio of the Square of the Distances, there is no Reason why this Action is not exerted at all Distances in the same Proportion; thus the Planets should attract each other mutually, and the Effects of this mutual Attraction are sensibly perceived in the Conjunction of Jupiter and Saturn.

From whence *Newton* concluded the mutual attraction of all the celestial bodies.

XVIII.

As Analogy enduces us to believe that the secondary Planets are in all Respects Bodies of the same Nature with the primary Planets, it is highly probable that they are also endued with the attractive Power, and consequently attract their Primaries in the same Manner they are attracted by them, and that they mutually attract each other. This is further confirmed by the Attraction of the Moon exerted on the Earth, the Effects of which are visible in the Tides and the Precession of the Equinoxes, as will appear in the Sequel: We may therefore conclude that the attractive Power belongs to all the Heavenly Bodies, and that it acts in all our planetary System in the inverse Ratio of the Square of the Distances.

XIX.

But what is the Cause which makes one Body revolve round another? for instance, the Earth and the Moon attracting each other with Forces decreasing in the duplicate Ratio of their Distances, why should not the Earth revolve round the Moon, instead of causing the Moon to revolve round the Earth; the Law which regulates Attraction does not therefore depend on the Distance alone, it must depend also on some other Element, in order to account for this Determination, for the Distance alone is insufficient, since it is the same for one and the other Globe.

From examining the Bodies that compose our planetary System, it is natural to conclude that this Law is that of their Masses; the Sun, round whom all the Heavenly Bodies turn, appears much bigger than any of them; Saturn and Jupiter are much bigger than their Satellites, and our Earth is much bigger than the Moon which revolves round it.

But as the Bulk and Mass are two different things, to be certain that the Gravity of the Celestial Bodies follows the Law of their Masses, it is necessary to determine those Masses.

But how can the Masses of the different Planets be determined? this *Newton* has shewn.

XX.

To trace the Road that conducted him to this Discovery.

Since the Attraction of all the Celestial Bodies on the Bodies which surround them follows the inverse Ratio of the Square of the Distances, it is highly probable that the Parts of which they are composed attract each other in the same Proportion.

The total attractive Force of a Planet is composed of the attractive Forces of its Parts; for supposing several small Planets to unite and compose a big one, the Force of this big Planet will be composed of the Sum of the Forces of all those small planets; and *Newton* has proved in (Prop. 74, 75 and 76,) that if the Parts of which a Sphere is composed, attract each other mutually in the inverse Ratio of the Square of the Distances, these

What cause makes one body revolve round another.

This cause appears to be the mass of the central body.

The knowledge of the masses of the planets necessary to determine this point.

Road that conducted *Newton* to this discovery.

entire Spheres will attract Bodies which are exterior to them, at whatever Distance they are placed in this same inverse Ratio of the Square of Distances; and of all the Laws of Attraction examined by *Newton*, he has found only two; namely, that in the inverse Ratio of the Square of the Distances, and that in the Ratio of the simple Distances, according to which Spheres attract external Bodies in the same Ratio in which their Parts mutually attract each other; from whence we see the Force of the Reasoning which made *Newton* conclude that since it is proved on one Hand from Theory, (Cor. 3. Prop. 74.) that when the Parts of a Sphere attract each other with Forces decreasing in the duplicate Ratio of the Distances, the entire Sphere attracts external Bodies in the same Ratio, and on the other, Observations evince that the Celestial Bodies attract external Bodies in this Ratio, it is obvious that the Parts of which the Heavenly Bodies are composed, attract each other in this same Ratio.

Newton examines (in Prop. 8. B. 3.) what the same Body would weigh at the Surfaces of the different Planets, and he found by means of (Cor. 2. Prop. 4.) in which he had demonstrated, that the Weights of equal Bodies revolving in Circles, are as the Diameters of those Circles, divided by the Squares of their periodic Times, therefore the periodic Times of Venus round the Sun, of the Satellites of Jupiter round this Planet, of the Satellites of Saturn round Saturn, and of the Moon round the Earth, and the Distances of those Bodies from the Centres about which they revolve being known, supposing also that they describe Circles, which may be supposed in the present Case, he discovers how much the same Body would weigh transferred successively on the Surfaces of Jupiter, Saturn and of the Earth.

He finds the weight of the same body upon the different planets at the same distance from their centres

Having thus found the Weights of the same Body on the Surface of the different Planets at the same Distance from their Centres, *Newton* deduces the Quantities of Matter they contain, for Attraction depending on the Mass and the Distance, at equal Distances the attractive Forces are as the Quantities of Matter in the attracting Bodies; therefore the Masses of the different Planets are as the Weights of the same Body at equal Distances from their Centres.

And proves that their quantities of matter are proportional to those weights.

XXI.

We may discover after the same Manner the Density of the Sun and of those Planets which have Satellites, that is, the Proportion of their Bulks and Masses, for *Newton*, (Prop. 72.) has proved, that the Weights of equal Bodies, at the Surfaces of unequal homogeneous Spheres, are as the Diameters of those Spheres; therefore if those Spheres were heterogeneous and equal, the Weights of Bodies at their Surfaces would be as their Density, supposing the Law of Attraction to depend only of the Distance,

From whence he deduces their densities.

and the Mass of the attracting Body; therefore the Weights of Bodies at the Surfaces of unequal and heterogeneous Spheres, are in the compound Ratio of their Densities and Diameters; consequently the Densities are as the Weights of the Bodies divided by their Diameters.

XXII.

The smallest
and densest
planets are
nearest the
sun.

From hence we find, that the smaller Planets are denser and placed nearer the Sun, for where all the Proportions of our System were laid down, we saw that the Earth, which is less and nearer the Sun than Jupiter and Saturn, is more dense than those Planets.

XXIII.

The reason
assigned by
Newton.

Newton deduces from thence, the Reason of the Arrangement of the Celestial Bodies of our planetary System, which is adapted to the Density of their Matter, in order that each might receive a Degree of Heat more or less according to its Density and Distance; for Experience shews us that the denser any Body is, the more difficultly does it receive Heat; from whence *Newton* concludes that the Matter of which Mercury is composed should be seven Times denser than the Earth, in order that Vegetation might take place; for Illumination, to which, ceteris paribus, Heat is proportional, is inversely as the Square of the Distance; but we know the Proportion of the Distances of the Earth and Mercury from the Sun, and from this Proportion we discover that Mercury is seven Times more illuminated, and consequently seven Times more heated than the Earth; and *Newton* discovered, from his Experiments on the Thermometer, that the Heat of our Summer Sun, seven Times augmented, would make Water boil; therefore if the Earth was placed at the Distance of Mercury from the Sun, our Ocean would be dissipated into Vapour; removed to the Distance of Saturn from the Sun, the Ocean would be perpetually frozen, and in both Cases all Vegetation would cease, and Plants and Animals would perish.

XXIV.

The densi-
ties of the
planets
which have
satellites on-
ly can be dis-
covered, the
moon ex-
cepted.

It easily appears, that the Masses and Densities of such Planets only as are attended by Satellites can be discovered, since to arrive at this Discovery we must compare the periodic Times of the Bodies revolving round those Planets, the Moon alone is to be excepted, of which mention will be made hereafter.

XXV.

Why the sun
is the centre
of the celest-
tial revolu-
tions.

Having determined the Masses of the Planets, we find that those Bodies which have less Mass, revolve round those which have a greater, and the greater Mass a Body has the greater is, ceteris paribus, its attractive Force; thus all the Planets revolve round the Sun, because the Sun has a much greater Mass than any of the Planets, for the Masses of the Sun, Jupiter, and Saturn are respectively as 1, 1100 and 3000; since therefore the Masses of these Planets exceed those of any other in our System, it follows that the Sun should be the Centre of the Motions of our planetary System.

XXVI.

If Attraction be proportional to the Masses, the Alteration caused by the Action of Jupiter on the Orbit of Saturn in their Conjunction, ought much to exceed that produced in the Orbit of Jupiter by the Action of Saturn, since the Mass of Jupiter is much greater than that of Saturn, and this Observation evinces; the Alteration in the Orbit of Jupiter in its Conjunction with Saturn, though sensible is considerably less than what is observed in the Orbit of Saturn.

The alterations which the planets mutually produce in their courses follow the ratio of their masses.

XXVII.

But if the Effect of Attraction, or the Space described by the attracted Body, depends on the Mass of the attracting Body, why should it not also depend on the Mass of the attracted Body? This Point surely deserves to be examined.

Experiment proves that all Bodies near the Surface of the Earth, when the Resistance of the Air is removed, descend with equal Velocities; for in the Air-pump, after exhausting the Air, Gold and Feathers fall to the Bottom in the same Time.

Newton has confirmed this Experiment by another, in which the smallest Difference becomes obvious to our Senses. He relates (Prop. 24. B. 2. and Prop. 6. B. 3.) that he composed several Pendulums of Materials entirely different; for instance of Water, Wood, Gold, Glass, &c. and having suspended them by Threads of equal Length, for a considerable Time their Oscillations were Synchronal,

XXVIII.

It admits therefore of no Doubt, that the attractive Force of our Earth is proportioned to the Masses of the Bodies it attracts, and at equal Distances it depends solely on their Masses, that is on their Quantities of Matter; hence if the terrestrial Bodies were transferred to the Orbit of the Moon, it having been proved already that the same Force acts on the Moon and on those Bodies, and that it decreases as the Square of the Distances. The Distances being supposed equal, it follows, that supposing the Moon deprived of her projectile Force, those Bodies and the Moon would fall in the same Time to the Surface of the Earth, and would describe equal Spaces in equal Times, the Resistance of the Air being taken away.

Attraction is proportional to the masses without any respect being had to the form or species of the attracting bodies.

XXIX.

The same Thing is proved of all the Planets having Satellites, for instance, of Jupiter and Saturn; if the Satellites of Jupiter, for example, were all placed at the same Distance from the Centre of this Planet, and deprived of their projectile Force, they would descend towards it and reach its Surface in the same Time; this follows from the Proportion between the Distances of the Satellites and their periodic Times.

XXX.

From the Proportion between the periodic Times and Distances of the primary Planets from the Sun, it may be proved in like Manner, that the Sun acts on each of them proportionally to its Mass, for at equal Distances their periodic Times would be equal, in which Case, supposing their projectile Force destroyed, they would all reach the Sun at the same Time; therefore the Sun attracts each Planet in the direct Ratio of its Mass.

XXXI.

This Truth is further confirmed by the Regularity of the Orbits which the Satellites of Jupiter describe round this Planet, for *Newton* has proved (Cor. 3. Prop. 65.) that when a System of Bodies move in Circles or regular Ellipses, these Bodies cannot be acted upon by any sensible Force but the attractive Force which makes them describe those Curves; now the Satellites of Jupiter describe round that Planet circular Orbits, sensibly regular and concentric to Jupiter, the Distances of these Moons and of Jupiter from the Sun should be considered as equal, the Difference of their Distances bearing no Proportion to the entire Distance; therefore if any of the Satellites of Jupiter, or Jupiter himself, were more attracted by the Sun in Proportion to its Mass than any other Satellite, then this stronger Attraction of the Sun would disturb the Orbit of this Satellite; and *Newton* says, (Prop. 6. B. 3.) that if this Action of the Sun on one of the Satellites of Jupiter was greater or less in Proportion to its Mass than that which it exerts on Jupiter in Proportion to his, only one thousandth part of its total Gravity, the Distance of the Centre of the Orbit of this Satellite from the Sun would be greater or less than the Distance of the Centre of Jupiter from the Sun, by the two thousandth part of its whole Distance, that is by a fifth Part of the Distance of the outermost Satellite of Jupiter from Jupiter, which would render its Orbit sensibly excentric; since then those Orbits are sensibly concentric to Jupiter, the accelerating Gravities of the Sun on Jupiter and on its Satellites, are proportional to their Quantities of Matter.

The same Reasoning may be applied to Saturn and its Satellites, whose Orbits are sensibly concentric to Saturn.

Experience and Observation therefore leads us to conclude, that the Attraction of the Celestial Bodies is proportional to the Masses, as well in the attracting Body, as in the Body attracted; that it is the Mass which determines a Body to revolve round another, that every Body may be considered indifferently, either as attracting or attracted; in fine, that Attraction is always mutual and reciprocal between two Bodies, and that it is the Proportion between their Masses which decides when this double Attraction shall or shall not be sensible.

Attraction
is always re-
ciprocal.

XXXII.

There is another Property of Attraction, by which it acts equally on Bodies whether at Rest or in Motion, and produces equal Accelerations in equal Times, from whence it follows that its Action is continued and uniform. Which sufficiently appears from the Manner gravity accelerates falling Bodies, and from the Motion of the Planets, which as we have shewn before, are only greater Projectiles regulated by the same Laws.

Attraction acts uniformly & continually whether the Bodies be at rest or in motion.

XXXIII.

Since the Proportion subsisting between the Masses of Bodies which attract each other determines how much one approaches towards the other, it is evident that the Sun having a much greater Mass than the Planets, their Action on him should be insensible. However the Action of the Planets upon the Sun, tho' too inconsiderable to be sensible, produces its Effect; and on Examination we find that the center round which each Planet revolves is not the center of the Sun, but the Point which is the common center of Gravity of the Sun and Planet whose revolution is considered. Thus the Mass of the Sun being to that of Jupiter as 1 to $\frac{1}{1027}$ and the distance of Jupiter from the Sun being to the Sun's semi diameter in a Ratio somewhat greater, it follows that the common Center of Gravity of Jupiter and the Sun is not far distant from the Surface of the Sun.

Effects of the Attraction of the planets on the sun

By the same way of reasoning we find that the common Center of Gravity of Saturn and the Sun falls within the Surface of the Sun, and making the same Calculation for all the Planets, *Newton* says (Prop. 12, B, 3.) that if the Earth and all the Planets were placed on the same Side of the Sun, the common Center of Gravity of the Sun and all the Planets would scarce be one of his Diameters distant from his Center. For tho' we cannot determine the Masses of Mercury, Venus and Mars, yet as these Planets are still less than Saturn and Jupiter, which have infinitely less Mass than the Sun, we may conclude that their Masses do not alter this Proportion.

XXXIV.

It is about this common Center of Gravity that the Planets revolve, and the Sun himself oscillates round this Center of Gravity in Proportion to the Actions of the Planets exerted on him. When therefore we consider the Motion of two Bodies whereof one revolves round the other, rigorously speaking we should not regard the central Body as fixed. The two Bodies, viz the central Body and that which revolves round it, both revolve round their common center of Gravity, but the spaces they describe round this common Center being in the inverse ratio of their Masses, the Curve described by the Body which has the least Mass is almost insensible: For this Reason the Curve described by the Body whose revolution is sensible is only considered, and the small Motion of the central Body, which is regarded as fixed, is neglected.

This effect consists in making the sun oscillate round the common center of gravity of our planetary system

XXXV.

The Earth and the Moon therefore revolve round their common Center of Gravity, and this Center revolves round the Center of Gravity of the Earth and the Sun. The Case is the same with Jupiter and his Moons, Saturn and his Satellites, and with the Sun and all the Planets. Hence the Sun according to the different Positions of the Planets should move successively on every Side around the common Center of Gravity of our planetary System.

This common center of gravity is at rest.

XXXVI.

This common Center of Gravity is at rest, for the different Parts of this System constantly corresponds to the same fixed Stars; now, if this Center was not at rest but moves uniformly in a straight Line, during so many thousand Years that the Heavens have been observed, there must have been remarked some Alteration in the Relation that the different Parts of our planetary System bear to the fixed Stars; but as no Alteration has been observed; it is natural to conclude that the common center of Gravity of our System is at rest. This Center is the Point where all the Bodies of our planetary System would meet if their projectile Forces were destroy'd.

Hence this center cannot be the center of the sun, which moves perpetually.

As the Center of Gravity of our planetary System is at rest, the Center of the Sun cannot be this Center of Gravity since it moves according to the different Positions of the Planets, though on Account of the small Distance between the Center of the Sun and the common Center of gravity of our planetary World it never sensibly recedes from its Place.

XXXVII.

Since Attraction is proportional to the Mass of the attracting Body, and that of the Body attracted, we should conclude that it belongs to every Particle of Matter, and that all the Particles of which a Body is composed attract each other; for if Attraction was not inherent in every Particle of Matter it would not be proportional to the Mass.

XXXVIII.

Answer to the objection founded on the attraction of terrestrial bodies not being sensible.

This Property of Attraction, of being proportional to the Masses, supplies us with an Answer to an Objection which has been alledged against the mutual Attraction of Bodies. If all Bodies it is said are endued with this Property of mutually attracting each other, why is not the Attraction which terrestrial Bodies exert on each other sensible? but it is easily perceived that Attraction being proportional to the Masses of the Attracting Bodies, the Attraction exerted by the Earth on terrestrial Bodies is far more intense than what they exert on each other, and of Consequence these partial Attractions are absorbed and rendered insensible by that of the Earth.

XXXIX.

It is sensible in some cases, as in the deviation of the

The Academicians who measured a Degree of the Meridian in Peru, imagined they perceived a sensible Deviation in the plumb Line occasioned by the Attraction of the Mountain Chimboraco the highest of the Cordilliers it is certain from Theory that the Attraction of this Mountain should affect the

Plumb Line and all Bodies in its Neighborhood : but it remains to know whether the quantity of the observed Deviation corresponds with that which should result from the Bulk of the Mountain for besides that these Observations do not determine the precise Quantity of the Deviation, on account of the errors inseparable from practice, Theory does not furnish any Method of estimating exactly the quantity of this Deviation, as the entire Magnitude, Density &c. of the Mountain are unknown.

XL.

The same reason that hinders us from perceiving the mutual Attraction of Bodies on the surface of the Earth, renders also the mutual Attraction of the heavenly Bodies very seldom sensible. For the more powerful Action that the Sun exerts on them, prevents this mutual Attraction from appearing. However in some cases it is perceivable, for instance in the conjunction of Saturn and Jupiter their Orbits are sensibly disturbed, the Attraction of those two Planets being too strong to be absorbed by that of the Sun.

As to the sensible Attractions of certain terrestrial Bodies, such as Magnetism and Electricity, they follow other Laws and probably arise from Causes different from the universal Attraction of Matter.

Newton demonstrates (Prop. 66.) that the mutual Attractions of two Bodies revolving round a Third, disturb less the Regularity of their motions when the Body round which they revolve is agitated by their Attractions, than if it was at rest; hence the inconsiderable Irregularities observed in the planetary Motions, is a further Proof of the mutual attraction of the celestial Bodies.

XLI.

The Irregularities in the Motion of any Planet arising from the Actions of the rest, are more or less considerable, in Proportion as the Sum of the Fractions composed each of the Mass and Square of the Distance of each of the other Planets, is more or less considerable with respect to the Mass of the Sun divided by the Square of its distance from the Planet, but as the Planes in which the Planets describe their Orbs are differently situated with respect to each other, the Directions of the Central Forces of which the Planets are the Origin, are each in different Planes, and they cannot be all reduced to fewer than Three, by the Rules of the Composition of Forces; each Planet therefore should be considered as actuated every instant by three Forces at the same Time, the first is a tangential Force, or a Force acting in the Direction of the Tangent of the Planets Orb, which is the Result of the Composition of all the Motions which the Planet was affected with the precedent Instant. The second is an accelerating Force, compounded of all the central Forces of the Planets, reduced to one in a right Line in a Plane whose Position is determined by the Center of the Sun, and by the Direction of the tangential Force; the Difference between this

plumb line
at the foot
of chimborazo.

Magnetism
and electricity
have different
causes from
the universal
attraction
of bodies

Manner of
determining
the irregularities
in the motion of
the planets
arising from
their mutual
attractions

Abstracting
from the
mutual at-
traction of
the planets
their aphelia
are at rest.

The slow
motion of
the aphelia
of the plan-
ets is a new
proof that
attraction
acts in the
inverse ratio
of the square
of the dis-
tances

compounded Force and the simple central Force which has no other Source but the Sun, is called the perturbing Force. The third Force is the deturbating Force, compounded of all the same central Forces of the Planets reduced to one in a Direction perpendicular to the Planes of their Orbits; this Force is very small in comparison of the two others, on account of the small Inclination of those Planes to one another, and because the Sun placed in the Intersection of all those Planes does no way contribute to the Production of this deturbating Force. If the Planets were only actuated by the two first Forces their Combination would serve to determine their Trajectories which would be each in a constant Plane, and if the perturbing Force vanished then they would be regular Ellipses, and consequently the Aphelia and Nodes of the Planets would be fixed (Prop. 14. B. 3. & Prop. 1. & 11. B. 1.) if not; these Trajectories might be considered as moveable Ellipses on account of the prodigious excess of the central Force of the Sun over the perturbing Force, it is thus *Newton* investigated the quantity and direction of the Motion of the Line of the Apfides of the Planets occasioned by the Action of Jupiter and Saturn, which according to his Determination follows the Sesquiplicate Proportion of the distances of the Planets from the Sun, from whence he concludes (Prop. 14. B. 3.) that supposing the Motion of the line of the Apfides of Mars in which this Motion is the most sensible to advance in a 100 Years $33^m 20^s$ in consequentia, the Aphelia of the Earth, Venus and Mercury would advance $17^m 40^s$ $10^m 53^s$ & $4^m 16^s$ respectively in the same Time.

This slow Motion of the Aphelia confirms the Law of universal Gravitation, for *Newton* has demonstrated (Cor. 1. Prop. 45.) that if the Proportion of the centripetal Force would recede from the Duplicate to approach to the Triplicate only the 60th Part, the Apfides would advance 3 Degrees in a Revolution, therefore since the Motion of the Apfides is almost insensible, Gravity follows the inverse duplicate Proportion of the distances.

But the deturbating Force which acts at the same Time causes the Planes of those moveable Ellipses to Change continually their Position; let there be supposed in the Heavens an immoveable Plane, in a mean Position between all those the Trajectory of the Earth would take in consequence of the deturbating Force, which may be called the true Plane of the Ecliptic, it is manifest that this Plane being very little enclined to the Plane of the Orbit of Each Planet, it is almost parallel to it, and consequently the Direction of the deturbating Force is always sensibly perpendicular to the true Plane of the Ecliptic, and it is easy to conceive that the effect of this Force produced in the Direction in which it acts, is either to remove the Planet from or to make it approach the true Plane of the Ecliptick, consequently to cause a Variation in the Inclination of the small Arc which the Planet des-

cribes that instant with the true Plane of the Ecliptick, the Position of the Planes of the Trajectories of the Planets varies therefore in Proportion of the Intensity of the deturbating Force, and in the Direction in which this Force acts; if for Example the Force tends to make the Planet approach the true Plane of the Ecliptic the Node advances towards the Planet with a Velocity, which tho' small increases diminishes or vanishes according as the intensity of the deturbating Force increases diminishes or vanishes, but in this Case the Node cannot advance or go meet the Planet without moving in an opposite Direction to that of the Planet, if therefore the heliocentric Motion is retrograde as in a great Number of Comets, that of the Nodes will be direct, the contrary would arrive if the deturbating Force tended to remove the Planet from the true Plane of the Ecliptic. *Newton* says that supposing the Plane of the Ecliptic to be fixed the Regression of the Nodes is to the Motion of the Aphelium in any Orbit of a Planet as 10 to 21 nearly (c).

Retrogradation of the nodes of the planets according to *Newton*.

It is therefore only by this Composition of Forces that all the Irregularities of the celestial Motions can be investigated, it is by discerning the particular Effects of each of those compounded Forces, and afterwards uniting them, that not only those Irregularities that have been observed can be determined, but those which will be remarked hereafter will be foretold. But it is easy to perceive how much sagacity and address to handle the sublimest Analysis these Reschearches require, and as it is almost impossible to combine at once the central Forces of more than three Bodies placed in different Planes, in order to discover the irregularities of the Motions of a Planet or Comet it is necessary to calculate succesively the Variations that each Planet taken seperately can cause in the central Force of which the Sun is the Focus. The Success that has attended the united Efforts of the first Mathematicians in Europe shall be explained hereafter.

Theory of the Figure of the Planets.

I.

The Planets have another Motion viz. their Rotation round their Axes, we have seen already, that this Motion of Rotation has only been discovered in the Sun, the Earth, Mars, Jupiter and Venus, and that Astronomers do not agree about the Time in which Venus turns round tho' they are unanimous with respect to its Rotation. But tho' it has not been discovered from Observation that Mercury, Saturn and the Satellities of Jupiter and Saturn turn round their Axes, from the uniformity that Nature Observes in her Operations, it is highly probable that those Planets revolve round their Axes, and that all the celestial Bodies partake of this Motion.

The cause of the rotary motion of the planets has not as yet been discovered.

This Rotation of the Planets round their Axes is the only celestial Motion which is uniform: this Motion does not appear to arise from Gravity, and its Cause has not as yet been discovered.

II.

The mutual attraction of the parts which compose the planets prevents them from being dispersed by the rotation.

The mutual Attraction of the Parts of which the Planets are composed binds them together, and prevents their being dispersed by this Rotation. For it is well known that all Bodies moving round acquire a centrifugal Force by which they endeavour to recede from the Center of their Revolutions; hence, were not the Parts of the Planets held together by their mutual Attractions, they would be dispersed and scattered by their Rotation. For supposing the Gravity of any one Part of the surface of the revolving Body destroyed, this Part instead of revolving with the Body would fly off in the direction of the tangent; therefore if Gravity did not counteract the Efforts of the centrifugal Force which the Parts of the celestial Bodies acquire in revolving round their Axes, this force would disperse their Parts.

III.

The rotational motion, raises the equators of the planets.

Tho' this mutual Attraction of the Parts of a Planet, counteracts the centrifugal Force, yet it does not destroy it, this Force still producing its Effect, in rendering the diameters of the revolving Body unequal, supposing it to be fluid; for the Planets being composed of Matter whose Particles at equal Distances are equally urged to the Center, they would be exact Spheres if they were at rest. But in consequence of the Motion of Rotation the Parts acquiring a centrifugal Force endeavour to recede from their Centers with Forces which increase as they are placed nearer the Equator of the revolving Body, since the centrifugal Forces of Bodies revolving in Circles, are as their Rays supposing the Time of Revolution to be equal: therefore supposing the Planets to be spherical and composed of fluid Matter, before they acquired a Motion of Rotation, that the Equilibrium of their Parts may be preserved during this Rotation, and that they may assume a permanent form it was necessary that the Column whose weight was diminished by the centrifugal Force should be longer than the Column whose Weight is not altered by the centrifugal Force, and therefore the Equatorial Diameter must exceed the Diameter passing thro' the Poles.

IV.

Method Newton pursued for determining the figure of the Earth.

Newton in (Prop. 19. B 3) determines the excess of the equatorial above the polar Column of the Earth, supposing as he does all thro' the Principia that the Gravity of Bodies near the surface of the Earth is the result of the Attraction, of all the Particles of which the Earth considered as Homogeneous is composed: he employs for Data in the Solution of this Problem, 1st the Semidiameter of the Earth considered as a Sphere and determined by Picard to be 19615800. Feet 2^d, the Length of the Pendulum vibrating seconds in the Latitude of Paris which is 3 Feet 8 $\frac{1}{2}$ Lines.

From the Theory of Oscillations and this Measure of a Pendulum vibrating seconds, he proves that a Body in the Latitude of Paris making the necessary Correction for the resistance of the Air, describes in a second 2174 Lines.

A Body revolving in a Circle at the Distance of 19615800 Feet from the Center, which is the Semidiameter of the Earth, in $23^h 56^m 4^s$ which is the exact Time of the diurnal Revolution, supposing its Motion uniform, describes in a second, an Arc of 1433, 46 Feet; of which the verse Sine is, 0,0523656 Feet, or 7, 54064 Lines; therefore in the Latitude of Paris the Force of Gravity is to the centrifugal Force, which Bodies at the Equator derive from the diurnal Rotation, as 2174 to 7, 54064. Adding therefore to the Force of Gravity, in the Latitude of Paris, the Force detracted therefrom by the centrifugal Force in that Latitude, in order to obtain the total Force of Gravity in the Latitude of Paris, *Newton* finds that this total Force is to the centrifugal Force under the Equator as 289 to 1 so that under the Equator the centrifugal Force diminishes the centrifugal Force by $\frac{1}{289}$.

Newton determines (Cor. 2. Prop. 91.) the Proportion of the Attraction of a Spheroid upon a Corpuscule placed in its Axe produced, to that of a Sphere, on the same Corpuscule, whose Diameter is equal to the lesser Axe of the Spheroid; employing therefore this Proportion and supposing the Earth homogeneous and at rest, he finds (Prop. 19. B. 3.) that if its Form be that of a Spheroid whose lesser Axe is to the greater as 100 to 101, the Gravity (g) at the Pole of this Spheroid will be to the Gravity (γ) at the Pole of a Sphere, whose Diameter is the lesser Axe of the Spheroid as 126 to 125.

In the same Manner supposing a Spheroid whose equatorial Diameter is the Axe of Revolution, the Gravity (V) at the Equator which is the Pole of this new Spheroid, will be to the Gravity (Γ) of a Sphere at the same Place having the same Axe of Revolution, as 125 to 126.

Newton shews afterwards that a mean proportional (G) between these two Gravities (V , Γ) expresses the Gravity at the Equator of the Earth; consequently the Gravity (G) at the Equator of the Earth, is to the Gravity (Γ) of a Sphere at the same Place, having the same Axe of Revolution, as $125\frac{1}{2}$, to 126, and having demonstrated (Prop. 72) that the Attraction of homogeneous Spheres at their Surfaces is proportional to their Rays, it follows that the Gravity (γ) at the Surface of the Sphere whose Diameter is the lesser Axe of the Spheroid, is to the Gravity (Γ) at the Surface of the Sphere whose Diameter is the great Axe of the Spheroid, as 100 to 101 wherefore by the Composition of Ratios $g \times \gamma \times \Gamma$ is to $\gamma \times G \times \Gamma$ or the Gravity (g) of the Earth, at the Pole, is to the Gravity (G) at the Equator as $126 \times 126 \times 100$ to $125 \times 125\frac{1}{2} \times 101$ that is as 501 to 500.

But he had demonstrated, (Cor. Prop. 91.) that if the Corpuscule is placed within the Spheroid, it would be attracted in the Ratio of its distance

from the Center; therefore the Gravities in each of the Canals corresponding to the Equator and to the Pole will be as the Distances from the Center of the Bodies, which are placed in those Canals; therefore supposing these Canals to be divided into Parts, proportional to the Wholes, consequently at Distances from the Center proportional to each other, by Transverse Planes, which pass at Distances proportional to those Canals. The Weights of each Part in one of those Canals, will be to the Weights of each correspondent Part in the other Canal, in a constant Ratio, consequently these Weights will be to each other in a constant Ratio of each Part, and their accelerative Gravities Conjointly, that is as 101 to 100, and 500 to 501, that is, as 505 to 501; therefore if the centrifugal Force of any Part of the Equatorial Canal be to the absolute Weight of the same Part as 4 to 505, that is, if the centrifugal Force detracts from the Weight of any Part of the Equatorial Canal $\frac{4}{505}$ Parts, the Weights of the Correspondent Parts of each Canal will become equal, and the Fluid will be in Equilibrio. But we have seen that the Centrifugal Force of any Part under the Equator, is to its Weight as 1 to 289, and not as 4 to 505; the Proportion of the Axes therefore must be different from that of 100 to 101, and such a Proportion must be found as will give the Centrifugal Force under the Equator, only the 289th Part of Gravity.

From whence he concludes the ratio of the axes of the earth to be that of 229 to 230.

But this is easily found by the Rule of Three; for if the Proportion of 100 to 101 in the Axes has given that of 4 to 505 for the Proportion of the Centrifugal Force to Gravity, it is manifest that the Proportion of 229 to 230 is requisite to give the Proportion 1 to 289 of the Centrifugal Force to Gravity.

V.

The flatness or the earth towards the poles would always result from the theory of centrifugal forces and that of fluids what hypothesis of gravity is assumed.

This Conclusion of *Newton*, that is, the Quantity of the Depression of the Earth towards the Poles, which he has determin'd is grounded on his Principle of the mutual Attractions of the Parts of Matter. But this Depression towards the Poles would also result from the Theory of Fluids, and that of Centrifugal Forces, tho' *Newton's* Discoveries concerning Gravity were rejected, unless very improbable Hypotheses concerning the Nature of primitive Gravity were adopted.

VI.

The measure of the degrees of the meridian in France

Notwithstanding the Authority of *Newton*, and although *Hugbans* in assuming a different Hypothesis of Gravity arrived, at the same Conclusion of the Depression of the Earth towards the Poles; and tho' all the Experiments made on Pendulums in the different Regions of the Earth, confirmed the decrease of Gravity towards the Equator, and consequently favoured the opinion of the Flatness of the Earth towards the Poles, yet the Measures of Degrees in France, which seemed to decrease as the Latitude increased still rendered the Figure of the Earth

uncertain. Hypotheses were formed on the Nature of primitive Gravity, which gave to the Earth, supposed at rest, a Figure whose Alteration agreed with the Theory of centrifugal Forces, and with the oblong Figure towards the Poles resulting from the actual Measures.

occasioned doubts with regard to the Figure of the Earth.

For the Question of the Figure of the Earth depends on the Law according to which primitive Gravity acts, and it is certain, for Example, that if this Force depended on a Cause which would make it draw sometimes to one Side and at other Times to another, and which increased or diminished without any constant Law, neither Theory nor Observation ever could determine this Figure.

VII.

To decide this Question finally it was Necessary to Measure a Degree under the Equator, and another within the polar Circle; if the French Astronomers gave Occasion to the Doubts raised concerning the true Figure of the Earth, yet in Justice to them it must be acknowledged, that it is to their indefatigable Industry we are indebted for the Confirmation of the Theory of *Newton*, with Respect to the Figure of the Earth, whose Depression towards the Poles is now universally allowed.

The measures of the meridian taken at the polar circle and at the equator confirms the theory of *Newton*.

VIII.

In determining the Ratio of the Axes of the Earth, *Newton* besides the mutual Attraction of the Parts of Matter supposes the Earth to be an Elliptic Spheroid, and that its Matter is Homogeneous; *Maclaurin* in his excellent Piece on the Tides which carried the Prize of the royal Academy of Sciences in 1740, was the first who demonstrated that the Earth supposed Fluid and Homogeneous, whose Parts attract each other mutually and are besides Attracted by the Sun and Moon, revolving about its Axis, would necessarily assume the Form of an Elliptic Spheroid, and demonstrated further, that in this Spheroid not only the Direction of Gravity was perpendicular to the Surface, and the Central Columns in Equilibrio, but that any Point whatsoever within the Spheroid was equally pressed on every Side; which last Point was no less Necessary to be proved than the two first, in Order to be assured that the Fluid was in Equilibrio, yet had been neglected by all those who before treated of the Figure of the Earth.

Two suppositions made by *Newton* in determining the figure of the earth.

Maclaurin verified the first.

The Case is not the same with regard to the second Supposition viz. the Homogeneity of the Matter of the Earth, for it is very possible (and *Newton* himself was of Opinion Prop. 20 B. 3) that the Density of the Earth increases in approaching the Center, now, the different Densities of the Strata of Matter composing the Earth should change the Law according to which the Bodies of which it is composed Gravitate, and of Consequence should alter the Proportion of its Axes.

It is probable that the second is false.

The ratio
of the axes
of the earth
decreases in
proportion
as gravity
increases at
the poles.

Clairaut improving on the Researches of *Maclaurin* has shewn that among all the most probable Hypotheses that can be framed concerning the Density of the interior Parts of the Earth considered as an Elliptic Spheroid, that adopting Attraction, there always subsists such a Connexion between the Fraction expressing the Difference of the Axes, and that which expresses the Decrease of Gravity from the Pole to the Equator, that if one of those two Fractions exceeds $\frac{1}{110}$ by any Quantity, the other will be exactly so much less; so that supposing, for Instance, that the excess of the equatorial Diameter above the Axe is $\frac{1}{173}$, a Supposition conformable with the actual Measures, we shall have $\frac{1}{173} - \frac{1}{110}$ or $\frac{1}{298}$ for the Quantity to be subtracted from $\frac{1}{110}$ in Order to obtain the total Abreviation of the Pendulum in advancing from the Pole to the Equator, that is to say, that this Abreviation or what comes to the same the total Diminution of Gravity, will be $\frac{1}{110} - \frac{1}{298}$, or $\frac{1}{141}$ nearly.

Now, as all the Experiments on Pendulums shew that the Diminution of Gravity from the Pole to the Equator, far from being less than $\frac{1}{110}$ as this Theory requires, is much greater, it follows, that the actual Measures in this Point are inconsistent with the Theory of the Earth considered as an Elliptic Spheroid.

It follows from the Theory of *Clairaut*, that admitting, the Suppositions the most natural we can conceive or imagine with regard to the internal Structure of the Earth considered as an oblate Elliptic Spheroid, that the Ratio of the Axes cannot exceed that of 229 to 230 since this Ratio is what arises from the Supposition of the Homogeneity of the Earth, and that it results from this Theory, that in every other Case Gravity increasing, the Depression towards the Poles is less.

Tho' the Earth supposed Fluid and Heterogeneous whose Parts attract each other mutually, assumes an Elliptic Form consistent with the Laws of Hydrostatics, yet it might equally assume an infinite Number of other Forms consistent with the same Laws, as *Dalambert* has demonstrated, and as a Variation in the Form would necessarily produce one in the Decrease of Gravity from the Pole to the Equator, and consequently in the Ratio of the Axes, it is highly probable that a Figure will be found that will conduct to a Result such as will reconcile Theory with Observation. The Recherches of this eminent Mathematician shall be explained hereafter.

Newton having computed the Ratio of the Axes of the Earth, determines the Excess of its Height, at the Equator above its Height at the Poles, in the following Manner. The Semidiameter ($b + c$) at the Equator being to the Semidiameter (b) at the Poles, as 230 to 229, $c = \frac{b}{229}$ and $2b = 458 c$. and the Mean Semidiameter according to *Picart's* mensuration, being 19615800 Paris Feet, or 3923, 16 Miles,

(reckoning 5000 Feet for a Mile,) $2 \times 19615800 = 2b + c$. consequently $459c = 2 \times 19615800$ and the Excess (c) of the Height of the Earth at the Equator, above its Height at the Poles, is 85472 Feet or 17 Miles $\frac{1}{16}$, and Substituting in the Equation $2 \times 19615800 = 2b + c$. for c its Value, there will result $459b = 2 \times 19615800 \times 229$, wherefore the Height (b) at the Poles will be 19573064 and the Height (b + c) at the Equator 19658536 Feet.

X.

After determining the Relation of the Axes of the Earth supposed Homogeneous, *Newton* investigates after the following Manner (Prop. 20 B. 3) what Bodies weigh in the different Regions of the Earth. Since he had proved that the Polar and Equatorial Columns, were in Equilibrio when their Lengths were to each other as 229 to 230 it follows that if a Body (B) be to another (b) as 229 to 230, and the one (B) be placed at the Pole, and the other (b) at the Equator, the Weight (W) of the Body (B) will be equal to the Weight (w) of the Body (b). but if those two Bodies be placed at the Equator the Weight (W) of the Body (B) will be to the Weight (w) of the Body (b) as 229 to 230, wherefore the Weight [W] of the Body [B] at the Pole will be to the Weight [W] of the same or of an equal Body at the Equator, as 230 to 229, that is reciprocally as those Columns. we see by the same reasoning, that on all the Columns of Matter composing the Spheroid, the Weights of Bodies should be inversely as these Columns, that is as their Distances from the Center: therefore supposing the Distance, of any Place on the Surface of the Earth, from the Center to be known, the Weight of a Body in this Place will be known, and consequently the Quantity of the Increase or Decrease of Gravity, in advancing towards the Poles or the Equator: but as the Distance of any Place from the Center decreases nearly as the Square of the Sine of the Latitude, or as the Versed Sine of double the Latitude as may easily be proved by Calculation, we see how *Newton* formed the Table given (Prop. 20 B. 3) where he lays down the Decrease of Gravity in advancing from the Pole to the Equator.

Example. The Latitude of Paris being $48^{\circ} 50'$ that of Places under the Equator $00^{\circ} 00'$ and that of Places under the Poles 90° , the versed Sines of double those Latitudes are 1134, 00000, and 20000, and the Force of Gravity (g) at the Poles being to the Force of Gravity (G) at the Equator as 230 to 229, the Excess (g — G or E) of the Force of Gravity at the Pole, is to the Force of Gravity (G) at the Equator as $230 - 229$ to 229, or as 1 to 229 but the Excess (e) of the Force of Gravity in the Latitude of Paris is to the Excess (E) of the Force of Gravity at the Poles as 1134 to 20000, wherefore by the Composition of Ratios, $e \times E$ is to EXG , or the Excess [e] of the Force of Gravity in the Latitude of Paris is to the Force of Gravity [G] at the Equator as 1×1134 to 229×20000 ,

What are the weights of bodies in the different regions of the earth.

that is, as 5667 to 2290000, and the Force of Gravity $[e+G]$ in the Latitude of Paris is to the Force of Gravity $[G]$ at the Equator as 5667 to 2290000, that is, as 2295667 to 2290000. By a like Calculus the Force of Gravity in any other Latitude is determined.

They are proportional to the lengths of synchro-
nal pendulums.

As Gravity is the sole Cause of the Oscillations of Pendulums, the slackning of these Oscillations proves the Diminution of Gravity, and their Acceleration proves that Gravity acts more powerfully; but it is demonstrated that the Celerity of the Oscillations of Pendulums is inversely as the Length of the Thread to which they are suspended, therefore when in Order to render the Vibrations of a Pendulum in a certain Latitude synchro-
nal with its Vibrations in another Latitude, it must be shortened or lengthned, we should conclude that Gravity is less or greater in this Region than in the other; *Hugbans* has determined the Relation which subsists between the Quantity a Pendulum is lengthned or shortened and the Diminution or Augmentation of Gravity; so that this Quantity being proportional to the Augmentation or Diminution of the Weight, *Newton* has given in his Table the Length of Pendulums instead of the Weights.

Example. The Length of the Pendulum in the Latitude of Paris being 3^f. 8^l. 56^l, the Gravity in the Latitude of Paris [2295667] is to the Gravity at the Equator [2290000] as the Length of the Pendulum in the Latitude of Paris [3^f. 8^l. 56^l] to the Length of the Pendulum at the Equator [3^f. 7^l. 684] By a like Calculus the Length of the Pendulum in any other Latitude is determined.

xi.

The degrees of latitude are in the same proportion.

The Degrees of Latitude decreasing in the Spheroid of *Newton* in the same Proportion as the Weights, the same Table gives the Quantity of the Degrees in Latitude commencing from the Equator where the Latitude is 0^d to the Pole where it is 90^d.

Example. The Length of a Degree $[d]$ at the Poles, being to the Length of a Degree $[D]$ at the Equator, as the Ray of the Circle which has the same Curvature as the Arc of the Meridian at the Pole, is to the Ray of the Circle which has the same Curvature as the Arc of the Meridian at the Equator of the Earth, that is, by the Property of the Ellipsis, as the Cube of 230 to the Cube of 229, that is, as 12167000 to 12008989, the Excess $[d-D]$ or $[E]$ of the Degree at the Pole is to the Degree $[D]$ at the Equator, as 158011 to 12008989; but the Excess $[e]$ of a Degree in the Latitude of Paris, is to the Excess $[E]$ of the Degree at the Pole, as 11334 to 20000 verse Sines of Double of those Latitudes. Wherefore by the Composition of Ratios $e \times E$ is to $B \times D$, or the Excess $[e]$ of a Degree in the Latitude of Paris is to the Length of the Degree $[D]$ at the Equator, as 895448337 is to 12008989000; and the Length $[e+D]$ of a Degree in the Latitude of Paris is to the Length of a

Degree [D] at the Equator, as 120985338337 to 120089890000; but the Length of a Degree in the Latitude of Paris, according to *Picard's*, Mensuration is 57061 Toises, wherefore the Length of a Degree at the Equator is 56637. By a like Calculus the Length of a Degree in any other Latitude is Determined.

XII.

Latitude of the Place.	Length of the Pendulum.	Measure of one Degree in the Meridian.
Deg.	Feet. Lines.	Toises.
0	3 . 7,458	56637
5	3 . 7,482	56642
10	3 . 7,526	56659
15	3 . 7,596	56687
20	3 . 7,692	56724
25	3 . 7,812	56769
30	3 . 7,948	56823
35	3 . 8,099	56882
40	3 . 8,261	56945
45	3 . 8,294	56958
50	3 . 8,327	56971
55	3 . 8,361	56984
60	3 . 8,394	56997
65	3 . 8,428	57010
70	3 . 8,461	57022
75	3 . 8,494	57035
80	3 . 8,528	57048
85	3 . 8,561	57061
90	3 . 8,594	57074
	3 . 8,756	57137
	3 . 8,987	57196
	3 . 9,044	57250
	3 . 9,162	57293
	3 . 9,329	57360
	3 . 9,372	57377
	3 . 9,387	57382

XIII.

Newton's Table gives the decrease of Gravity from the Pole to the Equator somewhat less than what results from actual Measures, but this Table is only calculated for the Case of Homogeneity; and he informs us at the End of

the Proposition where he gives this Table, that supposing the Density of the Parts of the Earth to increase from the Circumference to the Center, the Diminution of Gravity from the Pole to the Equator would also increase.

XIV.

He attributes this difference to the heat at the equator which lengthens the pendulum in those regions but later experiments have shewn that those differences cannot arise from the lengthening of the pendulum produced by the heat in those regions.

Altho *Newton* seems inclined to believe, from the Observations he relates in Prop. 20 on the lengthning of the Pendulum occasioned by the Heat in the Regions of the Equator, that these Differences arise from the different Temperature of the Places in which the Observations have been made, the great Care and Attention employ'd in preserving the same Degree of Heat by means of the Thermometer in the experiments made since *Newton's* Time on the Length of Pendulums in the different regions of the Earth proves that these Differences do not arise from this Cause, and that the Decrease of Gravity from the Pole to the Equator exceeds the Proportion assign'd by *Newton* in his Table.

In Effect the Lengths of the Pendulum Corrected by the Barometer and reduced to that of a Pendulum oscillating in a Medium without Resistance are under the Equator,

	439, 21 Lines,	
At Portobello Latitude, 9 Degrees,	439, 30	0, 09 Differences.
At little Goave Latitude, 18 Degrees,	439, 47	0, 26
At Paris Latitude, 48 ^d 50 ^m	440, 67	1, 46
At Pello Latitude, 66 ^d 48 ^m	441, 27	2, 06

Now the differences proportional to the Squares of the Sines of the Latitude, are 7, 24, 138, 206, which are less than what results from Experiment.

XV.

Method given by *Newton* for finding the ratio of the axes of any planet.

At the End of Prop. 19. B. 3. *Newton* shews how to find the Proportion of the Axes of a Planet whose Density and diurnal Rotation are known, employing for Term of Comparison the Ratio discovered between the Axes of the Earth; for whether the Bulk or Ray (r) of a Planet be greater or less than the Bulk or Ray (R) of the Earth, if its Density (d) be equal to the Density (D) of the Earth, and the Time (t) of its diurnal Rotation be equal to the Time (T) of the diurnal Rotation of the Earth, the same Proportion will subsist between the centrifugal Force and Gravity, and consequently between its Diameters as was found between the Axes of the Earth: But if its diurnal Rotation is more or less rapid than that of the Earth, the centrifugal Force of the Planet will be greater or less than the centrifugal Force of the Earth and consequently the Difference of the Axes of the Planet will be greater or less than the difference of the Axes of the Earth in the Ratio of $\frac{r}{tt}$ to $\frac{R}{TT}$

(Cor. 2. Prop. 4.) and if the Density of the Planet be greater or less than the Density of the Earth, the Gravity on this Planet will be greater or less than the Gravity on the Earth, in the Ratio of $d r$ to DR , and the Difference of the Axes of the Planet will be greater or less than the Difference of

the Axes of the Earth, in Proportion as the Gravity on the Planet is less or greater than the Gravity on the Earth consequently in the Ratio $\frac{1}{dr}$ to $\frac{1}{DR}$ wherefore if the Time of Rotation and Density of a Planet be different from that of the Earth, the Difference of the Axes of this Planet compared with its lesser Axis, is to $\frac{1}{2\frac{1}{2}}$, the difference of the Axis of the Earth compared with its lesser Axis, as $\frac{r}{t \times d \times r}$ to $\frac{R}{T \times D \times R}$ which gives $\frac{1}{2\frac{1}{2}} \times \frac{D \times T \times T}{d \times t \times t}$ for the expression of the Difference of the Axes of the Planet.

XVI.

Hence the Difference of the diameters of Jupiter, for instance whose diurnal Revolution and Density are known will be to its lesser Axis in the compound Ratio of the Squares of the Times of the diurnal Revolution of the Earth and Jupiter of the Densities of the Earth and Jupiter, and the Difference

Determination of the ratio of the axes of Jupiter according to this method.

of the Axes of the Earth compared with its lesser Axis, that is, as $\frac{29}{5} \times \frac{400}{49\frac{1}{2}} \times \frac{1}{229}$ to 1. that is, as 1. to $9\frac{1}{2}$ nearly: Therefore the Diameter of

Jupiter from East to West is to its Diameter passing thro' the Poles as $10\frac{1}{2}$ to $9\frac{1}{2}$ nearly. *Newton* adds that in this Determination he has supposed that the Matter of Jupiter was Homogeneous, but as it is probable on account of the Heat of the Sun that Jupiter may be denser towards the Regions of the Equator than towards the Poles, these Diameters may be to each other as 12 to 11, 13 to 12, or even as 14 to 13, and that thus Theory agrees with Observation, since Observation evinces that Jupiter is depressed towards the Poles, and that the Ratio of his Axes is less than that of $10\frac{1}{2}$ to $9\frac{1}{2}$ and is confined between the ratios of 11 to 12 and 13 to 14.

XVII.

This Method that *Newton* takes to explain a Depression towards the Poles of Jupiter less than that which results in the Case of Homogeneity seems very improbable, it is surprising that in Order to explain the flatness of the Figure of Jupiter, he has had recourse to a Cause whose Effect would be much more sensibly perceived on the Earth than in Jupiter, since the Earth is much nearer the Sun than Jupiter.

A very improbable reason assigned by *Newton* why the flatness of the figure of Jupiter is less than what results from theory.

The Proposition of *Clairaut* that the Flatness diminishes as the Density increases towards the Center, furnishes a natural Explication of this Phenomenon in supposing Jupiter denser towards the Center than at the Surface, an Hypothesis entirely consistent with the Laws of Mechanicks.

XVIII.

As the two Principles necessary for determining the Axes namely the diurnal Revolution and the Density, are known only in Jupiter, the Earth, and the Sun, these are the only celestial Bodies the Proportion of whose Axes can be discovered. How this Proportion has been discovered in the Earth

Why the ratio of the axes only of Jupiter the earth and the sun can be found.

The propor-
tion of the
axes of the
sun is too
inconsidera-
ble to be
observed.

and Jupiter has been already shewn; the Difference of the Axes of the Sun is to its lesser Axis in the compounded Ratio of the Square of 1 to $27\frac{1}{2}$ diurnal Revolution of the Earth to that of the Sun, of 400 to 100 Density of the Earth to that of the Sun, and $2\frac{1}{2}$, Difference of the Diameters of the Earth compared to its lesser Axis, to 1, that is, as $41\frac{1}{2}$ to 1, a Difference too inconsiderable to be observed.

Theory of the Precession of the Equinoxes.

I.

It was
thought for
a long time
that the
axis of the
earth always
preserved its
parallelism.

For many Ages it had been thought that the Axis of the Earth during its annual Revolution preserved the same Position, and this Supposition was very natural. For Theory shews that this Parallelism should result from the two known Motions of the Earth, the annual and diurnal Motion; and in Fact for a Number of Years this Parallelism is sensibly preserved. But from the Continuance, and accuracy of Astronomical Observations it has been discovered that the Poles of the Earth are not always directed to the same fixed Stars, and of Consequence that the Axis of the Earth does not always remain parallel to itself.

II.

Hyparchus
was the
first who
perceived
the revolu-
tion of the
poles of the
earth.

This Motion of the Axis of the Earth was first perceived by *Hyparchus*; and afterwards established by *Ptolomey* who fixed this Motion to a Degree in a hundred Years, so that the entire Revolution of the Sphere of the fixed Stars from whence *Ptolomey* derived this appearance, was completed in 36000 Years; and it was generally believed in his Time that at the Expiration of this Revolution called *the great Year*, the celestial Bodies would return to their primitive Position.

Ptolomey
fixed the
duration of
this revolu-
tion which
was called
the great
year.

The Arabs discovered that *Ptolomey* had made this Motion too slow, *Ullughbeig* fixed it to a Degree in 72 Years, and Modern Astronomers by fixing it to $51''$ annually have confirmed the Discovery of *Ullughbeig*; so that the Revolution of the Poles of the Earth is completed in 25920 Years.

III.

Ullughbeig
corrected
the time
assigned by
Ptolomey
for this re-
volution.
This regres-
sion causes
an apparent
motion in
the fixed
stars.

The equinoctial Points change their Places in the same Time and by the same Quantity as the Poles of the World, and it is this Motion of the Equinoctial Points which is called the Precession of the Equinoxes.

Tho' the fixed Stars are immovable, at least in respect of us, yet as the common intersection of the Equator and Ecliptic Recedes, it is necessary that the Stars which correspond to those Points should continually appear to change their Places, and that they should seem to advance eastward, from whence it arrives, that their Longitudes, which is reckoned on the Ecliptic

from the Beginning of *Aries*, or the vernal Intersection of the Equator and Ecliptic, continually increases, and the fixed Stars appear to move in *Consequencia*; but this Motion is only apparent and arises from the Regression of the Equinoctial Points in a contrary Direction,

v.

In Consequence of this Regression, all the Constellations of the Zodiac have changed their Places since the Observations of the first Astronomers; For the Constellation *Aries*, for Example, which in the Time of Hipparchus corresponded to the vernal Intersection of the Equator and Ecliptic, is now advanced into the Sign *Taurus*, and *Taurus* has passed into *Gemini*, &c. and thus they have taken the Place of each other, but the twelve Portions of the Ecliptic where these Constellations were formerly placed, still retain the same Names they had in the Time of *Hipparchus*.

vi.

Before *Newton* the physical Cause of the Precession of the Equinoxes was utterly unknown, and we shall now proceed to shew how he deduced this Motion from his Principle of universal Gravitation.

We have seen that the Figure of the Earth is that of an oblate Spheroid, Flat towards the Poles and elevated towards the Equator. In Order to explain the Precession of the Equinoxes, *Newton* premises 3 Lemmas, from whence he deduces (Prop. 39. B. 3.) that this Revolution of the equinoctial Points is produced by the combined Actions of the Sun and Moon on the protuberant Matter about the Earth's Equator.

vii.

In the first Lemma he supposes all the Matter by which the Earth considered as a Spheroid would exceed an inscribed Sphere, to be reduced to a Ring investing the Equator, and collects the Sum of all the Efforts of the Sun, on this Ring, to make it Revolve round its Axis which is the common Section of the Plane of the Ecliptic with the Plane passing thro' the Center of the Earth, and Perpendicular to the straight Line connecting the Centers of the Earth and the Sun. In the second Lemma he investigates the Ratio between the Sum of all those Forces, and the Sum of the Forces exerted by the Sun on all the protuberant Parts of the Earth, exterior to the inscribed Sphere. In the third Lemma he compares the Quantity of the Motion of this Ring, placed at the Equator, with that of all the Parts of the Earth taken as a Sphere.

viii.

To determine the Force of the Sun upon this Protuberant Matter about the Equator of the Earth, *Newton* assumes for *Hypothesis*, that if the Earth was annihilated, and that only this Ring remained, describing round the Sun the annual Orb, and revolving at the same Time by its diurnal Motion round its Axe, inclined to the Ecliptic in an Angle of $23^{\circ} 30''$, the Motion

It is the cause why the intersection of the equator and the ecliptic does not correspond to the same stars it did formerly, &c. that the constellations of the zodiac have changed their places.

Lemmas with which *Newton* sets out to deduce this motion from the principle of universal gravitation.

of the Equinoctial Points would be the same, whether the Ring was fluid or composed of solid Matter.

Newton after having investigated the Ratio of the Matter of this supposed Ring, that is, of the Protuberant Matter about the Equator, to the Matter of the Earth taken as a Sphere, and having found it [assuming the Ratio of the Axes of the Earth] to be as 459 to 52441, he proves that if the Earth and this Ring revolved together about the Diameter of this Ring, the Motion (R) of the Ring would be to the Motion (T), of the interior Globe, or to the Motion of the Earth round its Axis, in a Proportion compounded of the Proportion 459 to 52441 of the Matter in the Ring to the Matter in the Earth, and of the Number 1000000 to the Number 800000, or as 4590 to 419528, (a) and consequently that the Motion (R) of the Ring would be to the Motion (R+T) of the Ring and the Globe, in the Ratio of 4590 to 424118.

He found (Prop. 32. B. 3) that the mean Motion of the Nodes of the Moon in a Circular Orbit, is $20^d, 11^m, 46^s$, in *Antecedentia*, in a Sydereal Year; and he proved (Cor. 16 Prop. 66) that if several Moons revolved round the Earth, the Motion of the Nodes of each of those Moons would be as their periodic Times. from whence he concludes that the Motion (n) of the Nodes of a Moon revolving near the Surface of the Earth in $23^h, 56^m$, would be to $26^d, 11^m, 46^s$, Motion (N) of the Nodes of our Moon in a Year, as $23^h, 56^m$, the Time of the Earth's diurnal Rotation, to $27^d, 7^h, 43^m$, the periodic Time of the Moon, that is, as 1436 to 39343; and by the Cor. of Prop. 66 the same Proportions hold for the Motion of the Nodes of an Assemblage of Moons surrounding the Earth, whether these Moons were separate, and detached from each other, or if they coalesced supposing them liquified and forming a fluid Ring, or that the Ring became hard and inflexible.

Newton considers the protuberant matter about the equator of the earth as a ring of moons adhering to the globe of the earth.

He deduces from this supposition the manner that the attraction of the sun on the elevation at the equator causes the precession of the equinoxes.

Therefore, the protuberant Matter about the Equator of the Earth being considered as a Ring of Moons adhering to the Earth, and revolving along with it, since the Revolution (n) of the Nodes of such a Ring, is to the Revolution (N) of the Nodes of the Moon, as 1436 to 39343, (according to Cor. 16. Prop. 66.) and that the Motion (R) of the Ring is to the Sum of the Motions (T+R) of the Ring and the Globe to which it adheres, as 4590 to 424118; $n \times R$ is to $N \times T + R$, as 1436×4590 to 39343×424118 , or $\frac{n \times R}{T + R}$ is to N, as 1436×4590 to 39343×424118 ; but it is demonstrated that the Sum of the Motions T+R of the Ring and the Globe to which it adheres is to the Motion (R) of the Ring as the Revolution (n) of the Nodes of this Ring to half the annual Motion [$\frac{1}{2}P$.] of the Equinoctial Points of the Body composed of the Ring and Globe to which it ad-

(a) The ratio of the motion of the ring to the motion of the interior globe assigned by Newton, is 4590 to 485223. which is erroneous as shall be shewn hereafter.

heres, (b) wherefore the annual Motion (P.) of the equinoctial Points of the Body composed of the Ring and Globe to which it adheres, will be to the annual Motion of the Nodes (N) of the Moon, in the compounded Ratio of $1436 \times 4590 \times 2$ to 39343×424118 .

But *Newton* found (Lem. 2. B. 3.), that if the Matter of the supposed Ring was spread all over the Surface of the Sphere so as to produce towards the Equator, the same Elevation as that at the Equator of the Earth, the Force of the Matter thus spread to move the Earth, would be less than the Force of the equatorial Ring in the Ratio of 2 to 5; therefore the annual Regress of the equinoctial Points is to the annual Regress of the Lunar Nodes, as $1436 \times 4590 \times 2 \times 2$ to $39343 \times 424118 \times 5$, and consequently in a Sydereal Year it will be 22° , $58'$, $33''$ without any Regard being had to the Inclination of the Axis of the Ring, which Consideration causes still a Diminution in this Motion in the Ratio of the Cosine [91706] of this Inclination (which is $23\frac{1}{2}^{\circ}$) to the Radius (100000).

The mean annual Precession of the Equinoxes produced by the Action of the Sun will be therefore $21^{\circ} 6'$ nearly, supposing the Earth Homogeneous and the Depression towards the Poles $\frac{1}{11}$.

Simpson found from his Theory $21^{\circ} 6'$ (*Miscellaneous Tracts*) *D'Alambert* 23° nearly (*Recherches Sur la Precession des Equinoxes*) *Euler* 22° (*Mem. de Berlin Tom. 5, 1749*). And if this Quantity is greater by a third than what Observation indicates, it probably arises from the Earth's not being Homogeneous, as was supposed, the Researches of *Simpson*, *Euler*, and *D'Alambert* relative to this Object shall be explained hereafter.

IX.

In this Manner *Newton* determined the mean Quantity of the Motion of the equinoctial Points. But not without examining the different Varieties of the Action of the Sun on the protuberant Matter about the Equator supposed to be reduced to a Ring.

He shews in Cor. 18, 19 and 20 of Prop. 66 that by the Action of the Sun the Nodes of a Ring, supposed to encompass a Globe as the Earth, would rest in the Sygies, in every other Place they would move in *Antecedentia*, they would move swiftest in the Quadratures, that the Inclination of this Ring, would vary, that during each annual Revolution of the Earth, its Axe would Oscillate, and at the end of each Revolution would return to its former Position, but that the Nodes would not return to their former Places, but would still continue to move in *Antecedentia*.

(b) *Newton* supposes that the Sum of the Motions of the Ring and the Globe to which it adheres is to the Motion of the Ring, as the Revolution of the Nodes of this Ring is to the annual Motion of the Equinoctial Points of the Body composed of the Ring and Globe to which it adheres, in which he is mistaken as shall be shewn hereafter.

Irregularities in the motion of the equinoctial points produced by the action of the sun.

The action of the sun on the protuberant matter about the equator causes an annual nutation of the axis of the earth.

If the earth was elevated towards the poles and depressed towards the equator the equinoctial points would advance instead of retrograding.

Which proves the depression of the earth towards the poles.

The moon contributes to the production of the motion of the equinoctial points.

That the action of the moon on the protuberant matter about the equator is more powerful than that of the sun.

The greatest Inclination of the Ring should happen when its Nodes are in the Syngies, afterwards in the Passage of the Nodes to the Quadratures, this Inclination should diminish, and the Ring by its Effort to change its Inclination, impresses a Motion on the Globe, and the Globe retains this Motion, till the Ring, or the protuberant Matter about the Equator, (for it is the same Thing according to *Newton*) by a contrary Effort destroys this Motion, and impresses a new Motion in a contrary Direction.

Hence we see that the Axis of the Earth should change its Inclination with Respect to the Ecliptic, twice in its annual Course and return twice to its former Position.

Newton has shewn in Cor. 21 of Prop. 66 that the protuberant Matter about the Equator making the Nodes retrograde, the Quantity of this Matter increasing, this Regression, would increase, and would diminish when this Matter diminished; hence if there was no Elevation towards the Equator, there would be no Regression of the Nodes, and the Nodes of a Globe, which instead of been Elevated towards the Equator was depressed, and consequently would have its protuberant Matter about its Poles, would move in *Consequentia*.

And he adds, (Cor. 22 of Prop. 66) that as the Form of the Globe enables us to judge of the Motion of the Nodes, so from the Motion of the Nodes we may infer the Form of the Globe; and consequently if the Nodes move in *Antecedentia*, the Globe will be elevated towards the Equator, but on the Contrary depressed, if the Nodes move in *Consequentia*, which is a further Proof of the Flatness of the Earth towards the Poles.

xi.

We have hitherto considered only the Action of the Sun in explaining the Precession of the Equinoxes, and we have seen that in Consequence of this Action the equinoctial Points would recede annually $21^{\circ} 6'$. But the Moon by her Attraction Acts on the Earth and influence very sensibly this Phenomenon, its Action being to that of the Sun as $2\frac{1}{2}$ to 1 (c) if the Inclination of its Orbit to the Equator was always the same as that of the Ecliptic to the Equator, the Regression thence resulting would be to that arising from the Sun's Action as $2\frac{1}{2}$ to 1. But because its Nodes shift continually their Places, it happens that the Inclination of its Orbit to the Equator, on which depends its Effect varies continually, so that when the ascending Node is in Aries, the Inclination of the Moon's Orbit to the Equator a-

(c) The Proportion of the Force of the Sun to that of the Moon, assigned by *Newton* is 1 to 4, 4815, which he also assigns for the Proportion of the Precession of the Equinoxes produced by the Sun to that produced by the Moon but this Proportion does not agree with the Theories which depend on the Determination of the Mass of the Moon, and it appears from Computation as shall be shewn hereafter, that the Precession of the Equinoxes produced by the Sun and that produced by the Moon are not in the same Proportion as the Forces of those Luminaries.

amounts to $28^d \frac{1}{2}$, but when the ascending Node nine Years after, is in Libra it scarce amounts to $18^d \frac{1}{2}$ in each Revolution, which renders the Precession arising from the Action of the Moon very unequal during the Space of 18 Years, and Causes a Nutation in the Axis of the Earth, whereby its Inclination to the Ecliptic varies during the Revolution of the Nodes of the Moon; after which it returns to its former Position. This Nutation from Theory, amounts to 19^s , agreeable to Observation, the mean Precession arising from the Action of the Moon, to $35^s, 5$, consequently the Precession arising from the Action of the Sun to $14^s, 5$, and the greatest Difference between the true Precession arising from the Action of the Moon, and the mean Precession amounts to $17^s, 8$.

Nutation of the axis of the earth produced by the action of the moon.

Theory of the Ebbing and Flowing of the Sea.

I.

It is very easy to perceive the Connection between the Ebbing and Flowing of the Sea and the Precession of the Equinoxes. *Newton* deduces his Explication of the Ebbing and Flowing of the Sea, from the same Corollaries of Prop. 66, from whence we have seen he drew his Explication of the Precession of the Equinoxes; those two Phenomena are both one and the other a necessary Consequence of the Attractions of the Sun and Moon on the Parts which compose the Earth.

The explication of the ebbing and flowing of the sea, is deduced from prop. 66 & its cor. as is that of the precession of the equinoxes.

II.

Galileo imagined that the Phenomena of the Tides might be accounted for, from the Motion of Rotation of the Earth, and its Motion of translation round the Sun. But if this great Man had more attentively examined the Circumstances attending the Ebbing and Flowing of the Sea, he would have perceived that in Consequence of the diurnal Motion of the Earth, the Sea indeed would rise towards the Equator, and that the Earth would assume the Form of a Spheroid depressed towards the Poles, but this Motion of Rotation would never produce in the Waters of the Sea a Motion of Flux and Reflux, as *Newton* has demonstrated Cor. 19. Prop. 66. *Newton* Proves in this same Corollary, applying what he had demonstrated in Cor. 5 and 6 of the Laws of Motion, that the Translation of the Earth round the Sun has no Effect on the Motion of Bodies at its Surface, and consequently the Motion of Translation of the Earth round the Sun, cannot Produce the Motion of Flux and Reflux of the Sea.

Error of *Galileo* concerning the ebbing and flowing of the sea.

III.

On examining the Circumstances which attend the Ebbing and Flowing of the Sea, it was easy to perceive that those Phenomena depended on the Position of the Earth with Respect to the Sun and Moon; but it was not so, to discover the Manner those two Luminaries Produce those Phenomena and

The ebbing and flowing of the sea arises from the action of the sun and

moon on the
waters.

the Quantity that each contributes to their Production: we see but the Effects in which the Actions of those two Luminaries are so confounded, that it is only by the Assistance of *Newton's* Principles we are enabled to distinguish one from the other, and assign their Quantity. It was reserved for this great Man, to discover the true Cause of the Ebbing and Flowing of the Sea, and to reduce those Causes to Computation; we shall now trace the Road which conducted him to those Discoveries.

IV.

Road which
conducted
Newton to
assign the
quantity
that each of
those lumi-
naries contri-
bute to pro-
duce those
phenomena.

He begins by examining in Prop. 66. the Principle Phenomena which should Result from the Motion of three Bodies which attract each other mutually in the inverse Ratio of the Squares of the Distances, the small ones Revolving round the greater.

After having shewn in the first 17. Corollaries of this Prop. the Irregularities which the greater Body would Cause in the Motion of the lesser, which itself revolves round the third, and by this Means having laid the Foundation of the Theory of the Moon, he considers in Cor. 18 several fluid Bodies which revolve round a third, he afterwards supposes that those fluid Bodies all become contiguous so as to form a Ring revolving round the central Body, and proves that the Action of the greatest Body would produce in the Motions of this Ring the same Irregularities as in those of the solitary Body in whose Place the Ring was substituted; in fine Cor. 19. he supposes the Body round which this Ring Revolves to be extended on every Side as far as this Ring, that this Body which is solid contains the Water of this Ring in a Channel cut all round its Circumference, and that it revolves uniformly round its Axis, he then proves that the Motion of the Water in this Channel will be accelerated and retarded alternately by the Action of the greater Body and that this Motion will be swifter in the Sygies of this Water, and slower in its Quadratures, and finally that this Water will Ebb and Flow after the Manner of the Sea.

Newton applies this Prop. 66 and its Cor. to the Phenomena of the Sea (Prop. 24. B. 3.) and proves that they are a necessary Consequence of the combined Actions of the Sun and Moon on the Parts which compose the Earth.

V.

He afterwards investigates the Quantity, each of those Luminaries contribute, to the Production of those Phenomena. As this Quantity depends on their Distances from the Earth, the nearer they are to the Earth, the greater the Tides should be, *Cæteris Paribus*, when their Actions, conspire together: and according to Cor. 14. Prop. 66, those Effects are in the Inverse Ratio of the Cubes of their Distances from the Earth and the simple Ratio of their Masses.

Newton examines first the Action of the Sun on the Waters of the Sea, because its Quantity of Matter with Respect to that of the Earth is known. He observes that the Attraction of the Sun on the Earth is counterbalanced

as to the Totality by the centrifugal Force arising from the annual Motion of the Earth, which he considers as uniform and circular: But what is true as to the Totality is not so as to each particle of the Earth, that is, that the centrifugal Force of each of those Particles cannot be supposed equal to the Force with which the same Particle is Attracted by the Sun, since each Particle has the same centrifugal Force, and the Particles of the Earth which are nearer the Sun are more attracted than those which are remoter. Thus the Distance of the Earth from the Sun, being 22000 Semidiameters of the Earth, and the Law of Attraction, the inverse Ratio of the Squares of the Distances, the Attractive Force corresponding to the Point of the Earth nearest the Sun, to the Center of the Earth, and to the Point of the Earth remotest from the Sun, will be nearly as 11001, 11000 and 10999, and as the Sun's Attraction balances the centrifugal Force of each Particle of the Earth, this Force will be Proportional to 11000; if from the attractive Force of the Sun on each of those three Points, the centrifugal Force be Subducted, there will remain 1, 0, —1; which proves that the Center of the Earth is at Rest with Respect to the Motions of the Waters of the Sea, and that the two Extremities of the Diameter of the Earth directed towards the Sun, are actuated by equal Forces with opposite Directions, whereby the Parts tend to recede from the Center of the Earth.

If in the same Diameter there be taken two Points equally distant from the Center, those two Points will be likewise actuated by equal Forces with opposite Directions, whereby they tend to recede from the Center; but this Force will decrease as the Distance from the Center of the Earth. this Diameter of the Earth directed to the Center of the Sun may be called the Solar Axis of the Earth, if we now consider the Equator corresponding to this Axis, it is evident that each Point taken in the Plane of this Equator may be supposed equally distant from the Center of the Sun, and consequently that none of the Points of this Plane are affected by the Inequality between the centrifugal Force and attractive Force, and consequently their Gravity towards the Center of the Earth will not be diminished, therefore if we conceive two Canals full of Water the one passing thro' the demi solar Axis, and the other thro' a Ray at its Equator, which communicate at the Center of the Earth, the Water will ascend in the first and descend in the other, this will happen both in the one and the other demi solar Axis, and is the first Source of the Ebbing and Flowing of the Sea.

Each Particle of Water in the Canal of the demi solar Axis is attracted towards the Sun in the Direction of the Canal, but this Force acts on the Particles of Water in the other Canal, obliquely, it therefore should be resolved into two, one perpendicular to the Canal, and the other parallel to it. The first may be considered as perfectly destroyed by the centrifugal Force; but the other Force adds to the Gravity of each Particle in this Canal, this

First Source
of the ebbing
and
flowing of
the sea.

Second
source of
the ebbing
and flowing
of the sea.

small Force does not exist in the Canal of the demi solar Axe, and for this Reason the Water will descend in the Canal of the solar Equator, and will sustain that of the solar Axis to a greater Height. This is the second Source of the Ebbing and Flowing of the Sea.

From whence it appears that the Ascent of the Waters of the Sea does not arise from the total Action of the Sun, but from the Inequalities in that Action on the Parts of the Earth. *Newton* observes that in Consequence of this Action the Figure of the Earth (abstracting from its diurnal Motion) ought to be an elliptic Spheroid having for greater and lesser Axes the solar Axe and the Diameter of its Equator, and determines in the following Manner the Force of the Sun which produces the difference of those Axes.

Determina-
tion of the
force of the
sun, produc-
ing the eleva-
tion or de-
pression of
the waters
of the sea in
two points
diametrically
opposite.

He considers the Figure of the Earth (abstracting from its diurnal Motion) rendered Elliptic by the Action of the Sun, as a similar Effect to the Figure of the Orbit of the Moon, (abstracting from its excentricity) which he had shewn (Prop. 66. Cor. 5) to be rendered Elliptic and to have its Center in the Center of the Earth, by the same Action. He demonstrated (Prop. 25. B. 3) that the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (g) which draws the Moon towards the Earth, as the Square of the periodic Time (tt) of the Moon round the Earth, to the Square of the periodic Time (TT) of the Earth round the Sun, according to Cor. 17 of Prop. 66; but the Inequality (V) in the Action of the Sun on the Parts of the Earth being to its Action (G), as the Ray (r) of the Earth, to the Ray (R) of its Orbit, and the Force (G) of the Sun which retains the Earth in its Orbit, being to the Force (g) which retains the Moon in its Or-

bit, as $\frac{TT}{R}$ Ray of the Earth's Orbit divided by the Square of its periodic Time, to $\frac{b}{tt}$ Ray of the Moon's Orbit divided by the Square of its periodic Time (Cor. 2 Prop. 4), $V \times G$ is to $G \times g$, or the Inequality (V) in the Action of the Sun on the Parts of the Earth, is to the centripetal Force (g) of the Moon towards the Earth as $\frac{r \times R}{TT}$ to $\frac{R \times b}{tt}$, that is, as the Ray of the Earth divided by the Square of its periodic time round the Sun ($\frac{r}{TT}$) to the Ray of the Moon's Orbit, divided by the Square of its periodic Time round the Earth ($\frac{b}{tt}$)

Wherefore by the Composition of Ratios, $g \times V$ is to $F \times g$, or the Force (V) of the Sun disturbing the Motion of Bodies on the Surface of the Earth, is to its Force (F) with which it disturbs the Motion of the Moon, as $\frac{TT \times r}{TT}$ to $\frac{tt \times b}{tt}$ or as the Ray (r) of the Earth, to the Ray (b) of the Moon's Orbit, that is, as 1 to 60 $\frac{1}{4}$.

To compare now those two Forces with the Force of Gravity at the Surface of the Earth. Since the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (g), which would retain the Moon in an Orbit, described about the Earth quiescent at its present Distance ($60\frac{1}{2}$ Semidiameters of the Earth) as the Square of $27^d. 7^h. 43^m.$ to $365^d. 6^h. 9^m.$ or as 1000 to 178725, or as 1 to $178\frac{3}{4}$; and that the Force which retains the Moon in its Orbit, is equal to the Force (γ) which would retain it in an Orbit described about the Earth quiescent in the same periodic Time, at the Distance of 60 Semidiameters, according to Prop. 60, in which it has been demonstrated that the actual Distance ($60\frac{1}{2}$ Semidiameters) of the Centres of the Moon and Earth, both revolving about the Sun, and at the same Time about their common Centre of Gravity, is to the Distance (60 Semidiameters) of their Centres, if the Moon revolved about the Earth quiescent in the same periodic Time, as the Sum ($1+42$) of the Masses of the Moon and Earth, to the first of two mean Proportionals ($42\frac{1}{2}$) between that Sum and the Mass of the Earth. Consequently that the Force (γ) which retains the Moon in its Orbit is less than the Force (g) which would retain it in an Orbit described in the same periodic Time, about the Earth quiescent at the Distance $60\frac{1}{2}$ Semidiameters, in the Ratio of 60 to $60\frac{1}{2}$, (Cor. 2, P. 4); by the Composition of Ratios $F \times g$ is to $g \times \gamma$ or the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (γ) which retains the Moon in its Orbit, as $1 \times 60\frac{1}{2}$ to $178\frac{3}{4} \times 60$. but this Force (γ) which retains the Moon in its Orbit, (in approaching the Earth) increasing in the inverse Ratio of the Square of the Distance, is to the Force (G) of Gravity as 1 to 60×60 , wherefore $\gamma \times F$ is to $\gamma \times G$, or the Force (F) which draws the Moon towards the Sun, is to the Force (G) of Gravity as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 60 \times 178\frac{3}{4}$ or as 1 to 638092,6.

From whence *Newton* concludes [Prop. 36. B. 3.] that since the Ascent of the Waters of the Sea, and the Elliptic Figure of the Lunar Orbit [abstracting from its Excentricity] are similar Phenomena arising from the Solar Force, and that in descending towards the Surface of the Earth this Force decreases in the Ratio of $60\frac{1}{2}$ to 1. the Force of the Sun which depresses the Waters of the Sea in the Quadratures, or at the Solar Equator, is to the Force of Gravity as 1 to $638092,6 \times 60\frac{1}{2}$ or as 1 to 38604600. But this Force is double in the Sygies, or in the Direction of the Solar Axis of what it is in the Quadratures, and acts in a contrary Direction [Cor. 6. Prop. 66], wherefore the Sum of the two Forces of the Sun on the Waters of the Sea, in the Quadratures and Sygies, will be to the Force of Gravity as 3 to 38604600 or as 1 to 12868200. those two Forces united Compose the total Force which raises the Waters of the Sea in the Solar Canal, their Effect

Proportion
of the action
of the sun
on the wa-
ters of the
sea to the
force of gra-
vity.

being the same as if they were wholly employ'd in raising the Waters in the Syfigies, and had no Effect in the Quadratures.

VI.

Newton concludes from his theory that the sun raises the water of the sea to 2 feet.

Newton after having investigated the Force of the Sun which produces the Elevation of the Waters in the Solar Canal, determines in the following Manner the Quantity of this Elevation. He considers the Elevation of the Waters of the Sea arising from the Action of the Sun, as an Effect similar to the Elevation of the Equatorial Parts above the Polar Parts of the Earth, arising from the centrifugal Force at the Equator. Now the centrifugal Force (C) at the Equator being to the Force of Gravity (G) at the Surface of the Earth as 1 to 289, and the Force of the Sun (F) exerted on the Waters of the Sea being to the Force of Gravity (G), as 1 to 12868200, by the Composition of Ratios, $F \times G$ is to $C \times G$, or the Force (F) of the Sun exerted on the Waters of the Sea, is to the centrifugal Force (C) at the Equator, as 1×289 to 1×12868200 or as 1 to 44527; consequently the Elevation (85472 Feet) at the Equator produced by the centrifugal Force, is to the Elevation of the Waters in the Solar Canal produced by the Action of Sun, as 1 to 44527. which shews that the Elevation of the Waters in the Places directly under the Sun and in those which are directly opposite to them is 1 Foot, $11\frac{1}{16}$ Inches.

VII.

The ebbing and flowing of the sea arises from the motion of rotation of the earth and from the actions of the sun and moon.

Method of estimating the action of the sun on the waters of the sea in any place.

The fluid Earth would preserve a Spheroidal form its longest Diameter pointing to the Sun without any Ebbing or Flowing of its Waters, if it had no Motion of Rotation. It is therefore the Rotation of the Earth round its Axis joined to its oblong Figure which causes alternatly a Depression and Elevation of the Waters of the Sea. If the Axis of Rotation and the Solar Axis were the same, the Waters of the Sea would have no Motion of reciprocation, because each Point during the Rotation of the Earth would be constantly at the same Distance from the Solar Poles. But as those two Axes form an Angle, it is easy to perceive that each Point of the Surface of the Earth approaches and recedes alternatly from the Solar Poles and that twice in a Revolution, and the Waters will continually rise in this Point during its Approach to, and will fall continually during its Recess from these Poles. *Newton* investigated the Relation which subsists between the Elevation of the Waters in any Place above that at the Solar Equator and their Elevation in the Solar Canal; and found that the Square of the Radius [R] is to the Square of the Sine [ss] of the Altitude of the Sun in any Place, as the Elevation [S] of the Waters in the Solar Canal to their Elevation [ssS] in that Place.

VIII.

It is Manifest that what has been said with Respect to the Sun should be applied without Restriction to the Moon and all the Phenomena of the Tides

prove evidently that the Action of this Luminary on the Waters is considerably greater than that of the Sun, which at first View should seem the more surprising, as the Attractive Force of the Sun arising from its immense Bulk is so powerful as to Force the Earth to Revolve round it, whilst the Irregularities produced in its Orbit by the Action of the Moon are scarce sensible, but if we consider that the Motion of the Sea proceeds from its Parts being differently attracted from those of the rest of the Earth, because their Fluidity makes them receive more easily the Impressions of the Forces which Act on them, it will appear, that the Action of the Sun which is very powerful on the whole Earth attracts all its Parts almost equally on Account of its great Distance; but the Moon being much nearer the Earth Acts more unequally on the different Parts of our Globe, and that this Inequality should be much more sensible than that of the Sun; these inequalities being in the Inverse Ratio of the Cubes of the Distances of the Luminaries from the Earth, and in the simple Ratio of their Quantities of Matter.

The Elevation of the Waters of the Sea arising from the Action of the Moon, in the Direction of the lunar Axis, above their Height at the lunar Equator, being once determined, the Elevation of the Waters of the Sea in any Place above their Height at the lunar Equator, will be found, for in this Case, as in that of the Sun, the Square of the Radius (r) is to the Square of the Sine [tt] of the Altitude of the Moon in any Place, as the Elevation [L] of the Waters in the Direction of the lunar Axis, above their Height at the lunar Equator, to their Elevation [$tt L$] above the same Height, in that Place.

IX.

From the Combination of the Actions of the Sun and Moon on the Waters of the Sea there result two Tides, viz. the *solar Tides* and *lunar Tides*, which are produced independently of each other. Those two Tides by being confounded with each other appear to Form but one, but subject to great Variations, for in the Sygies the Waters are elevated and depressed at the same Time by both one and the other Luminary, and in the Quadratures the Sun raises the Waters where the Moon depresses them, and reciprocally the Sun depresses the Waters where the Moon raises them, [one being in the Horison when the other is at the Meridian] so that from the Actions of those Luminaries sometimes conspiring and at other Times opposed, there result very sensible Variations both with respect to the Height of the Tides and their Time.

X.

It is demonstrated that the Elevation of the Waters, produced by the conjoint Actions of the Sun and Moon, is sensibly equal to the Sum of the Elevations produced by the Actions of each seperately, wherefore the whole Elevation produced by the united Actions of the two Luminaries will

How is it possible that the attraction of the moon can have such influence on the waters of the sea & cause so little alterations in the motion of the earth.

The variations in the tides arise from the conjoint actions of the sun and moon.

be Expressed by $ss + ttL$; which shews that the Elevation of the Waters in any Place will continually increase until they attain their greatest Height, and then it is high Water, after which it will continually decrease during six Hours, and then it will be low Water; the Difference between those two Heights is called *the Height of the Tide*; from whence it appears that the Height of the Tides depends upon a great Number of Circumstances, viz. the Declination of each Luminary, the Age of the Moon, the Latitudes of Places and the Distance of the two Luminaries from the Centre of the Earth.

XI.

How New-
ton came
to estimate
the action
of the moon
on the wa-
ters of the
sea.

To examine the Variations in the Height of the Tides according to all those Circumstances, let us first suppose the Orbit of the Moon and that of the Sun in the Plane of the Equator, and let us further suppose them perfectly Circular, and let a Place be chosen at the Equator, in which Case we may suppose $s=1$ and $t=1$, which will happen at the appulse of the Luminaries to the Meridian in the Sygies, and the whole Elevation will be expressed by $S+L$; about six Hours after $s=0$ and $t=0$ nearly and the Waters will have no Elevation consequently the Height of the Tides in the Sygies will be expressed by $S+L$; but in the Quadratures at the appulse of the Moon to the Meridian $t=1$ and $s=0$, and the Elevation of the Waters will be expressed by L , about six Hours after $s=1$ and $t=0$ nearly, and the Elevation of the Waters will be expressed by S and the Height of the Tide will be expressed by $L-S$, consequently the Height of the Tides in the Sygies and Quadratures will be as $S+L$ to $L-S$. if therefore the Height of the Tides in the Sygies and Quadratures at the Time of the Equinoxes was determined from Observation, on the Coast of an Island situated near the Equator, in a deep Sea, and open on every Side to a great extent, the Ratio of L to S , the Effects of the Forces of the Sun and Moon, or the Ratio of those Forces which are proportional to those Effects, would be found.

As no such Observations have been made, *Newton* employs for determining the Ratio of those Forces the Observations made by *Sturmy* three Miles below Bristol. this Author relates that the Height of the Ascent of the Waters in the vernal and autumnal Conjunction and Opposition of the Sun and Moon, amounts to about 45 Feet, but in the Quadratures to 25 only, wherefore $L+S$ is to $L-S$ as 45 to 25 or as 9 to 5, consequently $5L+5S=9L-9S$, or $14S=4L$ and S is to L as 2 to 7.

To reduce this Determination to the mean State of the variable Circumstances; it is to be observed 1° that in the Sygies the conjoint Forces of the Sun and Moon being the greatest, it has been supposed that the corresponding Tide is also the greatest, but the Force impressed at that Time on the Sea being increased by a new Though a less Force still acting on it until it becomes too weak to raise it any more, the Tides do not rise to their greatest Height but some Time after the Moon has passed the Sygies, *Newton*

from the Observations of Sturmy concludes that the greatest Tide follows next after the Appulse of the Moon to the Meridian when the Moon is distant from the Sun about $18^{\circ} \frac{1}{2}$. the Sun's Force in this Distance of the Moon from Sygies being to the Force [S] in the Sygies, as the Cosine [7986355] of double this Distance, or of an Angle of 37 Degrees, to the Radius [10000000] in the Place of L+S in the preceding Analogy L+o, 7986355 S is to be Substituted. In the Quadratures the conjoint Forces of the Sun and Moon being least, it was also supposed that the least Tide happens at that Time, but the Sea looses its Motion by the same Degrees that it acquired it, so that the Tides are not at their least Height until some Time after the Moon has passed the Quadratures, and *Newton* from the same Observations of Sturmy concluded that the least Tide follows next after the Appulse of the Moon to the Meridian when the Moon is distant from the Quadratures $18^{\circ} \frac{1}{2}$. Now the Sun's Force in this Distance of the Moon from the Quadratures being to the Force [S] in the Quadratures, as the Cosine (7986355) of double this Distance or of an Angle of 37 Degrees, to Radius (10000000) in the Place of L-S in the preceding Analogy, L-o, 7986355 S is to be Substituted.

It is to be observed 2^o that the Orbit of the Moon was supposed to Coincide with the Plane of the Equator, but the Moon in the Quadratures, or rather $18^{\circ} \frac{1}{2}$ past the Quadratures, declines from the Equator by about $22^{\circ} 13'$, now the Force of the Moon in this distance from the Equator being to its Force (L) in the Equator, as the Square of the Cosine (8570327) of its Declination $22^{\circ} 13'$, to Radius (10000000) in the Place of L-o, 7986355 S in the preceding Analogy o, 8570327 L-o, 7986355 S is to be Substituted.

It is to be observed 3^o that the Orbits of the Sun and Moon were supposed to be perfectly Circular, and consequently those Luminaries to be in their mean Distances from the Earth. But *Newton* demonstrated that the lunar Orbit (abstracting from its Excentricity) ought to be an Elliptic Figure, having its Centre in the Centre of the Earth and the shortest Diameter directed to the Sun; and determined (Prop. 28. B. 3.) the Ratio of this shortest Diameter to the longest or the Distance of the Moon from the Earth in the Sygies and Quadratures to be as 69 to 70. To find its Distance when $18^{\circ} \frac{1}{2}$ Degrees advanced beyond the Sygies, and when $18^{\circ} \frac{1}{2}$ Degrees passed by the Quadratures, it is to be observed that in an Ellipsis if the longest Semidiameter be expressed by (a) its shortest by [b] and the Difference of the Squares of the longest and shortest Semidiameters by [cc] and the Sine of the Angle which any Diameter [y] makes with the longest Semidiameter by [s] $yy = \frac{aabb}{aa - sccc}$ wherefore substituting successively in this Expression 69 for [a] 70 for [b] for [s] 3173047 and 9483236 the Sines of $18^{\circ} \frac{1}{2}$ Degrees and $71^{\circ} \frac{1}{2}$ Degrees: those Distances will be 69,098747 and 69,897345 and the mean Distance will be $69^{\circ} \frac{1}{2}$ as equal to half the Sum

Reduction
of this estimation to
the mean
state of the
variable circumstances.

The force of the moon is to that of the sun as 4,5 to 1.

The force of the sun & moon united raises the waters of the sea to the height of 10 feet and even to 12 feet when the moon is perigee.

How Newton investigated the density and quantity of matter of the moon & what bodies weigh on her surface compared with the density and quantity of matter of the earth, and the weight of bodies on its surface.

of the the longest and shortest semidiameters. But the Force of the Moon to move the Sea is in the reciprocal triplicate Proportion of its Distance, and therefore its Forces in the greatest and least of those Distances are to its Force in its mean Distance, as 0,9830427 and 1,017522 to 1. consequently in the preceding Analogy, in the Place of $L + 0,7986355S$, we must put $1,017522L + 0,7986355S$, and in the Place of $0,8570327L - 0,7986355S$ we must put $0,9830427 \times 0,8570327L - 0,7986355S$; from whence we have $1,017522L + 0,7986355S$, to $0,9830427 \times 0,8570327L - 0,7986355S$ as 9 to 5, consequently $1,017522L \times 5 + 0,7986355S \times 5 = 0,9830427 \times 9 \times 0,8570327L - 0,7986355S \times 9$, and by transposition, S is to L , as $0,9830427 \times 0,8570327 \times 9 - 0,17522 \times 5$ to $0,7986355 \times 5 + 0,7986355 \times 9$, that is, S is to L as 1 to 4,4815 nearly.

XII.

Newton having thus determined the Force of the Moon to raise the Waters of the Sea, assigns the Quantity of this Elevation. The Force (1) of the Sun being to the Force (4,4815) of the Moon, as the Elevation (1 Foot 11 $\frac{1}{3}$ Inches) arising from the Action of the Sun, to the Elevation (8 Feet 7 $\frac{5}{12}$ Inches) arising from the Action of the Moon. So that the Sun and Moon together may produce an Elevation of about 10 $\frac{1}{2}$ Feet in their mean Distances from the Earth, and an Elevation of about 12 Feet when the Moon is nearest the Earth.

The Influence of the Moon on the Tides has enabled Newton to Estimate her Density, her Quantity of Matter, and what Bodies weigh on her Surface, compared with the Density and Quantity of Matter of the Earth, and the Weights of Bodies on its Surface. For since the Force (v) of the Moon to move the Sea is to the like Force (V) of the Sun as 4,4815 to 1, and v is to V as $\frac{g}{b^3}$ absolute Force of the Moon divided by the Cube of its Distance from the Earth to $\frac{G}{R^3}$ absolute Force of the Sun divided by the Cube of its Distance from the Earth (Cor. 14 Prop. 66); 4,4815 is to 1 as $\frac{g}{b^3}$ to $\frac{G}{R^3}$, but the absolute Force (g) of the Moon is to the absolute Force (G) of the Sun, as the Density of the Moon and Cube of its Diameter conjointly ($d \times q^3$) to the Density of the Sun and Cube of its Diameter conjointly ($D \times p^3$), and the apparent Diameter (31^m. 16^s) of the Moon being to the apparent Diameter (32^m. 12^s) of the Sun as $\frac{q}{b}$ to $\frac{p}{R}$, $\frac{1}{b^3}$ is to $\frac{1}{R^3}$ as $\frac{141583}{q^4}$ to $\frac{154508}{p^4}$

wherefore by the Composition of Ratios $\frac{g}{b^3}$ is to $\frac{G}{R^3}$ as $d \times 141583$ to $D \times 154508$, consequently 4,4815 is to 1 as $d \times 141583$ to $D \times 154508$ that is, as the Densities of the Moon and Sun and the Cubes of their apparent Diameters conjunctly, from whence it follows that the Density (d) of

the Moon is to the Density (D) of the Sun, as $\frac{4,4815}{141583}$ to $\frac{1}{154508}$ or as 4891 to 1000, but the Density (D) of the Sun is to the Density (c) of the Earth, as 1000 to 4000, consequently $D \times d$ is to $D \times c$, or the Density (d) of the Moon is to the Density (c) of the Earth as 4891×1000 to 4000×1000 or as 11 to 9, therefore the Body of the Moon is more Dense and more Earthly than the Earth its self. Density of the moon.

And since the true Diameter of the Moon [from the Observations of the Astronomers] is to the true Diameter of the Earth as 100 to 365, the Quantity of Matter in the Earth, is to the Quantity of Matter in the Moon as 1000000×11 to 48627125×9 , that is, as 1 to 39, 788. Quantity of matter in the moon.

And since the accelerative Gravity on the Surface of the Moon is to the accelerative Gravity on the Surface of the Earth as the Quantity of Matter in the Moon to the Quantity of Matter in the Earth, directly, and as the Square of the Distances from the Center inversely, they will be to each other as 1×13324 to $39,788 \times 1000$ that is as 1 to 3 nearly: consequently the accelerative Gravity on the Surface of the Moon will be about three Times less than the accelerative Gravity on the Surface of the Earth. Weight of bodies on its surface.

XIII

Daniel Bernouilly, in his Piece on the Tides which carried the Prize of the Academy of Sciences in the Year 1738, observes that the Method of estimating the Proportion of the Force of the Sun to that of the Moon by the greatest and least Heights of the Tides as employ'd by *Newton* is very uncertain; because in the Ports of England and France the Tides are not immediately produced by the Actions of the two Luminaries, but are rather a Consequence of the great Tides of the Ocean, as the Tides of the Adriatic Sea are a Consequence of the Small Tides of the Mediterranean, and that the primitive Tides may differ very sensibly in every Respect from the secondary Tides which is confirmed by Observation; the Proportion of the Spring and Neap Tides being found to be very different in the different Ports. At St. Malo's, for Example, the greatest and least Height of the Waters are to one another as 10 to 3, and below Bristol according to Sturmy they are to each other as 9 to 5. Bernouilly is of a different opinion & why.

He observes further that the Motion of Rotation of the Earth being very rapid with Respect to the Motion of the Sun and Moon; The Sea cannot every Instant assume its Figure of Equilibrium without any sensible Motion, hence the Waters which were raised by the combined Actions of the Luminaries tending on one Hand to conserve as much as possible by their Force of *inertia* the Elevation they had acquired, and on the other tending as they recede from the Moon to loose a Part of their Elevation, they will be less Elevated than they would be if the Earth was at Rest, and consequently the Neap Tides are greater and the Spring Tides less than what results from a

Computation founded on the Laws of Equilibrium, wherefore the great Spring Tides and Neap Tides are in a greater Ratio according to the Laws of Equilibrium than that of 9 to 5.

Force of
the moon
according to
Bernoulli.

Bernoulli supposes them to be to each other as 7 to 3, consequently that the Force (L) of the Moon is to the Force (S) of the Sun as 5 to 2. A proportion which answers better to the Observed Variations in the duration and interval of the Tides (Variations which receive no Alteration from the above mentioned secondary Causes) and to the other Theories which depend on a Determination of the Force of the Moon. Hence the Density of the Moon is to the Density of the Earth as 7 to 10, the Quantity of Matter in the Moon is to the Quantity of Matter in the Earth as 1 to 70, and finally the accelerative Gravity at the Surface of the Moon is to the accelerative Gravity on the Surface of the Earth as 1 to 5.

XIV.

Singular
figure of the
moon.

If the Moon's Body were Fluid like our Sea it would be elevated by the Action of the Earth in the Parts which are nearest to it and in the Parts opposite to these, and *Newton* enquires into the Quantity of this Elevation. He observes that the Elevation ($8\frac{1}{2}$) of the Earth produced by the Action of the Moon would be to the Elevation (E) of the Moon (if it had the same Diameter as the Earth) produced by the Action of the Earth as the Quantity of Matter in the Moon to the Quantity of Matter in the Earth, or as 1 to 39,788. and the Elevation (E) produced by the Action of the Earth in the Moon if it had the same Diameter as the Earth, is to the real Elevation (x) produced in the Moon by the Action of the Earth, as the Diameter of the Earth to the Diameter of the Moon or as 365 to 100. wherefore by the Composition of Ratios $8\frac{1}{2} \times E$ is to $E \times x$ or the Elevation of the Earth ($8\frac{1}{2}$) produced by the Action of the Moon is to the real Elevation of the Moon produced by the Action of the Earth as 1×365 to $39,788 \times 100$ or as 1081 to 100 or $x = 93$ Feet. consequently the Diameter of the Moon that passes through the Centre of the Earth, must exceed the Diameter which is perpendicular to it by 186 Feet. Hence it is, that the Moon always turns the same Side towards the Earth.

Effect of
the oblong
spheroidal
figure of the
moon.

In Effect *La Grange* in his Piece which carried the Prize of the royal Academy of Sciences in the Year 1764, supposing with *Newton* that the Moon is a Spheroid having its longest Diameter directed towards the Earth, has found that this Planet should have a libratory or oscillatory Motion about its Axis, whereby its Velocity of Rotation is sometimes accelerated and other Times retarded, and that then the Moon should always turn the same Side nearly towards the Earth, though it did not receive in the Beginning a Motion of Rotation whose Duration was equal to that of its Revolution. *La Grange* has demonstrated also that the Figure of the Moon might be such that the Precession of its equinoctial Points or the Retrogradation of the

Nodes of the lunar Equator, would be equal to the retrograde Motion of the Nodes of the lunar Orbit; and in this Case he found that the lunar Axis would have no sensible Nutation. The Action of the Sun in all those Inquiries, is almost insensible with respect to that of the Earth; it is the Earth which produces the Motion of the Nodes of the lunar Equator, by acting more or less obliquely on the lunar Spheroid; hence the Precession of the lunar Equator, and the Law of the Motion produced in the lunar Spheroid, differ very much from that which is observed in the Equator of the Earth. The Researches of this eminent Mathematician of *Turin*, shall be explained hereafter.

XV.

Newton having shewn that the Tides proceed from the combined Actions of the Sun and Moon, and determined the Quantity that each of those Luminaries contribute to their Production, enters into an Explanation of the Circumstances which attend the Phenomena of the Tides.

There has been observed in all Times, three Kinds of Motions in the Sea, its diurnal Motion, whereby it ebbs and flows twice a Day, the regular Alterations which this Motion receives every Month, and which follow the Position of the Moon with respect to the Sun, and those which arrive every Year and which depend on the Position of the Earth with respect to the Sun.

Three kinds of variations have been observed in the motion of the sea.

To deduce those Motions from their Cause, we are to observe that the Sea yielding to the Force of the Sun and Moon impressed on it in Proportion to their Quantity, acquires its greatest Height by a Force compounded of those two Forces; hence this greatest Height (even abstracting from the Force of *Inertia* of the Waters) should not be immediately under the Moon, nor immediately under the Sun, but in an intermediate Point, which corresponds more exactly to the Motion of the Moon than to that of the Sun, because the Force of the Moon on the Sea is greater than that of the Sun. To determine the Position of this Point, it is manifest that at High-Water in any Place, $ssS+ttL$ is a *Maximum*, and at Low-Water a *Minimum* or $Ssds+Ltdt=0$. But the instantaneous Increment (ds) of the Sine of the Altitude of the Sun, is to the corresponding Increment (dz) of the Sun's diurnal Arc, as the Cosine ($\sqrt{1-ss}$) of the Altitude of the Sun to Radius (1), or $ds=\sqrt{1-ss} \times dz$ and the corresponding Decrement ($-dt$) of the Sine of the Moon's Altitude, is to the corresponding Increment (dx) of the Moon's diurnal Arc, as the Cosine ($\sqrt{1-tt}$) of its Altitude to Radius (1), or $-dt=dx \times \sqrt{1-tt}=\frac{29}{30} dz \times \sqrt{1-tt}$, dx being to dz as 29 to 30, on account of the Motion of the Moon. Substituting those Values of ds and dt in the Expression $Ssds+Ltdt=0$, we will have $Ss\sqrt{1-ss}=\frac{29}{30} \times L \times t\sqrt{1-tt}$, or $\frac{S\sqrt{1-ss}}{t\sqrt{1-tt}}=\frac{29}{30} \frac{L}{s}$ from whence it appears that at the Time

Diurnal variations.

of high and low Water the Quantities $\sqrt{1-ss}$ and $t\sqrt{1-tt}$ are always in the constant Ratio of $29L$ to $30S$, or of 20×5 to 30×2 ; but the Quantity $\sqrt{1-ss}$ can never exceed $\frac{1}{2}$; consequently $t\sqrt{1-tt}$ can never exceed $\frac{37 \times 1}{29 \times 5}$ or $\frac{6}{29}$; and of course one of the Factors t or $\sqrt{1-tt}$ must be always very small, which proves that the Moon is near the Meridian at High-Water, and near the Horizon at Low-Water.

The waters of the Sea ought twice to rise and twice to fall every day.

The Waters of the Sea therefore should be elevated and depressed twice in the Space of a lunar Day, that is in the Interval of Time elapsed between the Passage of the Moon at the Meridian of any Place, and its Return to the same Meridian; for the conjoint Force of the Sun and Moon on the Sea, being greatest when the Moon is near the Meridian, it should be equal twice in 24 Hours 49 Minutes (a), when the Moon is near the Meridian of the Place above and below the Horizon; wherefore in each diurnal Revolution of the Moon about the Earth, there should be two Tides distant from each other, by the same Interval that the Moon employs to pass from the Meridian above the Horizon to that below it, which Interval is about $12^h 24^m$ hence the Time of High-Water will be later and later every Day.

XVI.

High-water does not immediately follow the Appulse of the Moon to the Meridian.

Since $t\sqrt{1-tt}$ can never exceed $\frac{6}{29}$, and consequently the Distance of the Moon from the Meridian 12 Degrees, the greatest Elevation of the Waters in any Place can never happen later than 48 lunar Minutes, or 50 solar Minutes after the Appulse of the Moon to the Meridian, if the Waters had no *Inertia*, and their Motion were not retarded by their Friction against the Bottom of the Sea. But from those two Causes this Elevation still happens two Hours and a Half or three Hours later

(a) Whilst the Heavens seem to carry the Sun and Moon round from East to West every Day, those Luminaries move in a contrary Direction, the Sun $59^m 8^s$, the Moon $13^d 10^m 35^s$ in a Day, consequently after their Conjunction the Moon continually recedes $12^d 11^m 26^s$ from the Sun towards the East each Day, until she is 130 Degrees from the Sun, or in Opposition, after which being to the West of the Sun, she continually approaches, and at length overtakes him in 29 Days and an Half. From whence it appears that this Planet, the Day of the new Moon, rises, passes at the Meridian and sets about the same Time as the Sun; the following Days she rises, passes at the Meridian, and sets later and later than the Sun, so that the mean Quantity of the Retardation of one rising compared with the following, of one Appulse to the Meridian compared with the following, &c. is about 49 Minutes. Seven Days and One-third after the Conjunction, the Moon being 90 Degrees to the East of the Sun, or in its first Quarter, she rises when the Sun is in the Meridian, passes at the Meridian when the Sun sets, and sets at Midnight. The following Days she comes sooner to the Meridian than the Sun to the opposite Meridian, but the Difference continually decreases to the Opposition, and then she rises when the Sun sets, passes at the Meridian at Midnight, and sets when the Sun rises. The following Days she comes later and later to the Meridian than the Sun to the opposite Meridian, the Difference increasing to the last Quarter when the Moon being 90 Degrees to the West of the Sun, rises at Midnight, passes at the Meridian at Six of the Clock in the Morning and sets at Noon. The following Days she rises, passes at the Meridian, and sets sooner than the Sun, the Interval decreasing to the Conjunction.

in the Ports of the Ocean where the Sea is open; for the Waters in consequence of their Force of *Inertia* receiving but by Degrees their Motion, and retaining for some Time the Motion they have acquired, the Motion of the Sea is perpetually accelerated during the six Hours which precedes the Appulse of the Moon to the Meridian, by the combined Actions of the Sun and Moon on the Waters, which increases in proportion as the Moon rises above the Horizon, and by the diurnal Motion of the Earth which then conspires with that of the Moon. This Motion impressed on the Waters retains during some Time its Acceleration, so that the Sea rises higher and higher until the diurnal Motion of the Earth which becomes contrary after the Appulse of the Moon to the Meridian, as also the combined Actions of the Luminaries which becomes weaker and weaker, diminishes gradually the Velocity of the Waters, in consequence of which they sink. It is easy to perceive that the Friction of the Waters against the Bottom of the Sea should also contribute to retard the Tides.

What are the Causes which retard the Tides.

In the Regions where the Sea has no Communication with the Ocean, the Tides are much more retarded, in some Places even 12 Hours, and it is usual to say in those Places, that the Tides precede the Appulse of the Moon to the Meridian. In the Port of *Havre-de-grace*, for Example, where the Tide retards 9 Hours, it is imagined that it precedes by 3 Hours the Appulse of the Moon to the Meridian; but in Reality, this Tide is the Effect of the precedent Culmination.

The Waters falling to the lowest when the Moon is near the Horizon, her Action on the Sea being then most oblique, it is manifest that Low-water does not exactly fall between the two High-waters which immediately succeed each other, but is so much nearer to that which follows, as the Elevation of the Pole in the proposed Place is greater, and the Moon has a greater Declination; that is, in proportion to the Interval between the rising and setting of the Moon and the horary Circle of six Hours after her Culmination.

Low-water does not exactly fall between the two Elevations which immediately succeed each other, and why.

XVII.

These are the principal Phenomena which attend the Tides depending on the Position of the different Parts of the Earth in its diurnal Revolution with respect to the Sun and Moon. We shall now proceed to explain the Variations in the Tides which happen every Month, and which depend on the Change of Position of the Moon with Respect to the Sun and the Earth.

The menstrual Variations.

XVIII.

In the Conjunction of the Sun and Moon, those Luminaries coming to the Meridian at the same Time, and in the Opposition when one comes to the Meridian the other coming to the opposite Meridian, they conspire to raise the Waters of the Sea. In the Quadratures on the

The greatest Tides happen at the new and full Moon.

The least in the Quadratures.

contrary the Waters raised by the Sun, are depressed by the Moon, the Waters under the Moon being 90 Degrees from those under the Sun; consequently the greatest Tides happen at full and new Moon, and the least at first and last Quarter.

XIX.

The greatest and least Tides do not precisely happen at that Time, and why.

The greatest and least Tides do not happen in the Syfigies and Quadratures, but are the Third or the Fourth in Order after the Syfigies and Quadratures, because as in other Cases so in this, the Effect is not the greatest or the least when the immediate Influence of the Cause is greatest or least. If the Sea was perfectly at Rest when the Sun and Moon act on it in the Syfigies, it would not instantly attain its greatest Velocity, nor consequently its greatest Height, but would acquire it by Degrees. Now as the Tides which precede the Syfigies are not the greatest, they increase gradually, and the Waters have not acquired their greatest Height until some Time after the Moon has passed the Syfigies, and she begins to counteract the Sun's Force and depresses the Waters where the Sun raises them. Likewise the Tides which precede the Quadratures are not the least, they decrease gradually and do not come to their least Height until some Time after the Moon has passed the Quadratures.

XX.

The greatest Elevation of the Waters happens sooner after the Appulse of the Moon to the Meridian whilst she passes from the Syfigies to the Quadratures, and later whilst the Moon passes from the Quadratures to the Syfigies.

The greatest Height of the Waters which by the single Force of the Moon would happen at the Moon's Appulse to the Meridian, and by the single Force of the Sun at the Sun's Appulse to the Meridian, abstracting from the external Causes which retard it; by the combined Forces of both must fall out in an intermediate Time, which corresponds more exactly to the Motion of the Moon than to that of the Sun, wherefore when the Moon passes from Conjunction or Opposition to Quadrature, this greatest Height answers more to the setting of the Moon. The Sun in the first Case coming sooner to the Meridian than the Moon, and in the latter the Moon coming later to the Meridian than the Sun to the opposite Meridian; and when the Moon passes from Quadrature to Opposition or Conjunction, this greatest Elevation answers more to the rising of the Moon. In the first Case, the Moon coming sooner to the Meridian than the Sun to the opposite Meridian, and in the latter, the Moon coming sooner to the Meridian than the Sun (b). To calculate those Variations in the Time of High-water which arise from the respective Positions of the Sun and Moon, let us suppose on a certain Day, the Sun and Moon to be in Conjunction at the Appulse of the Moon to the Meridian of any Place, and consequently that it is High-Water there at that Instant. The following Day at the

(b) See preceding Note

Time of High-Water in said Place, the Sum of the Distances ($z' + x'$) of the Sun and Moon from the Meridian will be $12^d. 30^m.$ and the Interval between the two Tides will be expressed in solar Hours by $360^d. + \text{Arc } z'$. Since the Arcs z' and x' are very small, they may be supposed without any sensible Error to coincide with their Sines ($\sqrt{1 - ss}$) ($\sqrt{1 - tt}$) and $\sqrt{1 - ss} + \sqrt{1 - tt}$ may be supposed equal to $\text{Sin. } 12^d. 30^m. = 0,21643$, and consequently $\sqrt{1 - tt} = 0,21643 - \sqrt{1 - ss}$, we may suppose also $s = 1$ and $t = 1$: after those Substitutions the Equation $\frac{s\sqrt{1 - ss}}{t\sqrt{1 - tt}} =$

$\frac{29}{30} \times \frac{L}{S}$ will be transformed into $\frac{\sqrt{1 - ss}}{0,21643 - \sqrt{1 - ss}} = \frac{29}{30} \times \frac{L}{S}$; and substituting $\frac{5}{2}$ for $\frac{L}{S}$ we will have $\frac{\sqrt{1 - ss}}{0,21643 - \sqrt{1 - ss}} = \frac{29}{12}$ which gives for $\sqrt{1 - ss}$ or for the Sine of the Arc z' required $\frac{29}{41} \times 0,21643 = 0,15308$ or $z' = 8^d. 48^m.$ or $35\frac{1}{2}$ solar Minutes, so that the whole Interval is $24^h. 35^m. \frac{1}{2}$.

Let us now suppose on a certain Day, the Sun and Moon to be in Quadrature at the Appulse of the Moon to the Meridian at the above mentioned Place, and consequently that it is High-Water there at that Instant; the following Day at the Time of High-water the Sum of the Distances ($z' + x'$) of the Sun and Moon from the Meridian (if it be the last Quadrature) will be $77\frac{1}{2}$ Degrees, and the Sum of the Distances ($z + z'$) of the Sun from the Horizon and Meridian being 90 Degrees, $z - x' = 12^d. 30^m.$, that is, $s - \sqrt{1 - tt} = 0,21643$ and $\sqrt{1 - tt} = s - 0,21643$. But in this Case $\sqrt{1 - ss}$ may be supposed $= 1$ and $t = 1$, wherefore $\frac{s\sqrt{1 - ss}}{t\sqrt{1 - tt}} = \frac{s}{s - 0,21643} = \frac{s}{s - 0,21643} = \frac{29}{12}$ which gives $s = 0,36920$ answering to $21^d. 40^m.$ or to $1^h. 26\frac{2}{3}$ Minutes, so that the whole Interval ($360^d. + \text{Arc } z$) is 25 Hours, $26\frac{2}{3}$ Minutes.

From whence it appears that High-Water should precede the Appulse of the Moon to the Meridian whilst she is passing from the Syfigies to the Quadratures, and should follow the Appulse of the Moon to the Meridian whilst she is passing from the Quadratures to the Syfigies; that the greatest Anticipation or Retardation should be about 50 solar Minutes, and that the Distance of the Sun and Moon from each other at the Time of the greatest Anticipation or Retardation is about 57 Degrees. But from external Causes High-Water happens in the Ports of the Ocean three Hours later, consequently in those Ports it should precede the third lunar Hour, and that by the greatest Interval the ninth Tide after the Syfigies, and this greatest Anticipation being repaired in the five subsequent Tides, it should follow by like Intervals the third lunar Hour, whilst the Moon is passing from the Quadratures to the Syfigies.

XXI.

The Tides are greater *ceteris paribus*, when the Moon is in Perigee than when she is in Apogee.

The annual Variations, the Tides are greater in Winter than in Summer.

The Tides depend upon the Declination of the Sun and Moon.

Finally, all other Circumstances being alike, the Tides are greatest in the same Aspects of the Sun and Moon, when they have the same Declination, when the Moon is in Perigee than when she is in Apogee. The Force of the Moon on the Waters of the Sea decreasing in the triplicate Ratio of her Distance from the Earth.

XXII.

The annual Variations of the Tides depend on the Distance of the Earth from the Sun, hence it is that in Winter the Tides are greater, all other Circumstances being alike, in the Syfigies, and less in the Quadratures than in Summer, the Sun being nearer to the Earth in Winter than in Summer.

XXIII.

The Effects of the Sun and Moon upon the Waters of the Sea depend upon the Declination of the Luminaries, for if either the Sun or Moon was in the Pole, any Place of the Earth in describing its Parallel to the Equator, would not meet in its Course with any Part of the Water more elevated than another, so that there would be no Tide in any Place; therefore the Actions of the Sun and Moon on the Waters of the Sea become weaker as they decline from the Equator, and *Newton* found (Prop. 37. B. 3.) that the Force of each Luminary on the Sea decreases in the duplicate Ratio of the Cosine of its Declination; hence it is, that the Tides in the solstitial Syfigies are less than in the equinoctial Syfigies, and are greater in the solstitial Quadratures than in the equinoctial Quadratures, because in the solstitial Quadratures the Moon is in the Equator, and in the other the Moon is in one of the Tropics, and the Tide depends more on the Action of the Moon than that of the Sun, and is therefore greatest when the Moon's Action is greatest.

The Spring Tides therefore ought to be the greatest, and the Neap Tides the least at the Equinoxes, and because the Sun is nearer the Earth in Winter than in Summer, the Spring Tides are greatest and the Neap Tides the least in Winter; hence it is, that the greatest Spring and least Neap Tides are after the autumnal and before the vernal Equinox.

Two great Spring Tides never follow each other in the two nearest Syfigies, because if the Moon in one of the Syfigies be in her Perigee, she will in the following Syfigie be in her Apogee. In the first Case her Action being greatest and conspiring with that of the Sun, the Waters will be raised to their greatest Height; but in the latter Case her Action being least, the Tide will be less.

XXIV.

The Time and Height of the Tides depend up-

The ebbing and flowing of the Sea depends also upon the Latitude of the Place; for the conjoint Actions of the Sun and Moon changing the Water upon the Earth's Surface into an oblong Spheroid, one of the

Vertices of its longer Axis describing nearly, the Parallel on the Earth's Surface, which the Moon, because of the diurnal Motion, seems to describe, and the other a Parallel as far on the other Side of the Equator. The whole Sea is divided into two opposite hemispheroidal Floods, one on the North Side of the Equator, the other on the South Side of it, which come by Turns to the Meridian of each Place after an Interval of 12 Hours. Now the Vertex of the hemispheroidal Flood which moves on the same Side of the Equator with any Place, will come nearer to it than the Vertex of the opposite hemispheroidal Flood which moves in a Parallel on the other Side of the Equator; and therefore the Tides in all Places without the Equator, will be alternately greater and less; the greatest Tide when the Declination of the Moon is on the same Side of the Equator with the Place, will happen about three Hours after the Appulse of the Moon to the Meridian above the Horizon, and the least Tide about three Hours after the Appulse of the Moon to the Meridian below the Horizon, the Height of the Tide in the first Case, being expressed by a Semidiameter of the elliptic Section of the Spheroid nearer the transverse Axe than in the latter Case, and consequently is greater; and the Tide, when the Moon changes her Declination, which was the greatest will be changed into the least, for then the hemispheroidal Flood which is opposite to the Moon, moves on the same Side of the Equator with the Place, and therefore its Vertex comes nearer to it than the Vertex of the hemispheroidal Flood under it. And the greatest Difference of those Tides will be in the Solstices, because the Vertices of the two hemispheroidal Floods in that Case describe the opposite Tropics, which are the farthest from each other of any two parallel Circles they can describe. Thus it is found by Observation, that the Evening Tides in the Summer exceed the Morning Tides, and the Morning Tides in Winter exceed the Evening Tides; and we learn (Pro. 24. B. 3.) that at *Plymouth*, according to the Observations of *Colepreſs* this Difference amounts to one Foot, and at *Bristol*, according to those of *Sturmy* to 15 Inches. *Newton* (*de Mundi Systemate*, page 58.) found, that the Height of the Tides decreases in each Place, in the duplicate Ratio of the Cosine of the Latitude of this Place. Now we have seen, that at the Equator, they decrease in the duplicate Ratio of the Cosine of the Declination of each Luminary; therefore in all Places without the Equator, half the Sum of the Heights of the Tides Morning and Evening, that is, their mean Height decreases nearly in the same Ratio. Hence the Diminution of the Tides arising from the Latitude of Places, and the Declination of the Luminaries may be determined.

on the Latitude of Places.

The Height of the Tides decreases in the duplicate Ratio of the cosine of the Latitude.

XXV.

The Height of the Tides depend likewise upon the Extent of the Sea in which they are produced, whether the Seas be entirely separated

The Height of the Tides depend on

the Extent
of the Seas.

rated from the Ocean, or communicate with it by a narrow Channel; for if the Seas be extended from East to West 90 Degrees, the Tides will be the same as if they came from the Ocean, because this Extent is sufficient that the Sun and Moon may thereby produce on the Waters of the Sea their greatest and least Effect; but if those Seas be so narrow, that each of their Parts are raised and depressed with the same Force, there can be no sensible Effect, for the Water cannot rise in any one Place without sinking in another; hence it is, that in the *Baltick-Sea*, the *Black-Sea*, the *Caspian-Sea*, and other Seas or Lakes of less Extent, there is neither Flood nor Ebb.

XXVI.

The Tides
in the Me-
diterranean
are scarce
sensible.

In the *Mediterranean-Sea*, which is extended from East to West only 60 Degrees, the Flood and Ebb are scarce sensible, and *Euler* has given a Method for determining their Quantity. Those small Tides are still rendered less by the Winds and Currents which are very great in this Sea; hence it is, that in most of those Ports, there are scarce any regular Tides, except in those of the *Adriatick Sea*, which having a greater Depth, the Elevation of the Waters are rendered more sensible; hence it is, that the *Venetians* were the first who made Observations on the Tides of the *Mediterranean*.

XXVII.

Causes
which in-
fluence the
Tides that
are indeter-
minable.

Besides the assignable Causes which serve to account for the Phenomena of the Tides, there are several others which produce Irregularities in those Motions which cannot be reduced to any Law, because they depend on Circumstances which are peculiar to each Place; such are the Shores on which the Waters flow, the Straits, the different Depths of the Sea, their Extent, the Bays, the Winds, &c. so many Causes which alter the Motion of the Waters, and consequently retard, increase, or diminish the Tides, and are not reducible to Calculation. Hence it is, that in some Places, the Flood falls out the third Hour after the Culmination of the Moon, and in other Places the 12th Hour; and in general, the greater the Tides are, the later they happen, because the Causes which retard them act so much longer.

If the Tides were very small, they would immediately follow the Culmination of the Moon, because the Action of the Obstacles which retard them would be rendered almost insensible; this is partly the Reason why the great Tides which happen about the new and full Moon, follow later the Appulse of the Moon to the Meridian, than those which happen about the Quadratures; the latter being less than the former.

XXVIII.

Velocity of
the Waters
of the Sea.

Euler relates that at *St. Malos*, at the Time of the Sygies, it is High-Water the sixth Hour after the Appulse of the Moon to the Meridian, and the Retardation increases more and more until at *Dun-*

kerk and *Ostend*, it happens at Midnight. From this Retardation the Velocity of the Waters may be determined, and *Euler* concludes from those, and other Observations, that they move at the Rate of eight Miles an Hour; but it is easy to perceive, that this Determination cannot be general.

XXIX.

The Tides are always greater towards the Coasts than in the open Sea, and that for several Reasons; first the Waters beat against the Shores, and by the Re-action, are raised to a greater Height. Secondly, they come with the Velocity they had in the Ocean where their Depth was very considerable, and they come in great Quantity, consequently meet with great Resistance whilst they flow on the Shores; from which Circumstance, their Height is still encreased. Finally, when they pass over Shoals, and run through Straights, their Height is greatly encreased, because being beat back by the Shores, they return with the Force they had acquired from the Effort they had made to overflow them. Hence it is, that at *Bristol*, the Waters are raised to so great a Height at the Time of the Sygies, for the Shores on this Coast, are full of Windings and Sand-Banks, against which the Waters beat with great Violence, and are much impeded in their Motion.

The Tides are greater towards the Coasts, and why.

XXX.

Those Principles serve to account for the extraordinary great Tides which are observed in some Places, as at *Plymouth*, *Mount St. Michael*, the Town of *Avranches* in *Normandy*, &c. where *Newton* says, the Waters rise to 40 or 50 Feet, and some Times higher.

Explication of several Phenomena of the Tides.

It may happen, that the Tide propagated from the Ocean, arrives at the same Port by different Ways, and that it passes quicker through some of those Ways than through the others; in this Case, the Tide will appear to be divided into several Tides, succeeding one another, having very different Motions, and no ways resembling the ordinary Tides. Let us suppose, for Example, that the Tides propagated from the Ocean, arrive at the same Port by two different Ways, one of which is a readier and easier Passage, so that a Tide arrives at this Port through one of those Inlets at the third Hour after the Appulse of the Moon to the Meridian, and another through the other Inlet, six Hours after, at the 9th Hour of the Moon. When the Moon is in the Equator, the Morning and Evening Tides in the Ocean being equal, in the Space of 24 Hours, there will arrive four equal Tides to this Port, but one flowing in as the other ebbs out, the Water must stagnate. When the Moon declines from the Equator, the Tides in the Ocean are alternately greater and less, consequently two greater and two lesser Tides would arrive at this Port by Turns, in the Space of 24 Hours. The two greatest Tides would make the Water acquire its greatest Height at a mean Time

betwixt them, and the two lesser would make it fall lowest, at a mean Time between those two least Tides, and the Water would acquire at a mean Time betwixt its greatest and least Height, a mean Height; thus in the Space of 24 Hours, the Waters would rise, not twice, as usual, but once only to their greatest Height, and fall lowest only once.

If the Moon declines towards the Pole elevated above the Horizon, its greatest Height would happen the third, the sixth, or the 9th Hour after the Appulse of the Moon to the Meridian; and if the Moon declines towards the opposite Pole, the Flood would be changed into Ebb.

XXXI.

Explication of the Circumstances attending the Tides at Batsham in the Kingdom of Tunquin.

All which happens at *Batsham* in the Kingdom of *Tonquin*, in the Latitude of 20d. 50m. North. The Day in which the Moon passes the Equator, the Waters have no Motion of flux and reflux: as the Moon removes from the Equator, the Waters rise and fall once a Day, and come to their greatest Height when the Moon is near the Tropics; with this Difference, that when the Moon declines towards the North-Pole, the Waters flow in whilst the Moon is above the Horizon, and ebb out whilst she is under the Horizon, so that it is High-Water at the setting of the Moon, and Low-Water at her rising. But when the Moon declines towards the South-Pole, it is High-Water at the rising, and Low-Water at the setting of the Moon; the Waters ebbing out during the whole Time the Moon is above the Horizon.

The Tide arrives at this Port by two Inlets, one from the *Chinese* Ocean, by a readier and shorter Passage between the Island of *Leuconia* and the Coast of *Canton*, and the other from the Indian Ocean, between the Coast of *Cochin-China* and the Island of *Borneo*, by a longer and less readier Passage; but the Waters arrive sooner by the readiest and shortest Passage; hence they arrive from the *Chinese* Ocean in six Hours, and from the *Indian* Ocean in 12 Hours, consequently the Tide arriving the third and ninth Hour after the Appulse of the Moon to the Meridian, there result the above Phenomena.

XXXII.

At the Entrance of Rivers the Ebb lasts longer than the Flood, and why.

At the Entrance of Rivers, there is a Difference in the Time of the Tides flowing in and ebbing out, arising from the Current of the River, which running into the Sea, retards its Motion of flux, and accelerates its Motion of reflux, consequently makes the Ebb last longer than the Flood, which is confirmed by Experience; for *Sturmius* relates, that above *Bristol*, at the Entrance of the River *Oundal*, the Tide is five Hours flowing in, and seven Hours ebbing out. Hence it is also, that all other Circumstances being alike, the greatest Floods arrive later at the Mouths of Rivers than elsewhere.

XXXIII.

It has been found, as has been already mentioned, that the Tides depend on the Declination of the Luminaries, and the Latitude of the Place; hence at the Poles there is no diurnal ebbing and flowing of the Waters of the Sea; for the Moon being at the same Height above the Horizon during 24 Hours, cannot raise the Waters; but in those Regions, the Sea has a Motion of flux and reflux depending on the Revolution of the Moon about the Earth every Month; in consequence of which the Waters are at the lowest when the Moon is in the Equator, because she is then always in the Horizon with respect to the Poles; and as the Moon declines either towards the North or South Pole, the Sea begins to ebb and flow, and when her Declination is greatest, the Waters are raised to their greatest Height at the Pole towards which she declines; and as this Elevation, which does not exceed ten Inches, is produced but by a very slow Motion, the Force of *Inertia* increases it very little, consequently is scarce sensible.

At the Poles there is no diurnal Tides but such as depend on the Revolution of the Moon about the Earth.

XXXIV.

It is only at the Poles that the Waters have no diurnal Motion; in the Frigid-Zone, there is one Tide every Day instead of two, as in the Torid-Zone, and in our Temperate-Zones; and it is easy to shew, that this Passage of two Tides to one, is not effected suddenly, but like all other Effects of Nature, is produced gradually. For we have seen, that the Morning and Evening Tides in our Temperate-Zones are unequal, not only as to their Height, but also as to the Time of their Duration; that the remoter the Place is from the Equator, the greater is this Inequality between the two Tides which immediately succeed each other, both as to their Height and the Time of their Duration, for the greatest Tide should last longer than the least; and notwithstanding which they both cease in $12^h. 24^m.$ nearly; therefore, in those Regions where the Moon after her Appulse to the Meridian above or below the Horizon, returns to it in this Interval, the least Tide will entirely vanish, and there will remain but the greatest Tide, which alone will fill up the Interval of $12^h. 24^m.$

But it is only at the Poles that there is no diurnal Tides, for in the Frigid-Zone there is one and why there are not two as in the other Regions of the Earth.

XXXV.

The Force of the Sun and Moon are sufficient to produce the Tides, but are incapable of producing any other sensible Effects here below; for the Force (S) of the Sun in its mean Distance, being to the Force (G) of Gravity, as 1 to 12868200, and the Force (S) of the Sun being to the Force (L) of the Moon, as 1 to 4,4815, by the Composition of Ratios $L \times S$ is to $S \times G$, or the Force (L) of the Moon in her mean Distance, is to the Force (G) of Gravity, as 4,4815 to 12868200, or as 1 to 2871400. And since $S+L$ is to L as 5,4815 to 4,4815, $S+L \times L$ is to $L \times G$ or the Sum of the Forces ($S+L$) of the Sun and Moon

Why the Sun and Moon producing the Tides, produce no other sensible Effects here below.

when they conspire together, and in their mean Distances from the Earth, is to the Force (*G*) of Gravity as $5,4815 \times 1$ to $4,4815 \times 2871400$, or as 1 to 2347565, and the Sum of the greatest Forces of the Luminaries, or at their least Distance from the Earth, is to the Force of Gravity, as 1 to 2032890. From whence it appears, that those Forces united, cannot deflect the Direction of Gravity, nor consequently the Pendulum, from the true Vertical the 10th Part of a Second, nor cause a Variation in the Length of the Pendulum beating Seconds, which would exceed the $\frac{1}{300}$ of a Line, &c.

THEORY of the REFRACTION of LIGHT.

I.

Explication
of the Re-
fraction of
Light deriv-
ed from the
Principle of
Attraction.

THE Effects which Bodies exert on each other by their Attraction, become sensible only when it is not absorbed by the Attraction of the Earth, and it appears that this mutual Attraction of Bodies becomes sensible only when they are almost contiguous, and that then it acts in a Ratio greater than the inverse Triplicate of the Distances. Now the Atmosphere, or the Mass of Air encompassing the Earth, acting on Light in a very sensible Manner, it is certain, that if Attraction be the Cause, it should follow this Ratio.

The Advantage of the Principle of Attraction consists in having no Need of any Supposition but only the Knowledge of the Phenomena, and the more accurate are the Observations and Experiments, the easier it is to apply this Principle to their Explication.

II.

The Sine of
Incidence
and Refrac-
tion are al-
ways in a
constant
Ratio.

It is well known, that Light traversing Mediums of different Densities, changes its Direction. *Snellius*, and after him *Descartes*, found from Experiment, that the Sine of Incidence and that of Refraction are always in a constant Ratio; and *Newton* employs the 14th and last Section of the first Book of the *Principia* in explaining the Reason why those Sines should be in a constant Ratio, and proving that this Ratio depends on the Principle of Attraction. It is in this Explication we shall follow *Newton*.

Every Ray of Light which enters obliquely into any Medium, is to be considered as a Body acted on at the same Time by two Forces, in order to apply to the Explication of their Effects the Principles of Mechanicks. *Descartes* and *Fermat* considered Light as a Body of a sensible Magnitude on which the Mediums act after the same Manner as they appear to do on other Bodies: and finding that the Mediums which Light traverses, produce in them Effects quite contrary to those which should result from the Principles of Mechanicks, they invented each an Hypothesis in order to reconcile, in this Case, the Laws of Mechanicks, which are incontestable, and the phisical Effects which are almost as certain. ♦

III.

It is well known, that the denser the Mediums are, the greater Resistance Bodies which penetrate them meet with in separating their Parts. Now, in this Case, the Angle of Refraction is greater than the Angle of Incidence, because the vertical Velocity of the Body being diminished by the Resistance of the Mediums, the horizontal Velocity influences more the Direction of the Diagonal which the Body in obeying the two Forces into which its Motion is resolved, describes; hence it is, that when the Resistance of the Medium is insurmountable, the Body, instead of penetrating the Medium, returns back by its Elasticity, and the Proportion between this Resistance and the vertical Velocity of the Body may be such, that the Body would lose all its vertical Velocity, and would slide on the Surface of the Medium if it had no Elasticity, and if the Surface of the Medium was a perfectly smooth Plane.

The Laws of Refraction of Bodies of a sensible Magnitude.

IV.

Now quite the contrary happens to the Rays of Light, the denser the Medium is which they traverse, the more the Sine of Incidence exceeds that of Refraction; therefore the vertical Velocity of the Rays is increased in this Case, which is quite the Reverse of what the Laws of Mechanicks seem to indicate.

The Laws of Refraction of Light different from that of Bodies of a sensible Magnitude.

Descartes, in order to reconcile them with Experiment, which he could not evade, maintained, that the denser the Mediums were, the easier Passage they opened to Light; but this Manner of accounting for this Phenomenon was rather rendering it doubtful than explaining it.

Fermat, finding the Explication of *Descartes* impossible, thought it more advisable to have Recourse to Metaphysics, and the final Causes. He asserted, that since Light does not arrive to us by the shortest Passage, which is the straight Line, it was becoming the Divine Wisdom, it should arrive in the shortest Time; this Principle, once allowed, it followed, that the Sines of Incidence and Refraction are to each other as the Facilities of the Medium to be penetrated.

Hypotheses of *Descartes* and *Fermat*.

V.

It is easy to see how Attraction solves this Difficulty; for this Principle evinces, that the progressive Motion of Light, not only is not less retarded in the more dense Medium, as *Descartes* pretended, but is really accelerated, and that by the Attraction of the more dense Medium when it penetrates it. It is not only when the Ray has arrived at the refracting Medium and at the Point of Incidence that it acts on it; the Incurvation of the Ray commences some Time before, and it increases in proportion as it approaches the refracting Medium, and even within this Medium to a certain Depth.

Attraction accounts for every Circumstance attending the Refraction of Light.

Attraction accounts for every Circumstance attending Light in its Passage through one Medium into another; for the vertical Velocity of

the Ray is increased in the more dense Medium, which it traverses until it arrives at the Point where the superior and inferior Parts of this Medium act with equal Force on it, then it continues to advance with the acquired Velocity until being on the Point of quitting it, the superior Parts of this Medium attract it with a greater Force than the inferior Parts. The vertical Velocity of the Ray is diminished thereby, and the Curve it describes at its Emerſion, is perfectly equal and ſimilar to the one it deſcribed at its Incidence, (the Surfaces which bound the refracting Medium being ſuppoſed parallel) and the Poſition of this Curve is directly oppoſite to that of the firſt. In fine, the Ray paſſes through Degrees of Retardation which are in the ſame Ratio, and in the ſame inverſe Order as the Degrees of Acceleration which it paſſed through at its Incidence.

VI.

Experiments of Newton which prove that the Refraction of Light depends on the Density of the Mediums thro' which it paſſes.

Newton, who was as ſuperior in the Art of making Experiments as in that of employing them, found on examining the Deviation of the Rays of Light in different Mediums, that the Attraction exerted on the Particles of Light follows the Ratio of the Density of thoſe Mediums, if we except thoſe which are greaſy and fulphurous. Since then the different Densities of thoſe Mediums is the Cauſe of the Refraction of Light, the more homogeneous Bodies are, the more transparent they will be; and thoſe which are moſt heterogeneous will be leaſt ſo, for the Light in traversing them, being perpetually reflected in different Directions within thoſe Bodies, the Quantity of Light which arrives to us is thereby diminished; hence it is, that when the Sky is clear, the Stars are ſo diſtinctly perceived, but when clouded, the Rays are intercepted, and cannot reach the Earth.

VII.

The Rays of Light have not all the ſame Degree of Refrangibility.

Newton alſo found, that every Ray of Light, however ſmall, is compoſed of ſeven Rays, which as long as they are united continue white, but reſume their natural Colour when they are ſeparated, and that thoſe Rays have not all the ſame Degree of Refrangibility, that is, in paſſing through one Medium into another of different Density, are inflected ſome more and others leſs; ſo that when they paſs through a Lens, thoſe Rays do not all meet the Axe at the ſame Diſtance, but ſome nearer and others farther off, and thus form as many diſtinct Pictures of the Object as there are Colours. The Eye only perceives the moſt vivid, but as the Pictures are not equal, the greateſt form round thoſe, ſeveral coloured Circles, which is called the Crown of Aberration. This Aberration is quite diſtinct from that which ariſes from the Defect of Reunion of the Rays cauſed by the ſpherical Figure of the Lenses.

The Aberration of Refrangibility in the Rays of Light is not ſenſible when their Refraction is inconfiderable; now the Rays parallel to the

optic Axe of a Lens, and those at a small Distance from this Axe, are very little inflected, and the Picture they form may be considered as simple, as not being surrounded by any coloured Circles. Hence it is, that Artists are under the Necessity of giving to the objective Glass an Aperture of a very small Number of Degrees of the Sphere of which this Glass forms a Part, and consequently of increasing the focal Distance of this Glass, and the Length of the Telescope, as often as they change the Proportion of the objective and ocular Glasses, in order to increase its magnifying Power. Those Obstacles to the Perfection of refracting Telescopes arising from the Nature of Light, and the Laws of Refraction, *Newton* was on the Point of removing; an Experiment he made opened the Way which leads to this Discovery, but he did not pursue it: the Experiment is as follows: *As often as Light, traversing different Mediums, is so corrected by contrary Refractions, that it emergeth in Lines parallel to those in which it was Incident, continues ever after white.* OPTICS, First B. Part II. Exp. 8.

Euler in 1747, meditating on this Subject, demonstrated, that this Assertion was false, and consequently that the Experiment was ill made. *Mr. Dolond*, an eminent *English* Optician, well versed in the Theory and Practice of his Art, repeated this Experiment after the same Manner that *Newton* described it; he constructed for this Purpose, with two Plates of Glass, a Kind of Port-folio, which being filled with Water, formed a Prism of Water, that by closing or opening the Glasses, was susceptible of all Kinds of Angles; he plunged into the Water of this Prism, whose Angle was turned downwards, another Prism of Chrystal, whose Angle was turned upwards, and by moving the Plates of Glass, he found that Inclination which was necessary to make the Objects observed through the two Prisms of Water and Glass appear exactly at the same Height as they did to the naked Eye; it was then manifest, that the Refraction of one Prism was destroyed by the Refraction of the other, yet the Objects were tinged with various Colours, which was quite contrary to what *Newton* had asserted. *Mr. Dolond* afterwards tried, by moving the Plates of his Prism of Water, whether there was not some possible Proportion between the Angles of the two Prisms capable of destroying the Colours, and found that there was such a Proportion, which widely differed from that which destroys the absolute Refraction. The Objects not coloured viewed through the Prisms thus combined, not appearing at the same Height as when viewed by the naked Eye. From whence it was easy to conclude, that the Aberration of the Rays arising from their different Degrees of Refrangibility, might be corrected by employing transparent Mediums of different Densities, and that the Rays would be refracted, but in a different Manner from what they would be in passing through one Medium. *Mr. Dolond* in 1759, discovered a Method

How the Method for correcting the Aberration arising from the different Refrangibility of the Rays was discovered.

answering this Purpose, which he has employed with Success in the Construction of achromatic Telescopes, and the most eminent Mathematicians have since exerted all their Skill in investigating the different Combinations for the focal Distances, and the Quantity of Curvature requisite to correct at once, the Aberration arising as well from the different Degrees of Refrangibility of the Rays, as from the circular Figure of the Lenses. Those Researches shall be explained hereafter.

VIII.

The Principle of Attraction serves to explain how Refraction is changed into Reflection.

The Principle of Attraction serves to explain why the Refraction is changed into Reflection at a certain Obliquity of Incidence, when the Rays of Light pass through a more dense Medium into a less dense one; for in the Passage of a Ray through a more dense Medium into another that is less, the Curve it describes is inflected towards the more dense Medium it has passed through; now the Proportion between its Obliquity and the Force which draws it towards this more dense Medium may be such, that its Direction may become parallel to the Surface of the Medium which it quits, before it has passed the Limits within which the Attraction of this Medium is confined; and in this Case, it is very easy to see, that it should return toward the refracting Medium it had quitted, describing a Branch of a Curve equal and similar, to the Curve which it described in passing through this Medium, and reassume after having again entered this Medium the same Inclination it had before it quitted it.

The Action of the Medium which Light traverses, may give the Rays the Obliquity they require in order to be reflected, and as the more the Mediums differ in Density the less is the Obliquity of Incidence requisite that the Rays may be reflected, the Rays will be reflected at the least Obliquity of Incidence when the contiguous Space or refracting Medium will be purged of Air, and when the Vacuum will be most perfect. And so it happens in the Air-Pump, in which the more the Vacuum is increased, the quicker a Ray is reflected at the superior Surface of a Prism placed therein. The Refraction is therefore changed into Reflection at different Incidences, according to the Density of the different Mediums, *Diamond* which is the most brilliant Body known, operates an entire Reflection when the Angle of Incidence is only 30 Degrees, and it is according to this Angle Jewellers cut their Diamonds, that they may lose the least Quantity of the Light they receive.

IX.

It is easy to perceive, that when a Ray of Light passes through a less dense Medium into a more compact one, the Refraction cannot be changed into Reflection let the Obliquity of Incidence be ever so great, for when the Ray is on the Point of quitting the less dense Medium, the other Medium which is contiguous to it, begins to act on it, and

increases continually its vertical Velocity, the Rays of Light therefore in their Passage through the different Couches of the Atmosphere, whose Density continually increases in approaching the Earth, are more and more curved; in consequence of which the celestial Objects appear more elevated than they really are, and that by how much the more their Rays are curved from their Entrance into the Atmosphere until they arrive to us, the Eye receiving the Impression of Light in the Direction which the Rays have when they enter the Eye.

This apparent Elevation of the heavenly Bodies above their true Height, is called Astronomical Refraction, and is greatest near the Horizon, where repeated Observations prove, that it amounts to 33'; hence it is, that in our Climates, the Sun appears to rise 3 Minutes sooner, and set 3 Minutes later than it really does, whereby the artificial Day is increased 6 Minutes by the Effect of Refraction. This Effect gradually increases in advancing towards the Frigid-Zone, and at the Pole, by the Refraction alone, the Day becomes 36 Hours longer; hence it is also that the Sun and Moon at their rising and setting appear oval, the inferior Margin of those Luminaries being more refracted than the superior one, or appear higher in Proportion.

Refraction increases the Length of the Day.

Newton has shewn how to determine the Law according to which Refraction varies from the Zenith to the Horizon; from his Theory it results, that the Radius (R) is to the Sine of 87^d . as the Sine of (z) the Distance from the Zenith, to the Sine of ($z-6r$) of this same Distance diminished by six Times the Refraction at this Distance, wherefore $R-\text{Sine } 87 : \text{Sine } 87 = \text{Sine } z - \text{Sine } (z-6r) : \text{Sine } (z-6r)$; and $R-\text{Sine } 87 : \text{Sine } z - \text{Sine } (z-6r) = \text{Sine } 87 : \text{Sine } (z-6r)$; but $R-\text{Sine } 87$ is to $\text{Sine } z - \text{Sine } (z-6r)$ as $3d. \times \text{Cos. } 88\frac{1}{2}$ to $6r \times \text{Cos. } (z-3r)$, Differences of the Arcs multiplied by the Cosines of the Arcs, which are the arithmetical Means between 90 and 87 , and between z and $z-6r$. Therefore the Sine of $88d. \frac{1}{2}$, that is of $90d$. diminished by the Triple of the horizontal Refraction, is to the Sine of the Distance z diminished by the Triple of the Refraction at that Distance, as the horizontal Refraction, is to the Refraction at the Distance z , and as the Cosine of $88d. \frac{1}{2}$ to the Cosine of the Arc z diminished by the Triple of the Refraction; therefore the Refraction at the Distance z , is equal to the horizontal Refraction multiplied by the Tangent of z diminished by the Triple of its Refraction, the whole divided by the Tangent of $88d. 21^m$. from whence it appears, that the Refractions are proportional to the Tangents of the Distances from the Zenith diminished by three Times the Refraction.

Rule for finding the Refraction at any distance from the Zenith.

Example. Let the Refraction at the Distance of 45 Degrees from the Zenith be required, which is known to be about 1^m . the Tangent of $88d. 21^m$. is to the Tangent of $44d. 57^m$. as the horizontal Refraction 33^m . is to $57'$, the Refraction at 45 Degrees Distance from the Zenith. By this Rule the following Table was constructed.

X.

Table of
Astronomi-
cal Refrac-
tion.

App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.
D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.	D. M. M. S.
0 0 33.0,0	4 0 11.51,1	8.30 6. 8,0	15.30 3.23,7	36 0 1.18,5	63 0 0.29,1								
0 5 32.10,4	4 10 11.28,9	8.40 6. 1,3	16. 0 3.16,9	37 0 1.15,7	64 0 0.27,8								
0 10 31.22,2	4 20 11. 7,9	8.50 5.54,8	16.30 3.10,5	38 0 1.13,0	65 0 0.26,5								
0 15 30.35,4	4 30 10.48,0	9. 0 5.48,5	17. 0 3. 4,5	39 0 1.10,4	66 0 0.25,3								
0 20 29.49,7	4 40 10.29,2	9.10 5.42,4	17.30 2.58,9	40 0 1. 7,9	67 0 0.24,1								
0 30 28.22,3	4 50 10.11,3	9.20 5.36,5	18. 0 2.53,6	41 0 1. 5,5	68 0 0.22,9								
0 32 28. 4,8	5 0 9.54,3	9.30 5.30,9	18.30 2.48,6	42 0 1. 3,3	69 0 0.21,7								
0 36 27.30,3	5 10 9.38,2	9.40 5.25,4	19. 0 2.43,9	43 0 1. 1,1	70 0 0.20,6								
0 40 26.59,7	5 20 9.22,8	9.50 5.20,0	19.30 2.39,4	44 0 0.59,0	71 0 0.19,5								
0 50 25.41,8	5 30 9. 8,0	10. 0 5.14,8	20. 0 2.35,1	45 0 0.57,0	72 0 0.18,4								
1 0 24.28,6	5 40 8.54,0	10.15 5. 7,3	20.30 2.31,0	46 0 0.55,0	73 0 0.17,3								
1 10 23.19,8	5 50 8.40,6	10.30 5. 0,1	21. 0 2.27,2	47 0 0.53,1	74 0 0.16,2								
1 20 22.15,2	6 0 8.27,8	10.45 4.53,2	21.30 2.23,6	48 0 0.51,2	75 0 0.15,1								
1 30 21.14,7	6 10 8.14,9	11. 0 4.46,6	22. 0 2.20,3	49 0 0.49,4	76 0 0.14,0								
1 40 20.17,9	6 20 8. 2,8	11.15 4.40,3	23. 0 2.13,7	50 0 0.47,6	77 0 0.13,0								
1 50 19.24,8	6 30 7.51,1	11.30 4.34,3	24. 0 2. 7,4	51 0 0.45,9	78 0 0.12,0								
2 0 18.35,0	6 40 7.40,3	11.45 4.28,6	25. 0 2. 1,6	52 0 0.44,2	79 0 0.11,0								
2 10 17.48,4	6 50 7.30,2	12.00 4.23,2	26. 0 1.56,2	53 0 0.42,6	80 0 0.10,0								
2 20 17. 4,5	7 0 7.20,5	12.20 4.16,1	27. 0 1.51,2	54 0 0.41,1	81 0 0. 9,0								
2 30 16.23,8	7 10 7.11,1	12.40 4. 9,4	28. 0 1.46,6	55 0 0.39,6	82 0 0. 8,0								
2 40 15.45,4	7 20 7. 2,1	13. 0 4. 3,0	29. 0 1.42,4	56 0 0.38,2	83 0 0. 7,0								
2 50 15. 9,4	7 30 6.53,4	13.20 3.56,9	30. 0 1.38,4	57 0 0.36,8	84 0 0. 6,0								
3 0 14.35,6	7 40 6.45,1	13.40 3.51,1	31. 0 1.34,6	58 0 0.35,5	85 0 0. 5,0								
3 10 14. 3,9	7 50 6.37,1	14. 0 3.45,5	32. 0 1.31,0	59 0 0.34,2	86 0 0. 4,0								
3 20 13.34,1	8 0 6.29,4	14.20 3.40,1	33. 0 1.27,6	60 0 0.33,0	87 0 0. 3,0								
3 30 13. 6,2	8 10 6.22,0	14.40 3.34,9	34. 0 1.24,4	61 0 0.31,7	88 0 0. 2,0								
3 40 12.39,6	8 20 6.14,8	15. 0 3.29,9	35. 0 1.21,4	62 0 0.30,4	89 0 0. 1,0								
3 50 12.14,6	8 30 6. 8,0	15.30 3.23,7	36. 0 1.18,5	63 0 0.29,1	90 0 0. 0,0								

THEORY of the SECONDARY PLANETS.

I.

THE first Phenomenon which the Secondary Planets offer to natural Philosophers, is their Tendency towards their Primaries, in observing the same Law as the primary Planets towards the Sun. This Tendency has been sufficiently established in treating of the primary Planets, abstracting at first, as was necessary in order to simplify the Question,

from all the Irregularities which the Planets, by their mutual Attractions produce in each others Motions, or which arise from the Action of the Sun. Having afterwards examined the Irregularities in the Motions of the primary Planets; but the Irregularities in the Motions of the secondary Planets deserve particularly to be considered, in order to shew after a more satisfactory Manner, the Universality of the Principle of Attraction, and the Harmony of the System to which it serves as a Basis.

The different Kinds of Motions observed for many Ages in the Moon, and the Laws of those Motions discovered by eminent Astronomers, furnished *Newton* the Means of applying his Theory with Success to this Planet. This great Man, who had made so many Discoveries in the other Parts of the System of the World, was resolved not to leave this Part unexamined; and though the Method he has pursued on this Occasion, is less evident, and less satisfactory than the Method he employed in explaining the other Phenomena; we are however much indebted to him for having made it the Object of his Inquiry.

II.

It is easy to perceive, that if the Distance of the Sun from the Earth and the Moon, was infinite with the respect to their Distance from each other, the Sun would not disturb the Motion of the Moon about the Earth; because equal Forces, whose Directions are parallel, which act on any two Bodies, cannot affect their relative Motions. But as the Angle formed by the Lines drawn from the Moon and the Earth to the Sun, though very small, cannot be esteemed as having no Quantity, from this Angle therefore is to be deduced the Inequality of the Action of the Sun on these two Bodies.

Taking therefore, as *Newton* has done, (Propos. 66.) in the straight Line drawn from the Moon to the Sun, a Line to express the Force with which the Sun attracts it; let this Line be considered as the Diagonal of a Parallelogram, one of whose Sides will be in the straight Line drawn from the Moon to the Earth, and the other a Line drawn from the Moon parallel to the straight Line which joins the Sun and the Earth, it is evident, that those two Sides of the same Parallelogram will express two Forces which might be substituted for the Force of the Sun on the Moon; and that the first of those two Forces which urges the Moon towards the Earth, will neither accelerate nor retard the Description of the Areas, nor consequently prevent her from observing the Law of *Kepler*, viz. the Areas proportional to the Times, but will only change the Law of the Force with which the Moon tends towards the Earth, and consequently will alter the Form of her Orbit. As to the second Force, that which acts in a Direction parallel to the Ray of the Orbit of the Earth, if it was equal to the Force with which the Sun acts on the Earth, it is easy to perceive that it would produce no Irregularity in the Motion of the Moon; but those Forces are only equal in those

Manner of having regard to the Inequality of the Force of the Sun, on the Earth and the Moon.

The Force of the Sun is resolved in two others.

One urges the Moon towards the Earth.

The other acts in the Direction of the Line drawn from the Earth to the Sun.

Points of the Moon's Orbit, where her Distance from the Sun becomes equal to the Distance of the Earth from the Sun at the same Time, which happens in the Quadratures; in every other Point of her Orbit those two Quantities being unequal, their Difference expresses the perturbing Force of the Sun on the Moon, not only preventing her from describing equal Areas in equal Times, but also from moving always in the same Plane.

III.

Measure
of the
perturbing
Forces of
the Sun.

We find in Prop. 66 of the first Book, only the general Exposition of the Manner of estimating the perturbing Forces of the Sun on the Moon: But in Prop. 25 of the third Book, we find the Calculation which determines their Quantity; we learn that the Part of the Force of the Sun which urges the Moon towards the Earth, is in its mean Quantity, the $\frac{1}{17828}$ of the Force with which the Earth acts on her when she is in her mean Distance. The other Part of the same Force of the Sun which acts in a Direction parallel to the Ray of the Orbit of the Earth, is to the first Part, as the Triple of the Cosine of the Angle formed by the straight Lines drawn from the Moon and the Earth to the Sun.

IV.

Acceler-
ation of the
Areas descri-
bed by the
Moon pro-
duced by
this Force.

Newton employs this Determination of the perturbing Forces (Prop. 26, 27, 28, 29.) for computing the monthly Inequality in the Moon's Motion, called her Variation, whereby she moves swifter in the first and third Quarter, and slower in the Second and Fourth, and which becomes most sensible in the Octants or 45 Degrees from the Sygies.

Newton, to determine this Inequality, abstracts from all the rest; he further supposes the Moon's Orbit to be circular, if the Sun was away, and he investigates the Acceleration in the Area which the Moon describes, produced by that one of the two perturbing Forces which acts in a Direction parallel to the Ray drawn from the Earth to the Sun. He found that the Area described by the Moon in small equal Portions of Time, to be nearly as the Sum of the Number 219,46, and the versed Sine of double of the Moon's Distance from the nearest Quadrature, (the Radius being Unity); so that the greatest Inequality in the Areas described by the Moon, arrives in the Octants or 45 Degrees from the Sygies, where this versed Sine is in its Maximum.

V.

The Action
of the Sun
renders the
Orbit of the
Moon more
contracted
between the

To determine afterwards the Equation or Correction in the mean Motion of the Moon arising from this Acceleration of the Area described by the Moon, he has Regard to the Change in the Form of the lunar Orbit, produced by the perturbing Force. He investigates the Quantity which the perturbing Force would render the Line passing through the Quadratures longer than that which traverses the Sygies. The

Data which he employs in solving this Problem, are the Velocities of the Moon in those two Points, which he had shewn how to determine in the foregoing Proposition, and the centripetal Forces corresponding to the same Points, which are both one and the other compounded of the Force with which the Moon tends towards the Earth, and the perturbing Forces of the Sun, which in the Sygies and Quadratures act in the Direction of the Ray of the Orbit of the Moon. Now the Curvatures in those Points, being in the direct Proportion of the Attractions, and in the Inverse of the Squares of the Velocities, by this Means he obtains the Ratio of the Curvatures, and from thence deduces the Ratio of the Axes of the Orbit, assuming for Hypothesis, that this Curve is an Ellipse, having its Centre in the Centre of the Earth, if the Sun be supposed to have no apparent Motion round the Earth; but when Regard is had to this Motion of the Sun, because the lesser Axe of the Ellipse is also carried about the Earth with the same Motion, as being always directed towards the Sun, that it is a Curve whose Rays are the same as those of the Ellipse, but the Angles they form are increased in the Ratio of the periodic Motion of the Moon to its synodical Motion. The first of those Motions being that in which the Moon is referred to a fixed Point in the Heavens; the other in which she is compared with the Sun. By the Means of those Suppositions, *Newton* found that the Axe which passes through the Quadratures, is greater than that which passes through the Sygies by $\frac{1}{87}$.

VI.

He afterwards computes in the same Hypothesis of the Moon's Orbit being circular, if the Sun was away, by the Principle of the Areas proportional to the Times, the Equation or Correction in the mean Motion of the Moon resulting not only from the Acceleration found in the foregoing Problem, her Orbit being supposed circular, but from the new Form of this Orbit. From the Combination of those two Causes, he finds an Equation or Correction which becomes most considerable in the Octants, and then amounts to $35^m. 10'$ when the Earth is in its mean Distance; and in the other Points of the Earth's Orbit, is to $35^m. 10'$, in the inverse Ratio of the Cube of the Distance from the Sun, because the Expression of the perturbing Force of the Sun, which is the Cause of all these Irregularities of the Moon, is divided by the Cube of the Earth's Distance from the Sun. This Correction in the other Points of the Moon's Orbit, is proportional to the Sine of double of the Distance of the Moon from the nearest Quadrature.

VII.

Newton passes from the Examination of the Variation of the Moon, to that of the Motion of the Nodes, (Prop. 30, 31.) In this Inquiry he supposes the Moon's Orbit to be circular if the Sun was away, and attributes to the Force of this Luminary no other Effect than to change

Sygies
than be-
tween the
Quadra-
tures.

Computa-
tion of the
Variation
of the
Moon.

Computa-
tion of the
Motion
of the
Nodes.

Which of
the two per-
turbating
Forces of
the Sun he
employs.

this circular Orbit into an Ellipse, whose Centre is in the Centre of the Earth, or rather into the Curve whose Construction we have already given by the Means of an Ellipse. Of the two perturbating Forces of the Sun, that which urges the Moon towards the Earth, acting in the Plane of the Orbit, cannot produce any Motion in this Plane; he therefore only considers that Force which acts parallel to the Line drawn from the Earth to the Sun, which he had shewn to be proportional to the Cosine of the Angle formed by the Lines drawn from the Moon and the Earth to the Sun, and we shall now explain how he employs this Force.

At the Extremity of the little Arc which the Moon describes in any small Portion of Time, he takes one equal to it, which would be that which the Moon would describe if the perturbating Force of the Moon ceased to act on her; and at the Extremity of this new Arc, he draws a Line parallel to that which joins the Centre of the Earth and the Sun, and he determines the Length of this straight Line, by the Measure already determined of the Force which acts in the same Direction as it; which being done, the Diagonal of the Parallelogram, one of whose Sides is the little Arc which the Moon would describe if the perturbating Force ceased to act, and the other, the Arc the Moon would describe if this Force acted alone, is the real Arc the Moon would describe. There remains therefore no more to be done than to determine, how much the Plane which would pass through this small Arc and the Earth, differs from the Plane which passes through the first Side and the Earth.

The two Sides already mentioned, being produced until they meet the Plane of the Orbit of the Earth, and having drawn from their Points of Concourse with this Plane, two straight Lines to the Centre of the Earth, the Angle which those two straight Lines form, is the Motion of the Node during the small Portion of Time which the Moon employs in describing this small Arc, which we have been considering. And *Newton* finds that the Measure of this Angle, and consequently the Velocity or the instantaneous Motion of the Node, is proportional to the Product of the Sines of three Angles, which express the Distance of the Moon from the Quadrature, of the Moon from the Node, and of the Node from the Sun.

Law of the
Motion of
the Nodes.

Regression
and Progres-
sion of the
Nodes
in each
Revolution.

VIII.

It follows from hence, that when one of those three Sines becomes negative, the Motion of the Nodes which before was retrograde, becomes direct. Wherefore when the Moon is between the Quadrature and the nearest Node, the Motion of the Node is direct; in all other Cases, its Motion is retrograde, but the retrograde Motion exceeding

the direct Motion, it happens that in each Revolution of the Moon, the Nodes are made to recede.

When the Moon is in the Syfigies, and the Nodes in the Quadratures, that is, 90 Degrees from the Sun, their Motion is $33'' 10''' 37^{iv} 12^v$, wherefore the horary Motion of the Nodes in every other Situation, is to $33'' 10''' 27^{iv} 12^v$, as the Product of the three Sines already mentioned to the Cube of Radius.

IX.

Supposing the Sun and the Node to be in the same Situation with respect to the fixed Stars, whilst the Moon passes successively through its several Distances with respect to the Sun. *Newton* investigates (Prop. 32. B. III.) the horary Motion of the Node, which is a Mean between all the different Motions resulting from the foregoing Formula, and this mean Motion of the Node is $16'' 33''' 16^{iv} 36^v$, when the Orbit is supposed circular, and the Nodes are in Quadrature with the Sun; in every other Situation of the Nodes, this Motion is to $16'' 33''' 16^{iv} 36^v$, as the Square of the Sine of the Distance of the Sun from the Node, is to the Square of the Radius. If the Orbit of the Moon be supposed to be an Ellipse, having its Centre in the Centre of the Earth, the mean Motion of the Nodes in the Quadratures is only $16'' 16''' 37^{iv} 42^v$, and in any other Situation of the Nodes, it depends likewise on the Square of the Sine of the Distance from the Sun.

In order to determine for any given Time, the mean Place of the Nodes, *Newton* takes a Medium between all the mean Motions already mentioned. He employs in this Inquiry, the Quadrature of Curves, and the Method of Series. By this Means he finds that the Motion of the Nodes in a sydereal Year, should be $19^\circ 18' 1'' 23'''$, which only differs $3'$ from that which results from astronomical Observations.

X.

The same Curve the Quadrature of whose Area determines the mean Velocity of the Nodes, serves also for finding the true Place of the Nodes for any given Time, (Prop. 33. B. III.)

The Result of his Computation is as follows: Having made an Angle equal to the Double of that which expresses the Distance of the Sun from the mean Place of the Nodes, let the Sides of this Angle be to each other, as the mean annual Motion of the Nodes, which is $19^\circ 49' 3'' 55'''$, to the Half of their true mean Motion, when they are in the Quadratures, which is $0^\circ 31' 2'' 3'''$, that is, as 38,3 to 1, which being done, and having completed the Triangle which will be given, since this Angle and its two Sides are given, the Angle of this Triangle opposite to the least of those Sides, will express with sufficient Accuracy, the Equation or Correction in the mean Motion of the Nodes for determining the true Motion required.

At the End of every Revolution the Nodes recede.

Formula which gives the horary Motion of the Nodes in any Situation.

Determination of the mean Motion of the Nodes.

Determination of the true Place of the Nodes for any given Time.

XI.

Variation
of the In-
clination of
the Moon's
Orbit.

Horary Va-
riation of
the Inclina-
tion.

From the Investigation of the Motion of the Nodes, *Newton* passes (Prop. 34. B. III.) to the Determination of the Variation in the Inclination of the Orbit of the Moon. By employing that one of the two perturbing Forces of the Sun which does not act in the Plane of the Orbit of the Moon, he obtains the Measure of the horary Variation in the Inclination of the Orbit of the Moon; this Variation, when the Orbit is supposed circular, being to the horary Motion of the Nodes, $33'' 10''' 3^{iv} 12^v$, (the Nodes being in the Quadratures, and the Moon in the Sygies) diminished in the Ratio of the Sine of the Inclination of the Orbit of the Moon to the Radius: as the Product of the Sine of the Distance of the Moon from the nearest Quadrature, the Sine of the Distance of the Sun from the Nodes, and the Sine of the Distance of the Moon from the Nodes to the Cube of Radius. And this Quantity diminished by $\frac{1}{69}$ is the Variation corresponding to the Orbit rendered elliptic by the perturbing Force of the Sun.

XII.

Method for
finding the
Inclination
of the
Moon's
Orbit for
any given
Time.

The horary Variation of the Inclination of the Orbit of the Moon being thus determined, *Newton* employing the same Method, and the same Suppositions by which he found the true Place of the Nodes for any given Time, determines (Prop. 35. B. III.) the Inclination of the Orbit for any given Instant of Time; the Result of his Computation is as follows.

Let there be taken from the same Point of a straight Line, assumed as a Base, three Parts in geometrical Proportion, the first expressing the least Inclination, the third the greatest; let there be afterwards drawn through the Extremity of the Second, a Line making with this Base an Angle equal to double the Distance of the Sun from the Node for the proposed Motion let this Line be produced until it meets the Semicircle described on the Difference of the first and third Lines in geometrical Proportion; which being done, the Interval comprised between the first Extremity of the Base, and the Perpendicular let fall from the common Section of the Circle and the Side of the Angle just mentioned, will express the Inclination for the proposed Time.

Determina-
tion of the
Latitude of
the Moon.

From hence is deduced the Moon's Latitude corrected; for in a Right-angled spherical Triangle is given, besides the Right-angle, the Hypothenuse, viz. the Moon's Distance from the Node, the Angle at the Node, viz. the Inclination of the Plane of the Moon's Orbit to the Plane of the Ecliptic, consequently the Side opposite to this Angle, which expresses the Latitude corrected, will be also given.

But there is a more simple Method for finding the Latitude of the Moon corrected. For the mean Latitude being computed, the Inclination of the Moon's Orbit to the Ecliptic being supposed constant and equal to $5^{\circ}.9'.8''$. the Equation or Correction of the Latitude will be

8' 50" multiplied by the Sine of double the Distance of the Moon from the Sun less the Distance from the Node.

XIII.

Newton, after having exposed the Method by which he calculated that Inequality in the Moon's Motion, called her Variation, and the Method he had followed in determining the Motion of her Nodes, and the Variation of the Inclination of her Orbit to the Ecliptic, gives an Account of what he says he deduced from his Theory of Gravitation, with respect to the Motion of the Apogee, the Variation of the Excentricity, and all the other Irregularities in the Moon's Motion. It is in the Scholium of Prop. 35. B. III. he delivers those Theorems, which serve as a Foundation to the Construction of the Tables of the Moon's Motion. The Substance of which is as follows.

What *Newton* says with regard to the other Irregularities of the Moon's Motion.

XIV.

The mean Motion of the Moon should be corrected by an Equation depending on the Distance of the Sun from the Earth. This Equation or Correction, called the annual one, is greatest when the Sun is in his Perigee, and least when in his Apogee. Its Maximum is 11' 51", and in the other Cases, it is proportional to the Equation of the Centre of the Sun. It is to be added to the mean Motion of the Moon in the six first Signs, counted from the Apogee of the Sun, and to be subtracted in the six other Signs.

Annual Equations of the Motion of the Moon, of the Apogee and of the Nodes.

The mean Places of the Apogee and of the Nodes should be also each corrected by an Equation of the same Kind, depending on the Distance of the Sun from the Earth, and proportional to the Equation of the Centre of the Sun. The Equation of the Apogee in its Maximum is 19' 43", and is to be added from the Perihelion to the Aphelion of the Earth; the Equation for the Node is to be subtracted from the Aphelion to the Perihelion of the Earth, and in its Maximum amounts to 9' 24".

XV.

The mean Motion of the Moon requires a second Correction, depending at once on the Distance of the Sun from the Earth, and on the Situation of the Apogee of the Moon with respect to the Sun; this Equation, which is in the direct Ratio of the Sine of double the Angle expressing the Distance of the Sun from the Apogee of the Moon, and in the inverse Ratio of the Cube of the Distance of the Sun from the Earth, is called the Semestrial Equation; it is 3' 45" when the Apogee of the Moon is in Octants with the Sun, and the Earth is in its mean Distance. It is to be added, when the Apogee of the Moon advances from its Quadrature with the Sun to its Syysie; and is to be subtracted when the Apogee passes from the Syysie to the Quadrature.

First semestrial Equation of the mean Motion of the Moon.

XVI.

Second
semestrial
Equation.

The mean Motion of the Moon requires a third Correction, depending on the Situation of the Sun with respect to the Nodes, as also on the Distance of the Sun from the Earth; this Correction or Equation, which *Newton* calls the second Semestrial Equation, is in the direct Ratio of the Sine of double the Distance of the Node from the Sun, and in the inverse Ratio of the Cube of the Distance of the Earth from the Sun; it amounts to $47''$ when the Node is in Octant with the Sun and the Earth in its mean Distance; it is to be added when the Sun recedes in Antecedentia from the nearest Node, and is to be subtracted when the Sun advances in Consequentia.

XVII.

After those three first Equations of the Moon's Motion, follows that which is called her Equation of the Centre; but this Equation cannot be obtained as that of the other Planets, by the Help of one Table, because her Excentricity varies every Instant, and the Motion of her Apogee is very irregular. In order therefore to obtain the Equation of the Centre of the Moon, the Excentricity and the true Place of the Apogee of the Moon is first to be determined, which is effected by the Help of Tables founded on the following Proposition.

Determina-
tion of the
Place of the
Apogee, and
of the Ex-
centricity.

A straight Line being taken to express the mean Excentricity of the Orbit of the Moon, which is 5505 Parts of the 100000 into which the mean Distance of the Moon from the Earth is supposed to be divided; at the Extremity of this straight Line assumed as a Base, an Angle is made equal to double of the annual Argument, or of double the Distance of the Sun from the mean Place of the Moon once corrected, as has been already directed. The Length of the Side of this Angle is afterwards determined by making it equal to $1172\frac{1}{2}$, half the Difference between the least and greatest Excentricity. The Triangle being then completed, the other Angle at the Base, expresses the Equation or Correction to be made to the Place of the Apogee already once corrected; and the other Side of the Triangle which is opposite to the Angle made equal to double of the annual Argument, will express the Excentricity corresponding to the proposed Time. The Equation of the Apogee being added to its Place already corrected, if the annual Argument be less than 90, or between 180 and 270, or being subtracted in every other Case, the true Place of the Apogee will be obtained, which is to be subtracted from the Place of the Moon corrected by the three Equations already mentioned, in order to have the mean Anomaly of the Moon. With this Anomaly, and the Excentricity, the Equation of the Centre by the usual Methods will be obtained, and consequently the Place of the Moon corrected for the fourth Time.

Equation of
the Centre,
or fourth
Correction
of the Place
of the
Moon.

The Equation of the Centre may be obtained without supposing the Excentricity variable, or a Motion in the Apogee, by applying to double

of the Angle at the Moon subtended by the mean Excentricity, or to the mean Equation of the Centre, the Equation $80' \sin (2 \text{ Dif. } \odot - m. \text{ An. } \odot)$ expressing the Variation produc'd by the Change of Excentricity, and Libration of the Apogee.

XVIII.

The Place of the Moon corrected for the fifth Time, is obtained by applying to the Place of the Moon corrected for the fourth Time, the Equation called the Variation which was already found, to be always in the direct Ratio of the Sine of double the Angle expressing the Distance of the Moon from the Sun, and in the inverse Ratio of the Cube of the Distance of the Earth from the Sun; this Equation, which is to be added in the first and third Quadrant (in counting from the Sun) and subtracted in the second and fourth is $35' 10''$ when the Moon is in Octant with the Sun, and the Earth in its mean Distance.

The fifth Equation of the Moon's Motion, is her Variation.

XIX.

The sixth Equation of the Motion of the Moon is proportional to the Sine of the Angle which is obtained by adding the Distance of the Moon from the Sun, to the Distance of the Apogee of the Moon from that of the Sun. Its Maximum is $2' 20''$, and it is positive when this Sum is less than 180 Degrees, and negative if this Sum be greater.

Sixth Equation.

XX.

The seventh and last Equation, which gives the true Place of the Moon in its Orbit, is proportional to the Distance of the Moon from the Sun; it is $2' 20''$ in its Maximum.

Seventh Equation.

XXI.

It is scarce possible to trace the Road which could have conducted *Newton* to all those Equations, except some Corollaries of Prop. 66. where he shews how to estimate the perturbing Forces of the Sun. It is easy to perceive, that of those two Forces, the one which acts in the Direction of the Ray of the Orbit of the Moon, being joined to the Force of the Earth, alters the inverse Proportion of the Square of the Distances, and consequently should change not only the Curvature of the Orbit, but also the Time which the Moon employs in describing it: But how did *Newton* employ those Alterations of the central Force, and what Principles did he make use of to avoid or surmount the extreme Complication and the Difficulties of Computation which occur in this Inquiry is what has not as yet been discovered, at least after a satisfactory Manner.

The Method *Newton* made use of in investigating the foregoing Corrections has not as yet been discovered.

We find, it is true, in the first Book of the *Principia*, a Proposition concerning the Motion of the Apfides in general, by which we learn, that if to a Force which acts inversely as the Square of the Distance, another Force which is inversely as the Cube of the Distance be joined, the Body will describe an Ellipse whose Plane revolves about the Centre

of the Forces. In the Corollaries of this Proposition, *Newton* extends his Conclusion to the Case in which the Force, added to the Force which follows the Law of the Square of the Distance, does not vary in the Triplicate, but in the Ratio of any Power of the Distance.

If therefore the perturbing Force of the Sun depended on the Distance of the Moon from the Earth alone, by the Help of this Proposition, the Motion of the Apsides of the Moon could be determined; but as the Distance of the Moon from the Sun enters into the Expression of this Force, it is only by new Artifices, and perhaps as difficult to be found as the Determination of the entire Orbit of the Moon: the Proposition of *Newton* concerning the Motion of the Apsides in general, can be applied to the Moon. Sensible of which, the first Mathematicians of the present Age, have abandoned in this, as in every other Point that regards the Theory of the Moon, the Road pursued by the Commentators of *Newton*, and have resumed the whole Theory from its very Beginning; they have investigated in a direct Manner, the Paths and Velocities of any three Bodies which attract each other mutually. The Success which has attended their united Efforts shall be explained hereafter.

XXII.

Theory of
the Satellites
of Jupiter
and those of
Saturn.

It is manifest, that the Satellites of Jupiter, considered separately, should be affected by the three Forces which actuate them, in the same Manner as the Moon; but their Number introduces a new Source of Inequalities, not only each of them is attracted by Jupiter and the Sun, but they attract each other mutually; and this mutual Attraction should produce very considerable Variations in their Motions; Variations so much the more difficult to be subjected to exact Computations, as they depend on their different Positions with respect to each other, which their different Distances and Velocities continually alter. However, the Laws of their Motions discovered by *Bradley*, *Wargentin* and *Maraldi*, have enabled the eminent Mathematicians of this Age, to surmount those Difficulties, and to apply the Solution of the Problem of the three Bodies to the Investigation of the Inequalities of the Motions of those Satellites, with almost the same Success as they had already done to those of the Moon.

As to the Satellites of Saturn, Astronomers have not been able to determine the Phenomena of their Motions with any Degree of Accuracy on Account of their great Distance; hence the Theory of those Planets is reduced to shew, that the Forces with which they act on each other, or that with which the Sun acts on them, and disturbs their Motions, are very inconsiderable when compared with the Force with which they tend towards their principal Planet; and that this Attraction is inversely proportional to the Squares of the Distances.

THEORY of the COMETS.

I.

THOUGH the Comets have in all Ages, drawn the Attention of Philosophers, yet it is only since the last Century and even since *Newton*, they can be said to be known. *Seneca* seemed to have foreseen the Discoveries which one Day would be made concerning those Bodies, but the Germ of the true Principles which he had sown, were stifled by the Doctrine of the Peripateticks, who, transmitting from Age to Age, the Errors of their Master, maintained that the Comets were Meteors or transient Fires.

The Peripateticks regarded the Comets as Meteors.

II.

Several Astronomers, but particularly *Ticho*, proved this Opinion to be erroneous, by shewing by their Observations, that those Bodies were situated far above the Moon, they destroyed at the same Time, the solid Heavens, invented by the scholastic Philosophers, and proposed Views concerning the System of the World, which were much more conformable to Reason and Observation. But their Conjectures were yet very far from that Point, to which the Geometry of *Newton* alone could attain.

Ticho proved that they were situated above the Moon.

III.

Descartes, to whom the Sciences are so much indebted, did not succeed better than his Predecessors in his Enquiries concerning the Comets; he neither thought of employing the Observations which were so easy for him to collect, nor Geometry to which it was so natural to have Recourse, and which he had carried to so great a Point of Perfection; he considered them as Planets wandering through the different Vortices, which, composed according to him, the Universe; and did not imagine that their Motions were regulated by any Law.

Descartes regarded them as Planets wandering from Vortex to Vortex.

IV.

Newton, aided by his Theory of the Planets, and by the Observations which taught him that the Comets descended into our planetary System, soon perceived that those Bodies were of the same Nature with the Planets, and subject to the same Laws.

Newton discovered that the Comets revolve about the Sun, and are subjected to the same Laws as the Planets.

Every Body placed in our planetary System, should, according to the Theory of *Newton*, be attracted by the Sun, with a Force reciprocally proportional to the Squares of the Distances, which combined with a Force of Projection, would make it describe a Conic Section about the Sun placed in the Focus. According therefore to this Theory, the Comets should revolve in a Conic Section about the Sun, and describe Areas proportional to the Times.

V.

Calculation and Observation, the faithful Guides of this great Man, enabled him to verify his Conjecture. He solved this fine Astronomico-geometrical Problem. Three Places of a Comet which is supposed to

He determines the Orbit of a Comet from three Observations.

move in a parabolic Orbit, describing round the Sun Areas proportional to the Times, being given, with the Places of the Earth in the Ecliptic corresponding to those Times, to find the Vertex and Parameter of this Parabola, its Nodes, the Inclination of its Plane to that of the Ecliptic, and the Passage of the Comet at the Perihelion, which are the Elements necessary for determining the Position and Dimensions of the Parabola.

This Problem, already of very great Difficulty in a parabolic Orbit, was so extremely complicated in the Ellipse and Hyperbola, that it was necessary to reduce it to this Degree of Simplicity. Besides the Hypothesis of a parabolic Orbit, answered in Practice, the same End as that of the Ellipse, because the Comets during the Time they are visible, describing but a very small Portion of their Orbit, move in very excentric Ellipses, and it is demonstrated that the Portions of such Curves which are near their Foci, may be considered without any sensible Error as parabolic Arcs.

VI.

Rules for determining the Elements of a Comet.

Preliminary Computations.

FIRST HYPOTHESIS.

Angle at the Comet.

Heliocentric Latitude.

The Result of his Solution of this important Problem is as follows. From the observed Distances of the Comet from the fixed Stars, whose right Ascensions and Declinations are known, deduce the right Ascension and Declination, and from thence the Longitude of the Comet reduced to the Ecliptic, and its Latitude, corresponding to each Observation: Compute the Longitude of the Sun at the Time of each Observation, take the Difference (A, A', A'') between the Longitude of the Comet and that of the Sun, corresponding to each Observation, which is the Elongation of the Comet reduced to the Ecliptic. Compute also the Distance (B, B', B'') of the Earth from the Sun at the Time of each Observation.

Those preliminary Calculations being performed, assuming by Conjecture, the Distances (Y and Z) of the Comet from the Sun, reduced to the Ecliptic at the Time of the first and second Observation, determine the true Distances by the Means of the two following Proportions, as the assumed Distance (Y or Z) of the Comet from the Sun in the first or second Observation, is to the Sine of the observed Elongation, (A or A') so is the Distance (B or B') of the Earth from the Sun at the Time of the first or second Observation, to the Sine of the Angle (C or C') contained by the straight Lines drawn from the Earth and the Sun to the Comet. Add this Angle (C or C') to the Elongation (A or A') their Sum will be the Supplement of the Angle of Commutation (D or D'). And then say as the Sine of the Angle of Elongation (A or A') is to the Sine of the Angle of Commutation (D or D'), so is the Tangent of the observed geocentric Latitude of the Comet corresponding to the first or second Observation, to the Tangent of the corresponding heliocentric Latitude of the Comet (E or E').

Each of the curt Distances Y and Z divided by the Cosine of the corresponding heliocentric Latitude E and E' gives the true Distances (V, V') of the Comet from the Sun. Vector Rays.

Find the Angle contained by those Distances thus: Add to (a) or subtract from the Places of the Earth, the corresponding Angles of Commutation (D, D') which will give the two heliocentric Longitudes (L, L') of the Comet, whose Difference (F) is the heliocentric Motion of the Comet in the Plane of the Ecliptic. Then say, *As Radius, is to the Cosine of the Motion (F) of the Comet in the Ecliptic, so is the Cotangent of the greatest of the two heliocentric Latitudes, to the Tangent of an Arc X.* Subtract this Arc X from the Complement of the least heliocentric Latitude, and call the Remainder X'. Then *the Cosine of the first Arc X, will be to the Cosine of the second Arc X', as the Sine of the greatest of the two Latitudes, to the Cosine of the Angle contained by the two vector Rays of the Comet.* Motion of the Comet in its Orbit.

Which being done, determine the Place of the Perihelion by the following Rule: subtract the Logarithm of the least vector Ray from that of the greatest, take half the Remainder, to whose Characteristic, 10 being added, it will be the Tangent of an Angle, from which subtracting 45°, the Logarithm of the Tangent of the Remainder, added to the Log. of the Cotangent of $\frac{1}{4}$ of the Motion of the Comet in its Orbit, will be the Logarithm of the Tangent of an Angle, to which $\frac{1}{4}$ of the Motion of the Comet in its Orbit being added, the Sum will be the Half of the greatest true Anomaly, and their Difference will be Half the least of the two true Anomalies. Double those Quantities to obtain the two true Anomalies, which will be both on the same Side of the Perihelion, when their Difference is the whole Motion of the Comet, but on different Sides of it, when it is their Sum, which is equal to the whole Motion of the Comet. True Anomalies.

Find the Perihelion Distance by adding twice the Logarithm of the Cosine of the greatest of the Halfs of the two true Anomalies, to that of the greatest of the two vector Rays, which will be the Logarithm of the Perihelion Distance required. Perihelion Distance.

Determine the Time which the Comet should employ in describing the Angle contained by the two vector Rays, by the following Rule: *To the constant Logarithm 1,9149328, add the Logarithm of the Tangent of half of each true Anomaly. Add the Triple of this same Logarithm of the Tangent to the constant Logarithm 1,4378116, the Sum of the two Numbers corresponding to those two Sums of Logarithms, will be the exact Number of Days corresponding to each true Anomaly in a Parabola whose perihelion Distance is 1. Take the Logarithm of the Difference or Sum of those two Numbers, according as the two Anomalies are situated on the same Side, or on different Sides of the Perihelion. To this Logarithm add the $\frac{3}{2}$ of the Log. of the perihelion Distance, the Sum will be Log. of the* Interval of Time employed in describing the Angle contained by the two vector Rays.

(a) According to the Position of the Comet with respect to the Signs of the Zodiac.

Time the Comet should employ to describe the Angle contained by the two vector Rays.

Second Sup-
position of
the first
Hypothesis.

If the Time thus found, does not agree with the observed Time, another Value is to be assumed, for the curt Distance (Z) corresponding to the second Observation, retaining the assumed Distance (Y) corresponding to the first, and the heliocentric Longitude and Latitude of the Comet from thence deduced, and all the Operations indicated in the foregoing Articles being repeated, another Expression will be found for the Interval of Time between the two Observations. Which if it approaches nearer the observed Time, the second Value assumed for the Distance (Z) is to be preferred to the first; if not, a third Value is to be assumed for this Distance, and by the Increase or Decrease of the Errors, the Value to be assumed for it, so that the Interval of Time calculated may agree with the observed one, will easily be discovered, and consequently a Parabola will be found, which answers the two first Observations, which may be called *first Hypothesis*.

SECOND
HYPOTHE-
SIS.

This Parabola answering the two first Observations would be the Orbit sought if it answered likewise the third Observation; but as this never happens, another Parabola is to be found which answers the two first Observations, by increasing or diminishing, at will, the curt Distance (Y) preserved constant in the first Hypothesis, and preserving it still constant, but varying the second assumed Distance (Z) until this second Parabola is obtained.

The third Observation calculated in those two Parabolas, will shew which of them approaches nearest the true Orbit sought. To calculate this third Observation in each Hypothesis, the Time of the Passage of the Comet at the Perihelion, the Inclination to the Ecliptic, and the Place of the Nodes of each Parabola is first to be determined.

Passage
at the
Perihelion.

To determine the Time of the Passage of the Comet at the Perihelion, find the Number of Days corresponding to one of the two true Anomalies; for Example, to that which corresponds to the first Observation in the Parabola whose perihelion Distance is 1, as before directed, the Logarithm of this Number of Days added to $\frac{3}{2}$ of the Logarithm of the perihelion Distance, will be the Logarithm of the Interval of Time elapsed between the first Observation and the Passage of the Comet at the Perihelion, which is to be added to or subtracted from the Time of the Observation, according, as it was made before or after the Passage of the Comet at the Perihelion.

Place of the
Node.

To determine the Place of the Node, say, *As the Sine of the second Arc X' is to the Sine of the first Arc X, so is the Tangent of the Motion of the Comet in the Ecliptic, to the Tangent of an Angle (R). Then the Radius, is to the Sine of the least Latitude, as the Tangent of the Angle R, to the Tangent of the Distance from the Node.* By the Means of this Dis-

tance from the Node, and the heliocentric Longitude of the Comet, the heliocentric Longitude of the Node is obtained. With which and the Distance measured on the Orbit of the Comet, the Place of the Perihelion is Determined. To find this Distance say, *As the Sine of Angle R, to Radius, so is this Distance measured on the Ecliptic, to the Distance required.* Inclination.

To determine the Inclination say, *As the Radius is to the Sine of the Angle R, so is the Cosine of the least Latitude, to the Cosine of the Angle of Inclination.*

The Elements of each Parabola being determined, the Place of the Comet seen from the Earth, answering to the third Observation, is computed in each, by the following Rules.

First, Take the Logarithm of the Difference between the Time of the third Observation, and the Time of the Passage of the Comet at the Perihelion; subtract from it $\frac{2}{3}$ of the Logarithm of the perihelion Distance, the Remainder will be the Logarithm of the Difference between the Time of the third Observation and the Time of the Passage of the Comet at the Perihelion of the Parabola, whose perihelion Distance is 1. Secondly, Find the true Anomaly corresponding to this Time, by solving the Equation $t^3 + 3t = \frac{b}{27,4038}$ (b) in which t expresses the Tangent of half the true Anomaly, and b the Time employed in describing it. Thirdly, When the Motion of the Comet is direct, add this true Anomaly to the Place of the Perihelion, if the third Observation was made after the Passage of the Comet at the Perihelion; But subtract it from the Place of the Perihelion if the Observation was made before the Passage at the Perihelion. And when the Motion of the Comet is retrograde, add the true Anomaly to the Place of the Perihelion, if the Observation was made before the Passage at the Perihelion; but subtract it from the Place of the Perihelion, if the Observation was made after the Passage at the Perihelion; by this Means, the true heliocentric Longitude of the Comet in its Orbit is obtained. Fourthly, Take the Difference between this Longitude and that of the ascending Node, which will be the true Argument of the Latitude of the Comet. Fifthly, say, *As the Radius is to the Cosine of the Inclination, so is the Tangent of the Argument of Latitude, to the Tangent of this Argument measured on the Ecliptic;* which added to the true Place of the Node, gives the heliocentric Longitude reduced to the Ecliptic. Sixthly, say, *As the Radius is to the Sine of the Argument of Latitude, so is the Sine of the Inclination of the Orbit of the Comet, to the Sine of its heliocentric Latitude,* which, when the Mo-

Rules for finding the heliocentric Longitude and Latitude of a Comet.

(b) The Equation $t^3 + 3t = \frac{b}{27,4038}$ may be solved thus: Make a Right-angled Triangle, one of whose Sides is expressed by 1. and the other by $\frac{b}{54,8077}$, calculate the Hypotenuse (H), find two mean Proportionals between $H + \frac{b}{54,8077}$ and $H - \frac{b}{54,8077}$ and their Difference will be the Value of t .

Rule for
finding the
curt Di-
stance.

Rules for
finding the
geocentric
Longitude
and Lati-
tude.

tion of the Comet is direct, is North or South, according as the Argument of Latitude is less or greater than six Signs; and when the Motion of the Comet is retrograde, it is North or South according as the Argument of Latitude is greater or less than six Signs. Seventhly, Add the Logarithm of the Cosine of the heliocentric Latitude to the Log. of the perihelion Distance, and subtract from this Sum the Log. of double of the Cosine of half the true Anomaly, the Remainder will be the Logarithm of the curt Distance corresponding to the third Observation. Eighthly, Take the Difference between the Logarithm of the curt Distance, and that of the Distance of the Earth from the Sun, add 10 to the Characteristic of this Difference, and it will be the Logarithm of the Tangent of an Angle; from which subtract 45^d . and to the Logarithm of the Tangent of the Remainder, add the Logarithm of the Tangent of the Complement of half the Angle of Commutation, the Sum will be the Logarithm of the Tangent of an Arc, which add to this Complement, if the curt Distance of the Comet from the Sun exceeds the Distance of the Earth from the Sun, but subtract from this Complement if the Distance of the Comet be less than that of the Earth; in order to obtain the Angle of Elongation, which added to or subtracted from the true Place of the Sun, according as the Comet seen from the Earth, is to the East or to the West of the Sun, will give the geocentric Longitude of the Comet. Ninthly, and lastly say, *As the Sine of the Angle of Commutation, is to the Sine of the Angle of Elongation, so is the Tangent of the heliocentric Latitude of the Comet to the Tangent of its geocentric Latitude.* The Longitude and Latitude thus found ought to agree with the observed ones, if the Parabola obtained was really the Orbit described by the Comet.

VII.

Example. Let it be proposed to find the Elements of the Parabola described by the Comet which was observed in *Europe*; the beginning of *March* 1742, with a very remarkable Tail, coming with extraordinary Rapidity from the southern Hemisphere, and afterwards advancing towards the North Pole, its heliocentric Motion being retrograde, and its Velocity and Splendor decreasing to the 6th of *May*, when it disappeared.

1742. mean Time.	Obs. Long. of the Comet.	Observ. Lat. North of the Comet.	Long. of the Sun. calcula- ted.	Log. of the Dis. of the E. from the Sun	Elong. of the Comet from the Sun.
	h. m. s. s. o. / //	o. / //	s. o. / //		o. / //
4 March at 16 9 50	9 16 0 40	34 45 37	11 14 27 44	9.996910	58 27 4 W.
28 . at 13 39 0	2 18 52 45	63 3 55	0 8 11 28	9.999840	
24 April at 9 39 0	3 1 5 33	50 32 50	1 4 27 16	0.003092	56 38 17 E.

I Supposition, $Y=0,879$, $Z=0,957$ of the mean Distance of the Earth from the Sun $=1$, then Angle $C=105^{\circ} 42' 48''$, $C'=61^{\circ} 31' 0''$, $C+A=164^{\circ} 9' 52''$, and $C'+A'=118^{\circ} 9' 17''$, wherefore Angle $D=15^{\circ} 50' 8''$, and Angle $D'=61^{\circ} 50' 43''$, consequently the heliocentric Latitudes, $E=12^{\circ} 31' 42''$ North and $E'=52^{\circ} 3' 38''$, and the Log. of the vector Rays, $V=9,954455$ $V'=0,192159$.

FIRST HYPOTHESIS.

Heliocentric Latitude and Longitude of the Comet.

The Angle of Commutation $D=15^{\circ} 50' 8''$, being added to $5^{\circ} 14' 27' 44''$, and Angle $D'=61^{\circ} 50' 43''$ subtracted from $7^{\circ} 40' 27' 16''$, the corresponding Longitudes of the Earth, gives the heliocentric Longitudes of the Comet, $L=6^{\circ} 00' 17' 52''$, and $L'=5^{\circ} 20' 36' 33''$; their Difference $F=27^{\circ} 41' 19''$ is the Motion of the Comet in the Ecliptic, the Arc X will be found $=34^{\circ} 37' 11''$, and Arc $X'=42^{\circ} 51' 7''$; consequently the Angle contained by the two vector Rays $=45^{\circ} 22' 8''$.

Angle contained by the two vector Rays.

The Log. of the greatest vector Ray, $0,192159$ less the Log. of the least, $9,954455=0,237704$, and its Half $10,118852$, 10 being added to its Characteristic, is the Tangent of $52^{\circ} 44' 38''$, from which 45° being subtracted, and to the Log. of the Tangent of the Remainder $7^{\circ} 44' 38''$, the Log. of Cotangem of $11^{\circ} 20' 32''$, the $\frac{1}{4}$ of the Motion ($45^{\circ} 22' 8''$), of the Comet in its Orbit being added, the Sum will be the Logarithm of the Tangent of $34^{\circ} 8' 5'' \frac{1}{2}$, whereby the Halfs of the two true Anomalies are found to be $22^{\circ} 47' 33'' \frac{1}{2}$, and $45^{\circ} 28' 37'' \frac{1}{2}$, consequently the least true Anomaly $=45^{\circ} 35' 7''$, and the greatest $=90^{\circ}$, $57' 15''$; and their Difference being equal to the Motion of the Comet in its Orbit, those two Anomalies are on the same Side of the Perihelion. The Log. of the perihelion Distance will be found $=9,883835$.

True Anomalies.

Perihelion Distance.

To determine the Time the Comet employed to describe the Angle contained by the two vector Rays, to the constant Log. $1,9149328$ adding $0,007233$ Log. of the Tangent of $45^{\circ} 28' 37'' \frac{1}{2}$, and to the constant Log. $1,438112$ adding $0,021699$ Triple of the Log. of this same Tangent. I find $83,592$ and $28,808$ for the Numbers corresponding to $1,922166$ and $1,459512$ Sums of those Logarithms, consequently $112,400$ Days is the Time corresponding to the true Anomaly $90^{\circ} 57' 15''$, in a Parabola whose perihelion Distance is 1 . By a like Process, I find the Number of Days $36,579$ corresponding to the true Anomaly $45^{\circ} 35' 7''$, in the same Parabola, I take the Difference $75,821$ of those Times, because the two Anomalies are situated on the same Side of the Perihelion, whose Logarithm $1,879780$ added to $9,825752$ the $\frac{3}{4}$ of the Log. of the perihelion Distance, is the Log. $1,705541$, to which corresponds $50,762$ Days, Time employed by the Comet to describe the Angle contained by the two vector Rays.

Interval of Time between the two Observations calculated.

Comparing this Time with the Interval $50,728 \frac{1}{2}$ between the two Observations, I find it exceeds it by $0,033$, I therefore make a Variation of $0,001$ in the Distance (Z), in order to discover which Way,

and by how much the Elements of the corresponding Parabola will be changed.

Second Supposition of the first Hypothesis.

II Supposition, $Y = 0,879$, $Z = 0,956$, and repeating the same Calculations as in the first Supposition, I find the heliocentric Latitudes $E = 12^\circ 31' 42''$, $E' = 52^\circ 1' 54'' \frac{1}{2}$, the Log. of the vector Rays, $V = 9,954455$, $V' = 0,191424$, the heliocentric Longitudes, $L = 6^\circ 0' 17' 52''$, $L' = 5^\circ 2' 43' 11''$. The Motion of the Comet in the Ecliptic $= 27^\circ 34' 41''$, and the Motion of the Comet in its Orbit $= 45^\circ 18' 13''$ the true Anomalies $45^\circ 32' 3''$, and $90^\circ 50' 16''$, the corresponding Days $36,529$ and $112,056$, the Log. of the perihelion Distance $= 9,883997$; finally the reduced Time employed in describing the Angle contained by the two vector Rays $50,594$ Days. From whence I find that by increasing Z by the Quantity $0,001$, I diminish the Time by $0,168$: And I say, $0,168 : 0,001 :: 0,033 \frac{1}{2} : 0,0002$. I diminish therefore Z by $0,0002$ to obtain a Parabola answering the Conditions required.

III Supposition, $Y = 0,879$, $Z = 0,9568$, and I find the heliocentric Latitudes, $E = 12^\circ 31' 42''$, $E' = 52^\circ 3' 16'' \frac{1}{2}$, the Log. of the vector Rays, $V = 9,954455$, and $V' = 0,192009$; the heliocentric Longitudes, $L = 6^\circ 0' 17' 52''$, and $L' = 5^\circ 2' 37' 53''$; the Motion of the Comet in the Ecliptic, $27^\circ 39' 59''$; and the Motion in its Orbit $45^\circ 21' 22''$; the true Anomalies $45^\circ 34' 28''$, and $90^\circ 55' 50''$; the corresponding Times $36,568 \frac{1}{2}$, and $112,330$ Days: The Log. of the perihelion Distance $9,883870$, and the Time reduced employed in describing the Angle contained by the two vector Rays, $50,728 \frac{1}{2}$ Days, agreeable to Observation.

SECOND HYPOTHESIS.

Having found a Parabola answering the two first Observations, I search for another, answering the same Observations, by making a Variation in the Distance (Y) preserved constant in the first Hypothesis.

First Supposition of the second Hypothesis.

IV Supposition, $Y = 0,878$, $Z = 0,957$, and I find the heliocentric Latitudes, $E = 12^\circ 42' 11''$, $E' = 52^\circ 3' 38''$, the Log. of the vector Rays, $V = 9,954257$, $V' = 0,192159$, the heliocentric Longitudes, $L = 6^\circ 0' 31' 54''$, and $L' = 5^\circ 2' 36' 33''$; the Motion of the Comet in the Ecliptic $= 27^\circ 55' 21''$, the Angle contained by the two vector Rays $= 45^\circ 17' 56''$, the true Anomalies $45^\circ 44' 56''$ and $91^\circ 2' 52''$, the corresponding Times $36,743$ and $112,680$, the Log. of the perihelion Distance $9,883115$, the reduced Time employed in describing the Angle formed by the two vector Rays $50,714$, which differs by $0,014 \frac{1}{2}$ from the observed Interval, consequently by diminishing Y by $0,001$, the Time is diminished by $0,048$. I say, $0,048 : 0,001 :: 0,014 \frac{1}{2} : 0,0003$.

Second Supposition of the second Hypothesis.

V Supposition, $Y = 0,8783$, $Z = 0,957$, I find the heliocentric Latitudes, $E = 12^\circ 39' 2''$, $E' = 52^\circ 3' 38''$ the Log. of the vector Rays, $V = 9,954316$, $V' = 0,192159$, the heliocentric Longitudes, $L = 6^\circ 0' 27' 40''$, $L' = 5^\circ 2' 36' 33''$, the Motion of the Comet in the Ecliptic $27^\circ 51' 7''$ the Angle contained by the two vector Rays $45^\circ 19' 20''$, the true Anomalies $45^\circ 41' 45''$ and $91^\circ 1' 5''$ the corresponding Times $36,689$,

and 112,590, the Log. of the perihelion Distance 9,883344, and the Time reduced employed in describing the Angle contained by the two vector Rays = 50,729 agreeable to Observation.

Having found two Parabolas answering the two first Observations, we are next to examine which approaches nearest the Orbit of the Comet sought, by calculating the third Observation in each; for which Purpose I calculate the Place of the Perihelion, the Time of the Passage at the Perihelion, the Inclination to the Ecliptic, and the Place of the Nodes of each Parabola.

To determine those Elements in the first Parabola, I find the Angle $R=23^{\circ} 40' 15''$, then the Distance of the Comet reduced to the Ecliptic at the first Observation from the ascending Node $5^{\circ} 25' 45''$, which added to the heliocentric Longitude of the Comet, the 4th of March, which is $6^{\circ} 0' 17' 52''$, because its heliocentric Motion is retrograde, gives the Place of the Node, in $6^{\circ} 5' 43' 37''$. The Distance of the Comet from the Node measured on its Orbit, which I find to be $13^{\circ} 38' 14''$, subtracted from the Place of the Node, gives the Place of the Comet in its Orbit, at the Time of the first Observation: and because it had then $45^{\circ} 34' 28''$ true Anomaly, I add them to its Place in its Orbit to obtain the Place of the Perihelion in $7^{\circ} 7' 39' 51''$. I add $\frac{3}{2}$ of the Log. of the perihelion Distance to that of $36,568\frac{1}{2}$ Days, Time corresponding to the least true Anomaly $45^{\circ} 34' 28''$, which gives 24,486 Days, for the Interval of Time elapsed between the first Observation, and the Instant of the Passage of the Comet at the Perihelion, which being subtracted from the 4th of March at $16^h 9' 50''$, or at $0,673\frac{1}{2}$, the Time of the first Observation, fixes the Instant of the Passage at the Perihelion to the 8th of February at 0,188. In fine, I find the Angle of Inclination of the Plane of the Ecliptic, and that of the Comet to be $66^{\circ} 56' 14''$.

Elements of the Comet calculated in the first and second Hypothesis.

The same Elements in the second Parabola are, the ascending Node in $6^{\circ} 5' 59' 6''$, the Place of the Perihelion in $7^{\circ} 7' 53' 42''$, the Inclination, $66^{\circ} 47' 14''$, and the Time of the Passage at the Perihelion, February the 8th, 151 $\frac{1}{2}$.

From those Elements I calculate the geocentric Longitude for the 28th of March, at 0,569 of the Day, in each Parabola. The Interval of Time elapsed between the Passage at the Perihelion in the first Parabola, and the Time of the Observation 28th March 0,569 is 48,381 Days. The Log. of the perihelion Distance, 9,883870, its Triple is, 9,651610, its Half, 9,825805, which being subtracted from 1,684675, Log. of 48,381 gives 1,858870, Log of 72,255 Days, which corresponds to $73^{\circ} 11' 7''$, or $2^{\circ} 13' 11' 7''$ Anomaly, which subtracted from the Place of the Perihelion $7^{\circ} 7' 39' 51''$, because the Comet being retrograde, the given Instant follows, that of the Passage at the Perihelion, which gives the true heliocentric Place of the Comet in its Orbit,

Geocentric
Longitude
of the Co-
met calculat-
ed in the first
and second
Hypothesis.

$4^s 24^{\circ} 28' 44''$, from $4^s 24^{\circ} 28' 44''$, subtracting $6^s 5. 43' 37''$, the Place of the ascending Node, the Argument of Latitude $10^s 18^{\circ} 45' 7''$ is obtained, which measured on the Ecliptic is $11^s 11^{\circ} 2' 47''$; consequently the heliocentric Longitude of the Comet is $5^s 16^{\circ} 46' 24''$, and the heliocentric Latitude, $37^{\circ} 20' 41''$ North because the Argument of Latitude of the Comet, which is retrograde, is greater than six Signs.

The true Place of the Sun the 28 of March, at $13^h 39^m$ is $0^s 8^{\circ} 11' 28''$, and the Log. of its Distance from the Earth, is 9,999841; therefore the true Place of the Earth seen from the Sun, is $6^s 8^{\circ} 11' 28''$, which exceeds $5^s 16^{\circ} 46' 24''$ by $21^{\circ} 25' 4''$, which is the Angle of Commutation. I find the Log. of the curt Distance, corresponding to the third Observation = 9,974915, I subtract 9,974915 from 9,999841, Log. of the Distance of the Sun from the Earth: The Remainder is 0,024926, which by adding 10 to its Characteristic, gives 10,024926, Log. of the Tangent of $46^{\circ} 38' 42'' \frac{2}{3}$, from which subtracting 45, the Log. of Tan. of Remainder, $1^{\circ} 38' 42''$, added to that of the Tangent of $79. 17' 28''$; (Complement of $10^{\circ} 42' 32''$, half of the Angle of Commutation $21^{\circ} 25' 4''$) the Sum is the Log. of the Tangent of $8^{\circ} 37' 39''$, which subtracted from $79. 17' 28''$; because the Distance of the Comet from the Sun, is less than that of the Earth from the Sun, gives $70^{\circ} 39' 49''$, or $2^s 10^{\circ} 39' 49''$, for the Angle of Elongation. By Means of a Figure representing the Ecliptic divided into 12 Signs, in which I place the Sun, the Earth, and the Comet, according to their Longitudes found by the above Calculations, I perceive that the Comet seen from the Earth, is to the East of the Sun. I therefore add the Angle of Elongation to the true Place of the Sun, which gives the true geocentric Longitude of the Comet, in $2^s 18^{\circ} 51' 17''$, which is less than the observed Longitude $2^s 18^{\circ} 52' 45''$ by $1' 28''$; by a like Process I find the geocentric Longitude of the Comet in the second Parabola, the 28 of March, in $2^s 18^{\circ} 45' 14''$, which is less than the observed Longitude, by $7' 31''$; consequently neither of the two Parabolas, is the Orbit of the Comet.

THIRD HY-
POTHESIS.

But because the Variations of the Orbits, are sensibly proportional to those made in the curt Distances, to obtain the two curt Distances which correspond to the Orbit sought. I make those two Proportions; (c) *As 6' 3'' Difference of the two Errors —1' 28'' and —7' 31'', Is to the least of the two 1' 28'' : So is 0,0007 and 0,0002, Corrections made to the two curt Distances Y and Z, to obtain two Parabolas answering the two first Observations, to 0,000235 and 0,000065, Corrections to be made to those Distances Y and Z, to obtain the Orbit required.*

To apply those Corrections, I observe, that since Y, supposed = to 0,879, gives an Error of —1' 28'', and Y supposed = to 0,8783, gives an Error of —7' 31'', by diminishing Y, the Error is increased; from whence I conclude, that 0,000235 is to be added to 0,879, to obtain

(c) I would have said as the Sum of the Errors &c. if the one was by excess and the other by defect.

the true Value of Y, which consequently will be 0,879235; in like Manner, I find that Z should be supposed = 0,956735.

vi Supposition, $Y=0,879235$, and $Z=0,956735$, and I find the heliocentric Latitudes, $E=12^{\circ}29'17''\frac{1}{2}$, $E'=52^{\circ}3'10''\frac{1}{2}$; the Log. of the vector Rays, $V=9,954504$, and $V'=0,191963$; the heliocentric Longitudes, $L=6^{\circ}0'14'37''$, and $L'=5^{\circ}2'38'19''$; the true Anomalies, $45^{\circ}32'0''$ and $90^{\circ}54'4''$; the corresponding Times 36,528 and 112,243 Days; the Log. of the perihelion Distance 9,884049; and the Time employed in describing the Angle contained by the two vector Rays, 50,729; the Place of the Node in $6^{\circ}5'38'29''$; the Place of the Perihelion, $7^{\circ}7'35'13''$, the Inclination of the Orbit, $66^{\circ}59'14''$; and the Time of the Passage at the Perihelion the 8th of February, at $4^h 48'$: In fine, from those Elements, I calculate the geocentric Longitude and Latitude the 28th of March, at $13^h 39'$, which I find, the one in $2^{\circ}18'53'18''$, the other $63^{\circ}3'57''$ North, agreeable to Observation. By these Rules the following Table was calculated, containing the Elements of all the Comets which have been observed with any Degree of Accuracy.

Geocentric Longitude and Latitude of the Comet calculated in the third Hypothesis.

Years.	Place of the ascending Node.	Inclination	Place of the Perihelion.	Perihelion Distance.	Time of the Passage at the Perihelion at Paris.	
	s o ' "	o ' "	s o ' "		d h ' "	
837	6.26.33.00	12.00.00	9.19.3.00	0,5800	March. 11.12.00	retr.
1231	0.13.30.00	6.5.00	4.14.48.00	0,9478	Jan. 30.7.00	dir.
1264	7.28.45.00	20.25.00	9.5.45.00	0,4108	July. 17.6.10	dir.
1299	3.17.8.00	68.57.30	0.3.20.00	0,3179	March. 31.7.38	retr.
1301	0.16.00.00	70.00.00	9.30.00.00	0,4467	Oct. 22.0.00	retr.
1337	2.6.22.00	32.11.00	0.20.00.00	0,6445	June. 1.1.00	retr.
1472	9.11.46.20	5.20.00	1.15.33.30	0,5427	Feb. 28.22.32	retr.
1532	2.20.27.00	32.36.00	3.21.7.00	0,5092	Oct. 19.22.21	dir.
1533	4.7.42.00	46.30.00	5.6.38.00	0,1525	May. 25.10.32	dir.
1556	5.25.42.00	32.6.30	9.8.50.00	0,4639	April. 21.20.12	dir.
1577	0.25.52.00	74.32.45	4.9.22.00	0,1835	Oct. 26.18.54	retr.
1580	0.18.57.20	64.40.00	3.19.5.50	0,5963	Novem. 28.15.9	dir.
1585	1.7.42.30	6.4.00	0.8.51.00	1,1094	Oct. 7.19.29	dir.
1590	5.15.30.40	29.40.40	7.6.54.30	0,5767	Feb. 8.3.54	retr.
1593	5.14.15.00	87.58.00	4.26.19.00	0,8911	July. 18.13.47	dir.
1596	10.12.12.30	55.12.00	7.18.16.00	0,5130	Aug. 10.20.4	retr.
1618	9.23.25.00	21.28.00	0.18.20.00	0,5131	Aug. 17.3.12	dir.
1618	2.16.1.00	37.34.00	00.2.14.00	0,3798	Novem. 8.12.32	dir.
1652	2.28.10.00	79.28.00	00.28.18.40	0,8475	Novem. 12.15.49	dir.
1661	2.22.30.30	32.35.50	3.25.58.40	0,4486	Jan. 26.23.50	dir.
1664	2.21.14.00	21.18.30	4.10.41.25	1,1026	Decem. 4.12.3	retr.
1665	7.18.2.00	76.5.00	2.11.54.30	0,1065	April. 24.5.24	retr.
1672	9.27.30.30	83.22.10	1.16.59.30	0,6975	March. 1.8.46	dir.

Table of the Elements of the Comets.

Years.	Place of the ascending Node.	Inclination	Place of the Perihelion.	Peri- helion Dif- tance.	Time of the Passage at the Perihelion at Paris.	
	s o ' "	o ' "	s o ' "		d h ' "	
1677	7.26.47.10	79. 3.15	4.17.37. 5	0,2806	May. 6. 0.46	retr.
1678	5.11.40.00	3. 4.20	10.27.46.00	1,1238	Aug. 26.14.12	dir.
1680	9. 2. 2.00	60.56.00	8.22.39.30	0,0061	Decem.18.00.15	dir.
1683	5.23.23.00	83.11.00	2.25.29.30	0,5602	July. 13. 2.59	retr.
1684	8.28.15.00	65.48.40	7.28.52.00	0,9601	June. 8.10.25	dir.
1686	11.20.34.40	31.21.40	2.17.00.30	0,3250	Sept. 16.14.42	dir.
1689	10.23.45.20	69.17.00	8.23.44.45	0,0168	Decem. 1.15. 5	retr.
1698	8.27.44.15	11.46.00	9.00.51.15	0,6913	Oct. 18.17. 6	retr.
1699	10.21.45.35	69.20.00	7. 2-31. 6	0,7440	Jan. 13. 8.32	retr.
1702	6. 9.25.15	4.30.00	4.18.41. 3	0,6459	March.13.14.22	dir.
1706	0.13.11.40	55.14.10	2.12.29.10	0,4258	Jan. 30. 4.52	dir.
1707	1.22.46.35	88.36.00	2.19.54.56	0,8597	Decem.11.23.39	dir.
1718	4. 8.43.00	30.20.00	4.01.30.00	1,1027	Jan. 14.23.48	retr.
1723	0.14.16.00	49.59.00	1.12.52.20	0,9876	Sept. 27.16.20	retr.
1729	10.10.32.37	76.58. 4	10.22.40.00	1,4261	June. 25.11. 6	dir.
1737	7.16.22.00	18.20.45	10.25.55.00	0,2229	Jan. 30. 8.30	dir.
1739	6.27.25.14	55.42.44	3.12.38.40	0,6736	June. 17.10. 9	retr.
1742	6. 5.38.29	66.59.14	7. 7.35.13	0,7657	Feb. 8. 4.48	retr.
1743	2.18.21.15	2.19.33	3. 2.41.45	0,8350	Jan. 10.20.35	dir.
1743	0. 5.16.25	45.48.20	8. 6.33.52	0,5205	Sept. 20.21.26	retr.
1744	1.15.46.11	47. 5.18	6.17.10.00	0,2225	March. 1. 8.13	dir.
1747	4.27.18.50	79. 6.20	9. 7. 2.00	1,2198	March. 3. 7.21	retr.
1748	7.22.52.16	85.26.57	7. 5. 0.50	0,8407	April. 28.19.34	retr.
1748	1. 4.39.43	56.59. 3	9. 6. 9.24	0,6553	June. 18. 1.33	dir.
1757	7. 4. 5.50	12.39. 6	4. 2.39.00	0,3391	Oct. 21. 9.42	dir.
1758	7.20.50. 9	68.19.00	8.27.37.45	0,2154	June. 11. 3.27	dir.
1759	4.19.39.24	78.59.22	1.23.24.20	0,7985	Novem.27. 2.28	dir.
1759	2.19.50.45	4.51.32	4.18.24.35	0,9660	Decem.16.21.13	ret.
1762	11.19.00.00	85.20.00	3.14.00.00	1,0090	May. 28.00.00	dir.
1763	11.26.17.00	72.42.00	2.24.43.00	0,4991	Novem. 1.18.39	dir.
1764	4. 0. 7.00	52.47.00	0.15.26.00	0,5567	Feb. 12.13.40	retr.
1766	8. 4.10.50	40.50.27	4.23.15.25	0,5053	Feb. 17. 8.50	retr.
1766	1.17.22.19	8.18.45	6.26. 5.13	0,6368	April. 17.10.26	dir.

Elements of the COMET of HALLEY, in its different Revolutions.

1456	1.18.30.00	17.56.00	10.1.00.00	0,5856	June 8. 22. 10.	retr.
1531	1.19.25.00	17.56.00	10.1.39.00	0,5670	Aug. 24. 21.27.	retr.
1607	1.20.21.00	17. 2.00	10.2.16.00	0,5868	Oct. 26. 3. 59.	retr.
1682	1.20.48.00	17.42.00	10.1.36.00	0,5825	Sept. 14. 21.31.	retr.
1759	1.23.49.00	17.39.00	10.3.16.00	0,5835	March 12. 13.41.	retr.

VIII.

Newton having thus solved the above-mentioned Problem, and applied it to all the Comets observed, deduced from thence a complete Confirmation of his Conjecture. For all the Places of the Comets calculated in the parabolic Orbits, whose Elements were delivered in the foregoing Table, compared with those immediately deduced from Observation, never differed sensibly, which will appear so much the more surprising, when we consider how difficult it is to attain to Precision in Observations of this Nature.

Newton verifies his Calculation by the Observations of a great Number of Comets.

IX.

As to the Duration of the Periods of the Comets, it cannot be deduced from the same Calculation, because as we have already hinted, their Orbits being so excentric that they may be taken for Parabolas without any sensible Error, very great Differences in their Duration would produce scarce any Alteration in the Arc of their Orbit, which they describe during the Time they are visible. However, it no less confirms the Theory of *Newton*, to have shewn, that in this Portion of their Orbit, they observe the Law of *Kepler*, that of the Areas being proportional to the Times, and that the Sun attracts them in the same Manner as all the other celestial Bodies, in the inverse Ratio of the Squares of the Distances.

The Duration of their Period cannot be deduced but from the History of the Appearances of the Comets in the same Circumstances, and at equal Intervals.

X.

Halley, on examining the famous Comet of 1680, having found that the Observations of a Comet recorded in History, agreed with it in very remarkable Circumstances, and that they had appeared at the Distance of 575 Years from each other, conjectured, that it might be but one and the same Comet, performing its Revolution about the Sun in this Period, he therefore supposed the Parabola to be changed into an Ellipse described by the Comet in 575 Years, and having the same Focus and Vertex with the Parabola. Calculating afterwards, the Places of the Comet in this elliptic Orbit, he found them to agree perfectly with those where the Comet was observed; so that the Variation did not exceed the Difference found between the calculated Places of the Planets, and what are immediately deduced from Observation, though the Motions of the Planets have been the Object of the Inquiries of Philosophers for thousands of Years.

Halley employs the Period of the Comet of 1680 to rectify its Orbit.

XI.

Besides the Comet of 1680, *Halley* found three others, which nearly agreed, those of 1531, of 1607, and of 1682, the three Parabolas were situated after the same Manner, the perihelion Distances were equal, and the Intervals of Time 75 or 76 Years; he conjectured that it might be but one and the same Comet, and that the Difference in their Inclinations and Periods, might arise from the Attractions of the su-

Effect of Attraction on the Comets.

perior Planets; for he observed, that the Comet in 1681, passed very near Jupiter; and it is certain, that the Comets receding farther from the Sun than the Planets, their Velocity and Tendency towards the Sun should thereby be considerably lessened in the superior Parts of their Orbits, and consequently should be more susceptible of the Modifications and Impressions of the Attractions, which the Planets in their Approach exert on them; from whence he concluded, that the following Apparition would be retarded, and announced the Return of this Comet for 1759. But these Considerations were too vague to be depended upon. To attain to Certainty in this Point, it was necessary to calculate the Situations of the Comet, and the Forces with which Jupiter and Saturn attract it during several Revolutions, and by the Help of those Forces, expressed in Numbers, to determine the total Effect of the Attractions of those Planets on the Comet. This *Clairaut*, and after him the first Mathematicians in *Europe* have effected, and have demonstrated that this Comet observed in 1531, 1607, and 1682, should have the unequal Periods of $913\frac{1}{2}$ and $898\frac{1}{2}$ Months and that the Period after which it would appear again in this Age, would be 919 Months, which the Event has justified. These Researches shall be explained hereafter.

XII.

Different Opinions concerning the Tails of Comets.

The Tails of Comets which formerly occasioned the Apparition of those Bodies to be regarded as portentous Omens, are now ranked in the Number of those ordinary Phenomena which raise the Attention of Philosophers alone. Some would have it, that the Rays of the Sun passing through the Body of the Comet, which they suppose to be transparent, produced the Appearance of their Tails, in the same Manner as we perceive the Space traversed by the Beams of the Sun passing through the Hole of a darkened Room: others imagined that the Tails were the Light of the Comet refracted in their Passage to the Earth, and producing a long Spectrum, as the Sun does by the Refraction of the Prism. *Newton* having mentioned those two Opinions, and refuted them, exposes a Third which he adopted himself: it consists in regarding the Tail of a Comet as a Vapour which rises continually from the Body of the Comet towards the Parts opposite to the Sun, for the same Reason, that Vapours or Smoke rise in the Atmosphere from the Earth, and even in the Void of the Pneumatic Pump. On Account of the Motion of the Body of the Comet, the Tail is incurved towards the Place through which the Comet passed, much in the same Manner as the Smoke proceeding from a burning Cole put in Motion.

Newton is of Opinion that they are Vapours exhaled from the Body of the Comet.

Confirmation of this Opinion.

What confirms this Opinion is, that the Tails are found greatest when the Comet has just past the Perihelion or least Distance from the Sun, where its Heat is greatest, and the Atmosphere of the Sun is most dense. The Head appears after this, obscured by the thick Vapour that

rises plentifully from it, but about the Centre, a Part more luminous than the rest appears, which is called the Nucleus.

A great Part of the Tails of the Comets should be dilated and diffused over the Solar System by this Rarefaction: some of it by its Gravity may fall towards the Planets, mix with their Atmospheres and repair the Fluids, which are consumed in the Operations of Nature.

Use of the
Tails of Co-
mets, ac-
cording to
Newton.

The Resistance which the Comets meet with in traversing the Atmosphere of the Sun when they descend into the lower Parts of their Orbits, will affect them, and their Motion being retarded, their Gravity will bring them nearer the Sun in every Revolution, until at length they are swallowed up in this immense Globe of Fire.

Some of the
Comets may
fall into the
Sun.

The Comet of 1680, passed at a Distance from the Surface of the Sun which did not exceed the sixth Part of his Diameter, and it is highly probable, that it will approach nearer in the next Revolution, and at length will fall into his Body.

XIII.

Let the Distance of any one of the primary Planets from the Sun $=1$ its periodic Time $=1$ the Force of the Sun exerted on it $=1$, the Distance of any Satellite from its Primary $=t$, and the periodic Time of the same Satellite $=r$; the Force (F) of the Sun on the Planet being to the Force (f) of any Planet on its Satellite as 1 to $\frac{r}{t^2}$ (Cor. 2. Prop. 4.)

Addition to
Article xx
of the Theo-
ry of the
primary Pla-
nets, where
it was shewn
how New-
ton deter-
mined the
Proportions
of the Mat-
ter in the
Sun, Jupi-
ter, Saturn,
and the
Earth.

and the Force (V) of this Planet on its Satellite if it was just as far from it as the Planet is from the Sun, being to its Force (f) exerted on it at its actual Distance from it, as r^2 to 1; by the Composition of Ratios $F \times f$ is to $V \times f$, or the Force (F) of the Sun on the Planet, is to the Force (V) of a Planet on its Satellite just as far from it as the Planet is from the Sun, as 1 to $\frac{r^3}{t^3}$.

Example. The Revolution of Venus round the Sun (5393^h) being to that of the fourth Satellite of Jupiter (400^h) as 1 to 0,0742716, $t=0,0742716$ and the Distance of Venus from the Sun 72333 being to the Distance of Jupiter from the Sun 520096 as 1 to 7,1903; and Radius being to the Sine of $8' 16''$ Elongation of the Satellite, or its Distance from Jupiter viewed from the Sun, as 7,1903 to 0,01729, $r=0,01729$; wherefore

$\frac{r^3}{t^3}=0,000937$ or $\frac{1}{1067}$, consequently the Weight of equal Bodies at equal

Distances from the Centre of the Sun and Jupiter, are to one another as

1 to $\frac{1}{1067}$.

The Revolution of Venus round the Sun 5393^h being to that of the fourth Satellite of Saturn 362^h as 1 to 0,0672475, $t=0,0672475$, and the

Distance of Venus from the Sun 72333, being to the Distance of Saturn from the Sun 954006 as 1 to 13,1890, and Radius being to the Sine of the Elongation of the Satellite or its Distance from Saturn, as 13,1890 to 0,1144, $r=0,1144$, wherefore $\frac{r^3}{r} = 0,000332$ or $\frac{1}{3021}$, consequently the Weights of equal Bodies at equal Distances from the Centres of the Sun and Saturn are to one another as 1 to $\frac{1}{3021}$.

The Revolution of the Earth round the Sun 365^d, 256 being to that of the Moon 27^d, 3215 as 1 to 0,748008, and the Distance of the Earth from the Sun being to that of the Moon from the Earth, as the Sine of the Parallax of the Moon to the Sine of the Parallax of the Sun, wherefore $\frac{r^3}{r} = \frac{1}{169282}$ consequently the Weights of equal Bodies at equal Distances from the Centres of the Sun and Earth are as 1 to $\frac{1}{169282}$.

Addition to Article XXI. of the Theory of the primary Planets, where it was shewn how Newton determined the Proportions of the Densities of the Sun, Jupiter, Saturn and the Earth.

To determine the Weights of Bodies on the Surfaces of the Sun, Jupiter, Saturn, and the Earth, or at the Distance of their Semidiameters from their Centres, those Semidiameters are to be investigated. First the apparent Diameter of the Sun in its mean Distance being found to be 32' 8" and that of Jupiter 37" $\frac{1}{4}$ (as determined from the Passage of those Satellites over its Disk) and the mean Distance of the Sun from Jupiter, being to the mean Distance of the Sun from the Earth as 520096 to 100000, and the true Diameters of Spheres, viewed under small Angles, being in the compound Ratio of those Angles, and the Distances conjointly, the true Diameter of the Sun will be to the true Diameter of Jupiter as 1928" $\times$ 100000 to 37" $\times$ 520096, or as 10000 to 997. Secondly, the apparent Diameter of Saturn being found to be 16", and the mean Distance of Saturn from the Sun being to the mean Distance of the Earth from the Sun as 954006 to 100000, the true Diameter of the Sun will be to the true Diameter of Saturn as 1928" $\times$ 100000 to 16" $\times$ 954006, or as 10000 to 791. Thirdly and lastly, the apparent Semidiameter of the Earth being found to be 10" 30'" as being equal to the Parallax of the Sun, the true Diameter of the Sun will be to the true Diameter of the Earth as 1928 to 21, or as 10000 to 109 nearly.

Now if we suppose a Body placed at a Distance from the Centre of the Sun equal to its Semidiameter, or on its Surface, the Force (F) of the Sun on this Body being to the Force (V) of Jupiter on an equal

Body at the same Distance from its Centre, as 1 to $\frac{1}{1067}$ and the Force

(V) of Jupiter on this Body, being to the Force (f), it would exert on it if it was placed on its Surface, inversely as the Squares of the

Distances, that is, inversely as the Squares of the true Semidiameters of the Sun and Jupiter, or as $\frac{1}{10000}$ to $\frac{1}{997}$; by the Composition of Ratios $F \times V$ is to $V \times f$ or the Weight (F) of a Body on the Surface of the Sun is to the Weight (f) of an equal Body on the Surface of Jupiter, as $\frac{1}{10000}$ to $\frac{1}{1067} \times \frac{1}{997}$ or as 10000 to 943, and consequently that the Density of the Sun is to the Density of Jupiter (the Densities being in the direct Ratio of the Weights and inversely as the Diameters) as 100 to 94 $\frac{1}{2}$. In the same Manner it will be found secondly, that the Weight of a Body on the Surface of the Sun is to the Weight of an equal Body on the Surface of Saturn as $\frac{1}{10000}$ to $\frac{1}{3021} \times \frac{1}{791}$ or as 10000 to 529, consequently that the Density of the Sun is to the Density of Saturn as 100 to 67. Thirdly and lastly, That the Weight of a Body on the Surface of the Sun, is to the Weight of an equal Body on the Surface of the Earth as $\frac{1}{10000}$ to $\frac{1}{169282} \times \frac{1}{109}$ or as 10000 to 435, consequently that the Density of the Sun is to the Density of the Earth as 100 to 400. Which Determination on examining the Process of the Computation will appear not to depend on the Parallax of the Sun but on the Parallax of the Moon, and is therefore truly defined.

XIV.

Such is the Plan of the immortal Discoveries of the most eminent Philosophers, and of Sir Isaac Newton in particular, whose Efforts and Sagacity we cannot sufficiently admire, which shine through the Whole of those Strokes of Genius, which characterise an Inventor, and a Mind fertile in Resources, that no Man possessed in so eminent a Degree. Aided by the Succours that the analitic Art furnishes in greater Abundance, it is not surprizing that some more Steps have been made in a vast and difficult Career that he has opened to us, that all the Irregularities that have been perceived in the Heavens, have been explained and demonstrated; that a great Number of others, which on Account of their Smallness and Complication had escaped the most exact Observers, have been foreseen and unfolded; that it has been proved, that the Return of the Comet which was observed in 1531, 1607, and 1682, ought to have had the unequal Periods of 913 $\frac{1}{2}$ and 898 $\frac{1}{2}$ Months, which was found to be so, and that the Period after which it would appear again in this Age, would be 919 Months; which the Event has justified. That the Course and Laws of the Winds, the ebbing and flowing of the Sea, as far as they depend on the attractive Action of the Sun and

CONCLUSION.

Recapitulation of the Improvements the Principia have received to this Day.

Moon, have been accurately determined. That the Nature and Laws of Magnetism, the Theory of Light and Laws of Vision, the Theory of Sound and Laws of Harmony, &c. have been accurately investigated.

New Edition
of the *Prin-*
cipia, with
the Improve-
ments they
have receiv-
ed to this
Day.

Such is the Plan of the MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY, which the Nobility and Gentry of the Kingdom of Ireland pursuant to their Resolution of the 4th of February 1768, have ordered to be published for the Use of the Mathematical School established under their immediate Inspection. Previous to which, in the Month of November, 1764, a Copy of the Chapter of the Theory of the primary Planets, as a Specimen of the whole Plan, was delivered to Dr. *Hugh Hamilton*, to have his Opinion of the same, which he returned in six Months after, with this Answer, That the above Piece was printing by Subscription at Cambridge, under the Title of *Excerpta quædam ex Newtoni Principiis*, with References to the Doctor's Treatise on Conic Sections.

PLAN of the Art of making Experiments and that of employing them.

Experimenta rerum naturalium ita sunt exhibenda, ut in his nobiles adolescentes studio suavissimo atque utilissimo humanæ mentis historiam, præclaræque artium inventa, quibus naturam et ornare et adjuvare, ediscere possunt.

TO illustrate Sir *Isaac Newton's Principia*, and thereby to enable Youth to make a Progress in the Knowledge of the Works of Nature, to improve to Advantage its Powers and Forces, and render them subservient to the Purposes of Life, they are initiated in the Art of making Experiments and Observations. For these Purposes the School is furnished with a complete Collection of the best executed Machines adapted for experimental Inquiries; they are instructed in the Management and Use of these Machines; they are taught how to ascertain the Difference between the Result from Theory and from Experiment, and how to employ this Difference, for determining the Alterations arising from external Causes, in order to shew them how Experiment not only serves to confirm Theory, but conducts to new Truths, to which we cannot attain by Theory alone. As to the Phenomena for the Discovery of whose Causes Theory affords little or no Assistance, for Instance, those of Chemistry, Electricity, &c. they are taught how to examine and consider them in different Lights, arrange them in Classes, and explain the one by the other as far as the Nature of the Subject will allow.

Course of
Experi-
ments for
illustrating
the *Princi-*
pia.

FIRST CLASS.

Machines for making Experiments on Motion, Gravity, and the Equilibrium of solid Bodies.

I.

A Machine for demonstrating the Theory of central Forces.

This Machine is so contrived, that by its Assistance may be solved experimentally, the Problems which appear the least susceptible of such a Solution; the Velocities and Masses may be varied at will, Friction is so diminished

as to cause no sensible Error, the Times are marked by Sounds, and the Spaces described by an Index.

A Glass Globe mounted on an Axis so that it may be turned round with any Degree of Velocity.

This Machine shews the Effects of central Forces on Fluids of different Specific Gravities, and on Solids, which circulate in the same Medium.

A terrestrial Globe which turns on its Axis with any given Velocity.

The Surface of this Globe is flexible, its Concavity is filled with a Matter somewhat fluid, and is so contrived, that its two Poles are capable of moving towards each other, so that by turning the Globe, the centrifugal Force raises the Equator of the Globe, and shews the Figure which modern Discoveries attribute to the Earth.

A graduated Rule adapted to a Glass Tube within which a small Cylinder is put in Motion. Second, A Plane upon which two Bodies describe in the same Time unequal Spaces. Third, A Globe of Cork of of three Inches Diameter, with a Ball of Lead of the same Weight.

By the Assistance of the three last Articles are explained the Properties of Motion, viz. Direction, Velocity, Quantity of Motion, &c.

A small Cystern divided into two equal Parts by a Partition upon which is mounted a double Pendulum, shewing in what Ratio different Mediums exert their Resistance.

A Machine with which is demonstrated the Doctrine of the Collision of Bodies.

The Parts of this Machine are made with the utmost Care, the Masses are in given Proportions, and the Effects remain visible after the Experiment by the Means of an Index.

A CHRONOMETER or Instrument for measuring small Intervals of Time.

The Pendulum which constitutes the principal Part of this Instrument may be lengthened or shortened according to a Scale accurately divided for the vibrating Minutes, Seconds, Thirds, and the different Times of Musick.

A small Billiard-table with its Appendages.

The Appendages of this Machine are Hammers suspended in such a Manner, that the Quantity of Motion may be regulated by the Velocity, or by the Mass, and so as to exhibit the Motion of a Body impelled by Forces acting in different Directions, and known Proportions.

A Machine for shewing the Motion of a Body which receives at the same Time an Impulse in a perpendicular and horizontal Direction.

Another Machine for shewing the Motion produced by two Forces acting on a Body in Directions forming an Angle, but which constantly remain in the same Ratio.

A Machine for shewing the Acceleration of Bodies which fall freely. Secondly, a Kind of Balance for making the same Kind of Experiments.

These two last Machines not only shew that the Motion of Bodies is accelerated in their Descent, but also renders sensible the Law of this Acceleration.

Experiments for illustrating the Theory of central Forces.

Experiments for illustrating the Doctrine of the Collision of Bodies.

Experiments for illustrating the Composition and Resolution of Forces.

Experiments for illustrating the Doctrine of the Motion of heavy Bodies.

A Machine for shewing the Line a Body describes when abandoned to its Weight after having received an Impulsion in an horizontal Direction.

A Machine for shewing the Motion of a Body abandoned to its Weight after having received an Impulsion upwards, but oblique to the Horizon.

As the Curve which results from this Motion depends on the Obliquity of the Direction, the Machine is constructed so that the Degree of Obliquity may be varied at will.

A Machine which serves to compare the Velocity of a Body which in its Discent describes a Cycloid with that of another tending to the same Point along an inclined Plane.

A Machine for shewing in what Ratio several Forces act on the same Body.

A Machine for explaining the Laws of Elasticity.

Experiments for illustrating the Nature and Properties of the Center of Gravity

Two Cones joined together by their Bases, and which ascend an inclined Plane. 2d. A Cylinder which ascends an inclined Plane.

Those two Machines serve for proving experimentally, that a Body cannot remain at rest when its Centre of Gravity is not supported. The Plane on which the double Cone moves is formed of two Rulers inclined to each other and to the Horizon, and this double Inclination may be varied at pleasure as the Experiment may require.

A small Carriage with its Appendages.

This Model with the Parts which accompany it, shews the respective Advantages of broad or narrow Wheels, of large or small ones, and what renders Carriages more or less liable to be overturned.

A Machine for shewing the Properties of the inclined Plane.

Experiments for illustrating the Theory of simple Machines, the inclined Plane, the Wedge, the Screw, the Lever.

This Machine is so constructed that the Inclination of the Plane may be varied from the horizontal Line to the vertical, and that the Power may act in any desired Direction.

A Machine for shewing the Nature and Properties of the Wedge.

What forms the Wedge in this Machine are two Planes inclined to each other, the Degree of Inclination can be varied at pleasure, as also the Power, the Weight and the Base of the Wedge.

A Screw which can be taken to Pieces to shew the Principles of its Construction.

A Machine for shewing the Nature of the three Species of Levers.

A large Beam accurately divided, mounted on a Foot, for shewing the Properties of the Lever.

The Power, the Weight, and the Prop or Fulcrum are moveable, and may be easily placed so as to be to each other in any given Proportions.

Two Figures in Equilibrio on a Pivot, for shewing the Art of Chord or Wire-dancing.

A large Brass Pulley, in which the Circumference and the diametral

Lines have only been left, in order to shew that the Pully may be considered as an Assemblage of Levers of the first Species.

At the Back of the Supporter, there is fixed a Lever of the same Species with those which constitute the Diameters of the Pully, to serve as a Proof by the Application of the same Power and Weight.

A Pully whose Axis is moveable in a perpendicular Direction, and which serves to shew the Action of the Power, and of the Weight on this Axis, in different Cases.

A Block with two Pullies. 2d. A Block with four Pullies; another Block whose Pullies are fixed on the same Axis.

All those combined Pullies are of Metal or Ivory, turned on their Axis with great Precision, and all possible Care has been taken to diminish the Friction.

An Assemblage of several Toothed Wheels and Pinions, for shewing that both the one and the other like the Pullies, may be considered as Levers.

At the Back of the Supporter, are fixed an Assemblage of Levers which correspond in the same Manner as the Diameters of the Wheels on the other Side, to serve as a Proof by the Application of the same Power and Weight.

A Model of Archimedes's Screw, whose Effects are rendered sensible by the Motion of several small Balls of Ivory, which are raised successively.

A Model of an Endless Screw, which drives an Axis. 2d. A Model of a Press. 3d. A Model of a Capstan. 4th. A Model of a Crane. 5th. A Model of an Engine for driving Piles.

A Jack, of a particular Construction, for raising great Weights. A common Balance, for shewing the Defects to which this Machine is liable, and how they may be remedied.

A large Roman Balance, contrived for making the Experiments of Sanctorius.

This Machine is so constructed, that a Person may weigh himself without the Assistance of another.

A Model of a Screen for winnowing Corn by the Means of an artificial Wind, and several Screens of a particular Construction.

A Model of a Saw for cutting at the same Time several Flints, Agates, Cornelians, &c. and at one Stroke, to form Tables for Snuff-Boxes, and other Works. An horizontal Turning Leath, adapted for grinding Glasses for Telescopes, Microscopes, &c.

A Model of a common Wind-Mill. 2d. A Model of a Polish Wind-Mill. 3d. A Model of a Water-Mill for extracting Oil. 4th. A Model of a Water-Mill for winnowing and grinding Corn, drawing up the Sacks, and boulding the Flour.

Models for shewing the Application of simple Machines in the more compounded ones. The Capstan, the Crane, the Pile-driver, Wind-mills, Water-mills, &c.

As all those Models are intended to shew the Application of simple Machines in the more compounded ones, Care has been taken to leave exposed or to cover with Glass, the Pieces destined for Motion, and the Proportion of the Parts have been carefully observed.

A Machine for shewing the Effects of Friction, in Machines more correct, and of a more extensive Use than any hitherto invented.

SECOND CLASS.

Machines for making Experiments on the Motion, Gravity and Equilibrium of Fluids.

II.

A large Cistern lined with Lead, with a Cock to it, which serves for making several hydrostatical Experiments.

Two large cylindrical Glasses mounted on a common Base, between which is erected a Stem which carries a Beam of a Balance.

This Machine is very commodious in several Operations which regard the Weight or Equilibrium of Fluids.

Experiments for shewing the Properties of Fluids.

A small Bottle with a Glass Stopper, and heavier in this State than a Quantity of Water of the same Bulk.

A Glass Tube, a Part of which rises perpendicularly, and the other forms several Flexions for shewing the Height of Fluids in Vessels which have a Communication with each other.

A small Barrel with a Cock to it, and a bent Tube which serves for demonstrating the same Principle, with some curious Applications.

A Glass Vessel, partly filled with a coloured Fluid, to which is adjusted a large Glass Tube, and a small sucking Pump, which serves to shew that Columns of the same Fluid are of the same specific Gravity.

A long Tube of Glass with a Cock at the lower Extremity, and mounted on a graduated Ruler, to which is adjusted a Pendulum which beats Seconds.

This Machine serves to shew how the Parts of a Fluid press each other, and in what Ratio the Effluxes thereof are performed.

A Bladder filled with a coloured Fluid, to which is fitted a Glass Tube, which serves to shew that Fluids exert their Pressure in all Directions.

A Vessel whose Bottom bursts by the Pressure of a small Quantity of a Fluid.

Experiments for shewing the Pressure of Fluids upon the Bottoms and Sides of the Vessels that contain them.

A large Machine, which serves to shew the Pressure of Fluids on the Bottoms and Sides of Vessels which contain them.

This Machine consists of several fine Vessels of Glass, which are adjusted successively on a common Base, the Piston which serves as a Bottom, is sufficiently moveable as not to cause any sensible Error by Friction, the Columns of the Fluid remain always at the same Height, and the Power acts uniformly.

An Hydrometer with six small cylindrical Vases, which are filled with different Fluids.

Two small Cruets, mounted each on a Pedestal, which serve for the Experiments by which Water is apparently changed into Wine, and Wine into Water.

Two Vases of different Forms, which serve to make a heavier Fluid assume the Place of a lighter in the same Vessel, without mixing.

A Vessel perfectly cylindrical of Copper, with a Solid of the same Metal, and of the same Figure, which fills it exactly, for shewing how much a Body immersed in a Fluid, loses of its Weight.

A Vase of Glass suspended to the Arm of a Balance, for making Experiments of the same Kind.

Two Balls, one of Ivory, and the other of Lead of the same Weight, prepared to be suspended to the Arm of the Balance just mentioned, for shewing, that what a Body loses of its Weight when immersed in a Fluid, is proportional to its Bulk.

A cylindrical Vase of Glass filled with Water, with several human Figures of Enamel, of which some are lighter and the others heavier than a like Portion of the Fluid in which they are immersed.

A Machine for shewing that the relative Gravity of a Body immersed in a Fluid, is changed when the Fluid is condensed or rarified.

This Machine renders palpable by a very quick Operation, the Effects which the different Temperatures of the Air produce in the different Kinds of Thermometers hitherto invented.

A human Figure of Enamel, which is made to move in Water by Compression. 2d. Two large Tubes of Glass mounted in a Frame, in which two Figures move by a Compression which is not perceived by the Spectator.

A Model of the Diving Bell, and the Appurtenances of a Diver.

An hydrostatic Balance, with all its Appendages.

A Model of a curious Machine for raising up Vessels that are sunk.

A Water Level. A simple Syphon. 2d. A Fountain Syphon mounted on a Pedestal. 3d. A Syphon with its Vase to be placed in Vacuum. 4th. A double Syphon. 5th. A Syphon whose Branches are moveable by the Means of a Joint. 6th. Tantalus's Cup. 7th. A large Syphon whose Branches are moveable, necessary in Experiments made with the Air-Pump.

All those different Species of Syphons are of Glass, that the Motion of the Fluids may be more easily perceived.

A Model of a Sucking-Pump. 2d. A Model of a Lifting-Pump. 3d. A Model of a Sucking and Lifting Pump. 4th. A Model of the Engine under London-Bridge, that raises Water by Forcing-Pumps.

Experiments for illustrating the Action of Fluids upon Bodies immersed in them.

Experiments for illustrating the Effect produced by the Pressure, &c. of Fluids.

5th. A Model of a new Pump whose Sucker has no Friction, an intermitting Fountain, *Hiero's Fountain*.

All those Models of Pumps and Fountains are of Glass, in all those Parts in which the Action passes, and the Motion of the Valves and Suckers, are easily perceived.

Experiments on Ice and artificial Congelations.

Several Cisterns and other Vases for making Experiments on Ice, and artificial Congelations. 2d. An Assortment of different Salts and Fluids for congealing Water with a Vase, in which without Ice, a Cold capable of freezing, may be produced.

THIRD CLASS.

Machines for making Experiments on the Air.

III.

A double barrelled Air-Pump mounted on a very solid Base.

The Pistons are put in Motion by a Handle. Instead of Valves Stop-Cocks are made Use of, which are opened and shut, and that by the same Motion which raises and lowers the Pistons; there is affixed to the Pump a whirling Machine, for the Experiments where it is necessary.

A single barrelled Air-Pump, mounted on a solid Base.

Experiments for shewing the Nature and Properties of the Air.

In the Construction of the whirling Machine, which serves as an Appendage to this Pump, Care has been taken, that the Axis of the great Wheel may move along its Frame, in order to straiten the Chord, and that the horizontal Pulley, which receives the whirling Axis, may be raised or lowered as the Height of the Receiver may require.

A large Receiver fitted for making Experiments on Bodies put in Motion in Vacuo. 2d. A Receiver of less Size fitted for the same Uses. 3d. A long and narrow Receiver fitted also for the same Uses.

Those Vases are fitted for the above Uses, by the Means of a Brass Box, filled with a Sort of prepared Leather, through which passes a Steel Axle-Tree, which communicates the Motion within the Receiver without letting the Air enter.

Experiments on Fire in Vacuo. Electrical Experiments in Vacuo.

An Apparatus necessary for making the Experiments on Fire in Vacuo.

An Apparatus for making electrical Experiments in Vacuo.

A large Receiver fitted for operating in Vacuo; a tall narrow Receiver fitted for the same Uses.

Those Vases are fitted for the above Uses, by Means of a Brass Box prepared as above, through which passes a Shaft of Metal, whose Extremity is fitted for receiving different Sorts of Pincers, and other Instruments.

Four Cruets mounted on one common Pedestal, and suspended so as to have their Contents poured out in Vacuo, which serve for mixing different Fluids therein. 2d. Two Cruets suspended in the same Manner.

This Machine is so contrived, that the Cruets may be raised or lowered, and brought nearer to each other, as may be required.

An Apparatus for essaying Inflammations in Vacuo.

A Receiver composed of several Pieces, very tall, at the upper End of which, a Machine is adapted with which may be repeated six Times, the Experiment of the descent of Bodies in Vacuo, when the Air is but once exhausted.

Experiments for shewing the Descent of Bodies in Vacuo.

A large Vase of Glass adjusted to a Receiver, and disposed for depriving Fishes in Water of Air.

A large Globe of Glass, joined to a Receiver by a Neck, to which is adapted a Stop-Cock, for making Experiments on the Vapours in the Air. 2d. Two Vases of Comparison having for a common Base a small Receiver, for similar Uses.

Experiments for shewing that the Air is filled with Vapours.

A Receiver, to which are adapted two Barometers, one of Mercury and the other of coloured Water.

Two large Receivers with a hollow Button at Top. 2d. Two Receivers of a middle Size. 3d. Four small Receivers. 4th. A Machine very commodious for sealing up Vases hermetically, &c.

Six small truncated Barometers of different Lengths, mounted each on a small Base, to which a Scale is adapted. 2d. Six small gage Tubes, for compressed and rarified Air.

Experiments for ascertaining the Degree of Compression and Rarefaction of the Air.

These Gage Instruments are more commodious for Use than any hitherto made, and it is well known of what Importance it is in making Experiments, to be assured of the Degree of Rarefaction, or of the Condensation of the Air.

A Receiver for making Experiments on burnt, or infected Air.

Experiments on burnt and infected Air.

Two large Copper Hemispheres, to one of which is adapted a Ring, and to the other a Stop-Cock.

A Fountain Bottle, and a Vase to place it in, with several spouting Pipes, which are successively adjusted on it.

A small Receiver for applying the Hand to the Air-Pump.

Experiments for shewing the Spring of the Air and its Applications.

A Receiver of very thick Glass for bursting a Bladder.

A Supporter, and a small Vase of Glass to place Eggs under a Receiver of the Air-Pump.

A small Receiver with a sharp edged Brim, to cut an Apple, or any like Body.

A large Glass Tube, at the Top of which is adjusted, a Wooden Vase for proving the Porosity of Vegetables.

A Tube of Crystal whose Bottom is of Leather, covered with Mercury, to shew that animal Substances are porous.

A Bladder suspended in a Receiver. 2d. A Bladder in a cylindrical Vase of Metal charged with a great Weight.

A Machine for compressing Air.

This Machine is of sufficient Strength to remove all Apprehensions of Danger, and is sufficiently large to place all such Bodies with which Experiments are made by the Means of an Air-Pump; it is constructed in such a Manner, that what passes within, may easily be perceived, and the Air is compressed with great Ease by Means of a Lever which puts the Piston of the Pump in Action.

Experiments for shewing the Pressure of the Air.

A small forcing Pump with Valves for compressing Air in certain Experiments.

A Glass Vase prepared for compressing Air on Liquors.

A Fountain of Compression of Copper.

A Tube which contains Water without Air.

A Kind of round Bellows, furnished with a long Tube for shewing the powerful Efforts of Fluids.

Two Hemispheres of Copper for the Machine of Compression.

An Air-Gun.

This Air-Gun is furnished with a condensing Syringe in the Butt, and is charged with Balls by a Receiver which contains 10. They may easily be taken out without letting the Air escape. At each Shot only one goes off and one Charge of Air is sufficient for them all, and the last pierces an Oak Plank half an Inch thick.

A Model of a Bellows, in which the Air is excited by the circular Motion of several Vans. 2d. A Model of a Bellows whose Effect depends on a Fall of Water.

Experiments for illustrating the Theory of Sounds.

A Glass Bell suspended, with a small Hammer put in Motion by a Screw, adapted for Experiments on Sound.

A small Bell mounted on Clock-Work, with a Tricker, for Experiments on Sound in Vacuo.

• An accoustique Tube of a parabolic Figure. 2d. A Speaking-Trumpet.

A graduated Monochord. 2d. Glasses of several Tones.

A large Column which imitates the Noise of Rain and Hail.

Glass Tears, with some Instruments necessary for the Experiments to which they are applied.

Capillary Tubes of different Sizes and Lengths.

FOURTH CLASS.

Machines for making Experiments on Fire.

IV.

Experiments for shewing the Operations of Chimiſtry.

A Lamp Furnace for shewing the ordinary Operations of Chimiſtry. *With this Machine Distillations are performed in Balneo Mariæ, in the Sand Bath, with the Cucurbit and with the Retort.*

An Assortment of Vessels of Glass for the Lamp Furnace.

A Table of an Enameller with a Bellows and Lamp; Pieces of Enamels and Tools, requisite for this Art.

Inclined Planes which turn round by the Action of two lighted Candles. Second, a Lantern which turns round.

Several Fluids which ferment with Heat and Ebullition. Second, several Fluids which ferment without Heat. Third, several Fluids which fermenting, burst into Flame, and the Vases necessary for those Experiments.

Experiments on Fermentation.

Fulminatory Substances and Instruments, necessary for performing Experiments on them.

Burning Powders. 2d. Powders for accelerating the Fusion of Metals. Third, several Dissolvents of Metals.

Experiments on the Dissolution of Metals.

The Urinous Phosphorus. 2d. Urinous Phosphorus dissolved in different Kinds of Oils. 3d. Luminous Calcinations.

A Glass Vessel, by which may be exhibited a Shower of Fire, produced by the Fall of Mercury in Vacuo.

Papin's Digester.

A large Copper Æolipile with a long Neck, to which is adapted an accurate Stop-Cock, which serves for condensing Air in Vases, when there is Reason to apprehend that the Moisture of other Air introduced may hurt the Experiment. 2d. A smaller Æolipile for ordinary Uses. 3d. An Æolipile for forming a Fountain of Fire, with the Spirit of Wine. 4th. An Æolipile mounted on a Carriage which recoils during the Experiment.

Experiments for explaining the Effects of Fire-arms, Fire-works, &c.

A small recoiling Cannon for explaining the Nature of Rockets.

FIFTH CLASS.

Machines for making Experiments on Light and Colours.

v.

A large Case, the Sides of which are of Glass adapted for the Experiments on Refraction.

In the two lesser Sides of this Case are adjusted, concave and convex Surfaces. It can be raised, lowered, or turned round on its Pedestal, and is furnished with a Lamp which in case of Necessity, supplies the Place of the Rays of the Sun.

Experiments for illustrating the Theory of Refraction of Light.

A triangular Box of Glass, whose Sides form with each other different Angles, mounted on a graduated Circle, with an Index for determining the Angles of Refraction.

Two Prisms of solid Crystal. 2d. A large solid Prism mounted on a Pedestal, so that it can be raised, lowered, inclined, and turned round its Axis. 3d. A Prism similar to the former, mounted vertically on a Pedestal, so that it can be raised, lowered, and turned round its Axis. 4th. A Right-angled triangular Prism. 5th. A large triangular Prism of Rock Crystal mounted on a graduated Circle, with an Index.

Experiments for illustrating the Theory of Colours.

A large folding Table with its Appendages, adapted for making Experiments on Light.

Six Frames covered with waxed Cloth, for rendering a Room per-

fectly dark, with a Tablet and Circles of Metal for opening Passages to the Rays of the Sun, of different Magnitudes and Figures.

A plain Mirror of Metal mounted on a Stem which can be lengthened and shorted, and on which the Mirror can be raised, lowered, inclined, and turned round, for introducing the Rays of the Sun into a darkened Room. 2d. A Mirror of Glass mounted as the former, and for the same Uses.

Four Glasses of different Colours, mounted in Tortoise Shell. 2d. Four Mirrors of Glass mounted in the same Manner.

A large Glass Lens of six Feet Focus Length, mounted on a Pedestal whose Stem can be lengthed or shortened. 2d. A Glass Lens of a shorter Focus mounted, so that it can be raised, lowered or inclined.

A Frame, in which is adjusted a Glass Lens between two vertical Planes, for shewing that some Rays of Light unite in a shorter Focus than others.

This Machine is so contrived, that the Experiment may be made upon any Ray separately, and may be adjusted to the Motion of the Sun.

A large concave Glass mounted. 2d. A large multilateral Glass mounted. 3d. Two Polyhedrons of very pure Glass. 4th Two concave Mirrors of Glass.

A very large convex Glass, composed of two curved Glasses mounted on a Pedestal, for making Experiments on the Refraction of Light through different Fluids.

A large vertical Plane for receiving the Image of the Sun when it has passed through the Prism. 2d. A smaller Plane, to which is adapted, an excentric Circle for making the Rays of Light of different Colours, pass successively.

A Cloth six Feet square spread on a Frame, which can be raised and lowered for receiving the Images produced by the Magic Lanthorn, and the Camera Obscura.

An artificial Eye with Spectacles for different Ages, for shewing how the Defects of Sight are remedied by the Help of Glasses.

A Cornea of an Insect adapted to a small Microscope for shewing that the Eyes of those Animals, for the most Part, are Multipliers.

An Assortment of Fluids for Experiments on the Colours which result from their Mixture.

Invisible Ink, the Writing of which appears and disappears several Times, when heated at the Fire. 2d. Sympathetic Ink.

A large Mirror of Metal, concave on one Side, and convex on the other, mounted on a Pedestal. Two convex Mirrors of Paste-board silvered over, with their Appendages, for some catoptrical Experiments.

A cylindrical Mirror of Metal, with thirty Anamorphoses. 2d. A conic Mirror of Metal, with six Anamorphoses. A pyramidal Mirror of Metal, with four Anamorphoses.

Experiments for shewing the different Refrangibility of the Rays of Light.

Experiments for illustrating the Laws of Vision.

Experiments for illustrating the Doctrine of the Reflection of Light.

To all those Mirrors is adapted a Machine for regulating the Point of View.

A Picture, commonly called the magical one, on account of the Effect of the multilateral Glass, for dioptrical Anamorphoses.

A Magic Lantern, enlightened by the Rays of the Sun. 2d. A Magic Lantern enlightened by a Lamp and a concave Mirror.

Although this Machine is become very common, it is not however despicable; the most eminent Philosophers of the present Age, have not thought it unworthy of a Place among their Machines, and have given ample Descriptions of it. The above mentioned one, presents a Sight so much the more agreeable, as the Objects appear animated, and are perfectly well designed.

A Camera Obscura of a new Construction, with a Stool and Table, and other Conveniencies for designing.

A kind of Telescope for observing Objects which present themselves at Right-angles to the Tube. 2d. A Newtonian Telescope, with which the Objects are viewed sideways, or in a Line which forms an acute Angle with the incident Rays of those Objects. 3d. A catoptrical Telescope two Feet long, which magnifies the Objects 300 Times. 4th. An Achromatic Telescope 12 Feet long.

A portable Microscope, with the Instruments necessary for observing. 2d. A larger Microscope, with a greater Number of Instruments and Lenses for increasing or lessening its magnifying Power. 3d. A Microscope which has six different Degrees of magnifying Power, with Mirrors of Reflection and Lenses for increasing the Light; it is mounted so that it can be moved in all Directions with great Ease, and has a Machine of a new Contrivance for fixing it at its true Point. The Drawer of its Chest contains every Thing necessary for the different Observations to which it may be applied.

A double Lens mounted in Tortoise Shell for Observations on Insects, and other Operations where the Microscope is not commodious.

An Apparatus for making Experiments on the Transparency and Opacity of Bodies, consisting in Squares of polished Glass, limpid Liquors of different Densities, &c.

SIXTH CLASS.

Machines for making magnetic and electrical Experiments.

VI.

A small Table one Foot long, and eight Inches broad.

A Magnet cut, but not mounted. 2d. A Magnet cut and suspended in a little Boat of Ebony. 3d. A Magnet mounted and adjusted to a whirling Machine. 4th. An artificial Magnet mounted on a Pedestal of Ebony.

A Box filled with the Filings of Iron. 2d. A Basin with little Swans and Frogs of Enamel. 3d. A Box filled with small Ends of Iron and Brass Wire. 4th. A Box filled with small Iron Rings. 5th. A Box containing several Iron Balls, and some Cylanders of the same Metal.

Experiments for illustrating the Theory of the Construction of optical Instruments, the Magic-Lantern, Camera Obscura, reflecting and refracting Telescopes, Microscopes, &c.

Experiments on Magnetism.

Two large magnetic Needles of polished Iron, placed one at the Top of the other, and mounted on a Pedestal. 2d. A Dipping-Needle mounted on a Pedestal.

A square Rod of polished Iron two Feet and a half long. 2d. A round Rod of polished Iron two Feet long. 3d. A thin Plate of polished Iron eighteen Inches long. 4th. A Stand of varnished Wood.

A Brass Circle garnished with Pivots, for placing twelve small Steel Needles.

A Glass Vase mounted on a Pedestal for placing a magnetic Needle in Water.

A Machine which serves for trying the Force of a Magnet.

A Dial Compass. 2d. A truncated Compass for determining the Meridian of a Place, &c. 3d. A Sea Compass, several Steel Needles of different Sizes adapted for magnetic Experiments.

Experiments on Electricity.

A large Tube of Crystal. 2d. Two smaller ones and not so thick. 3d. A large Glass Tube very thick, two Feet long. 4th. A Glass Tube three Feet and a half long, with a Stop-Cock, to be applied to the Air-Pump.

A thick square Rod of polished Glass, about eighteen Inches long. 2d. A round solid Rod of Crystal.

A large Globe of Crystal adjusted to a whirling Machine. 2d. A Globe of Crystal, the Inside of which is laid over with Sealing-Wax, to which is adapted a Stop-Cock to be applied to the Air-Pump, and afterwards to a whirling Machine.

A large Stand, whose Tablet is made of Sealing-Wax. 2d. A Glass Stand fourteen Inches high. 3d. A Stand of Crystal of a different Form from the preceding one, for containing Fluids, and Bodies of a round Figure.

A Stick of Sealing-Wax one Inch Diameter, and one Foot long. 2d. A Tube of Sealing-Wax of the same Diameter and Length as the Stick.

A Stick of Sulphur one Inch Diameter, and eighteen Inches long. 2d. A Globe of Sulphur three Inches Diameter. 3d. A Cone of Sulphur covered with a Vase of Crystal of the same Figure. 4th. A Cone of Sealing-Wax covered as the former. 5th. A small Globe of Amber and another of Gum.

Six small Cups of Ivory. 2d. A small polished Copper Pyramid for making Experiments on the Communication of Electricity.

A Suspensory garnished with Ribbands of different Colours. 2d. A Suspensory garnished with silk Twist for communicating Electricity to living Bodies. 3d. Thread Twist, with a Wooden Ball, for communicating Electricity a great Way off.

A Cake of Rosin and Gum weighing eight Pounds. A Cake of Rosin weighing twelve Pounds.

A Pallet of Paste-board covered with Gause, and garnished with Gold Leaf, Balls of Cotton and the Down of Feathers.

A Receiver without a Bottom for the Experiments of Transmission.

A Box containing six Rackets of Gause of different Colours. 2d. A

Box containing Plates of different Metals, Wood, Paste-board and Glass.

A Glass garnished with a Circle of Metal for containing Water.

A Bar of Iron one Inch square and three Feet long.

A small Globe of Chrystal mounted so that it can be rubed in Vacuo, to which is adapted a Stop-cock to be applied to the Air-pump.

A compleat Assortment of every Thing necessary for electrical Experiments, either in Air or in Vacuo.

Plates of Brass, Part of which has been beat cold, the other when tempered in Fire.

A large Paste-board covered on one Side with Leaf Gold, and on the other with Leaf-Silver, for shewing the Ductility of those Metals.

A Metal composed of Iron and Antimony, the Filings of which burst into Flame by the Friction of the File. 2d. Sounding Lead. 3d. An Amalgama of Tin and Mercury for colouring the Inside of Glass-Vessels.

SEVENTH CLASS.

Machines of Cosmography.

VII.

A large Planetarium five Feet and a Half Diameter, with all its Appendages for shewing the different Motions of the Planets, and the Relations of the celestial Bodies with the Earth.

A Box containing the Pieces necessary for explaining what concerns the Motions and Relations of the Sun, the Earth and the Moon.

This Box only supposes a Table five Feet Diameter, in the Middle of which it is fastened.

Two Globes, one celestial and the other terestial, one Foot Diameter, constructed on the latest Observations, coloured and varnished, mounted on four pillared Pedestals, with Meridians and Horizons of a particular Kind of Paste-board.

Two Armillary Spheres, of the same Diameter as the Globes, the one according to the Ptolemaic, the other according to the Copernican System, coloured and varnished, mounted on Pedestals of Ebony.

A small terestial Globe, three Inches and a half Diameter, coloured and varnished, with a Meridian and Quadrant of Altitude.

Two Globes, one terestial and the other celestial, 18 Inches Diameter, coloured and varnished, mounted on pillared Pedestals, with Meridians, horary Circles, Compasses of Brass, engraved and polished.

The same Globes varnished and polished, with Meridians, horary Circles, Brass Compasses, mounted on a turning Pedestal of a new Construction.

Experiments on the Transmission of Electricity.

Experiments for illustrating the Theory of the primary and secondary Planets.

Experiments for illustrating the Doctrine of the Sphere.

The celestial Globe is of an azure blue. The Figures of the Constellations are perceived as Shades, the principal Circles of the Sphere are marked in Silver, as also on the terrestrial Globe; the Stars are raised in Gold, each in their proper Size, so that at one View, the natural State of the Heavens is perceived without Confusion.

Two large Planispheres, mounted on a Frame with Gold Stars, and garnished with Meridians and Horizons.

A white Globe one Foot Diameter, mounted on a Stand, with some Instruments belonging to it.

A new Dial, which serves for tracing the Meridian of a Place.

An astronomical Quadrant two Feet Radius, with two Divisions of Nomius; a moveable and immoveable Telescope, and an exterior Micrometer. 2d. An astronomical mural Quadrant four Feet Radius.

A Sextant four Feet Radius. 2d. A Sextant one Foot Radius for taking corresponding Altitudes.

Observations shewing the Use of astronomical Instruments the Quadrant, the Sextant, the Meridian-telescope, the Parallatic-telescope, the Micrometer, &c.

A Quadrant two Feet and a half Radius, with a Transom and double Joint, for measuring Angles on Land.

A meridian Telescope or a passage Instrument, four Feet long, and its Axis two Feet. 2d. A parallatic Telescope with its Axis, which serves for following the Parallel of a Star. 3d. An equatorial Telescope moveable by the Means of several graduated Circles, with its objective Micrometer. 4th. A Telescope moveable on an Axis, with an horizontal and vertical Circle graduated, and an Helioscope.

A Micrometer, to be applied to a moveable Telescope for measuring the Diameters, the Differences of the right Ascensions and Declinations of the celestial Bodies. 2d. A Micrometer to be applied to an astronomical Quadrant. 3d. An achromatic Micrometer.

An Octant 18 Inches Radius, for observing the Altitudes and Distances of the Moon from the Stars on Sea.

A Clock adapted for astronomical Observations, whose Pendulum is so composed as to correct the Dilatation to which Metals are liable. 2d. A Telescope conducted by a Clock for designing the Spots of the Moon, &c.

EIGHTH CLASS.

Machines of Meteorology.

Meteorologic Observations.

A large Thermometer, constructed on the Principles of Reaumur. 2d. A Thermometer constructed on the same Principles mounted to accompany a Barometer. 3d. A Thermometer, constructed on the same Principles, to be exposed in open Air.

A portable Thermometer one Foot long, constructed on the same Principles. 2d. A portable Thermometer contrived so as to be plunged into Fluids, in order to determine their Degree of Heat or Cold. 3d. A Thermometer constructed with Mercury, for Experiments where the Heat exceeds that of boiling Water.

The Thermometer of *Florence*. 2. A Thermometer of Air with Mercury. 3d. A Thermometer of Air, with coloured Liquor.

A kind of Pyramid, garnished with several Thermometers of Water, Oil, Spirit of Wine, salt Water, Mercury, for shewing the Dilatability of each of those Fluids.

A large Thermometer filled with coloured Water, for shewing the Dilatability of Glass.

A double Barometer. 2d. The Barometer of *Bernouilly*. 3d. A Barometer bent in its upper Part.

Those three Machines serve for shewing the Means employed for rendering the Variation in the Weight or Spring of the Air more sensible.

The Barometer shortened, by the Opposition of the two Columns of Mercury to one Column of Air. 2d. The Barometer shortened, by a Remainder of Air in the upper Part. 3d. The Barometer of *Amonton*.

Those Machines, serve for shewing the Methods employed for rendering the Barometer portable.

The simple and luminous Barometer mounted, to accompany the Thermometer, constructed on the Principles of *Reaumur*.

This Barometer differs from the common ones by the Manner it is filled, by the Form of the Vase in which it is plunged, and the Exactitude of its Effects.

The same Barometer rendered portable in any Direction, or in any kind of Carriage. 2d. The same Barometer rendered portable in a walking Cane.

This Barometer has this Advantage, that the inferior Surface of the Mercury is seen, which is well known to be of Use.

A Dial Hygrometer very sensible. 2d. An Hygrometer of another Construction.

A Pyrometer, or Machine for measuring the Action of Fire on Bodies, whose Dilatation is not immediately perceived.

In the Construction of this Machine, every Imperfection to which it has been hitherto liable is removed, the Degree of Heat is easily regulated, and every Precaution necessary, has been taken to hinder the Dust or the Humidity to spoil the Polish or the Motion of the Pieces.

An Anemometer, or Machine for discovering the Direction and Velocity of the Wind, with the Time during which it continues.

Observations shewing when the Density of the Air is diminished either by the Expansion produced by Heat or by Causes which diminish its Weight.

Observations shewing when the Density of the Air is diminished by the Causes which diminish its Weight.

Experiments for shewing the Dilatation of Metals.

CONCLUSION.

Such is the Plan of the Collection of Machines which the Nobility and Gentry of the Kingdom of *Ireland* have purchased, and whose Construction and Application to Experimental Inquiries, they have ordered to be described, and published, for the Use of the Mathematical School established under their immediate Inspection, pursuant to their Resolution of the 4th February, 1768.

PLAN of the System of the Moral World.

—*Servare modum, finemque tueri,
Naturamque sequi, patriæque impendere vitam,
Non sibi sed toti genitum se credere mundo.*

LUCAN.

I.

Origin of
civil Society.

MEN in the State of Nature, being apt to allow no other Rule for determining the Difference which might arise among them, but what is common to the brute Creation, namely, superior Strength. The Establishment of civil Society should be considered as a Compact against Injustice and Violence, a Compact intended to form a Kind of Balance between the different Parts of Mankind; but the moral Equilibrium, like the physical one, is rarely perfect and durable. Interest, Necessity, and Pleasure, brought Men together, but the same Motives induce them continually to use their Endeavours to enjoy the Advantages of Society, without bearing the Charges necessary to its Support: and in this Sense, Men, as soon as they enter into Society, may be said to be in a State of War; Laws are the Ties, more or less efficacious, intended to suspend their Hostilities, but the prodigious Extent of the Globe, the Difference in the Nature of the Regions of the Earth and its Inhabitants, not allowing Mankind to live under one and the same Government, it was natural that Men should divide themselves into a certain Number of States, distinguished by the different Systems of Laws which they are bound to obey. Had all Mankind united under one Government, they would have formed a languid Body, extended without Vigour on the Surface of the Earth. The different States are so many strong and active Bodies, which lending each other mutual Assistance, form but one, and whose reciprocal Action supports the Life and Motion of the Whole.

II.

The different Forms
of Government in the
World.

All the States with which we are acquainted, partake of three Forms of Government, *viz.* the Republican, Monarchical, and Despotic. In some Places Monarchy inclines to Despotism, in others the Monarchical is combined with the Republican, &c. Those three Species of Government are so entirely distinct, that properly speaking, they have nothing in common: We should therefore form of those three, so many distinct Classes, and endeavour to investigate the Laws peculiar to each; it will be easy afterwards to modify those Laws in their Application to any Government whatsoever, in proportion as they relate more or less to those different Forms.

In the different States, the Laws should be conformable to their Nature, that is, to what constitutes them, and to their Principle, or to

that which supports and gives them Vigour. The Law relative to the Nature of Democracy is first explained; it is shewn how the People in some respects are Monarchs, and in other Subjects; how they elect and judge their Magistrates, and how their Magistrates decide in certain Cases, &c. Then the Laws relative to the Nature of Monarchies are unfolded; the Degrees of delegated Power and intermediate Ranks that intervene between the Monarch and the Subject, the Duties of the Body to be appointed, the Guardian of the Laws to mediate between the Prince and the Subject are properly settled: In fine, it is proved, that the Nature of Despotism requires, that the Tyrant should exert his Authority, either in his own Person, or by some other who represents him; afterwards the Principles of the three Forms of Governments is pointed out; it is proved, that the Principle of Democracy is the Love of Equality, whereby is meant, not an absolute, rigorous, and consequently chimerical Equality, but that happy Equilibrium which renders all its Members equally subject to the Laws, and equally interested in their Support: That in Monarchies, where a single Person is the Dispenser of Distinctions and Rewards, the Principle is Honour, to wit Ambition and the Love of Esteem; and in Despotism, Fear. The more vigorously those Principles operate, the greater the Stability of the Government; and the more they are relaxed and corrupted, the more it inclines to Destruction.

The System of Education, suitable to each Form of Government, follows: It is proved, that they ought to be conformable to the Principle of each Government: That in Monarchies, the principal Object of Education should be the Art of pleasing; as productive of Refinement of Taste; Urbanity of Manners, an Address that is natural, and yet engaging, whereby Civil Commerce is rendered easy and flowing. In despotic States, the principal Object should be to inspire Terror and implicit Obedience; in Republics all the Powers of Education are required; every noble Sentiment should be carefully instilled; Magnanimity, Equity, Temperance, Humanity, Fortitude, a noble Disinterestedness, from whence arises the Love of our Country.

The Laws relative to the Principle of each Government next occur; it is shewn, that in Republics, their principal Object should be to support Equality and Oeconomy; in Monarchies to maintain the Dignity of the Nobility, without oppressing the People; in Despotic Governments, to keep all Ranks quiet. Then the Differences which the Principles of the three Forms of Government should produce in the Number and Object of the Laws, in the Form of Judgments and Nature of Punishments is explained; it is proved, that the Constitution of Monarchies being invariable, in order that Justice may be rendered in a Manner more uniform and less arbitrary

The Laws derived from the Nature of Democracies.

The Laws derived from the Nature of Monarchies.

The Laws derived from the Nature of Despotism.

In what consist the Principles of the three Forms of Government.

The Laws of Education relative to the Principle of each Form of government.

The Laws derived from the Principle of each Form of Government.

More civil Laws and Tribunals are required, which are accurately described; that in temperate Governments, whether Monarchical or Republican, criminal Laws cannot be attended with too many Formalities; that the Punishments should not only be proportioned to the Crime, but as moderate as possible; that the Idea annexed to the Punishment, frequently will operate more powerfully than its Intensity; that in Republicks, the Judgment should be conformable to the Law, because no Individual has a Right to alter it; in Monarchies, the Clemency of the Sovereign may abate its Rigour; but the Crimes should be always judged by Magistrates appointed to take Cognizance of them. In fine, that it is principally in Democracies, that the Laws should be severe against Luxury, Dissoluteness of Manners, and the Seduction of the Sex.

Advantages
peculiar to
each Form
of Govern-
ment.

The Advantages peculiar to each Government, is, in fine, enumerated; it is proved, that the Republican is better suited to small States, the Monarchical to great Empires; that Republicks are more subject to Excesses, Monarchies to Abuses; that in Republicks the Laws are executed with more Deliberation, in Monarchies with more Expedition. As to despotic Governments, to point out the Means necessary for its Support, is in effect to sap its Foundation; the Perfection of this Government is its Ruin; and the exact System of Despotism is at once the severest Satire, and the most formidable Scourge of Tyrants.

III.

Liberty is
the Preroga-
tive of every
temperate
Govern-
ment.

Is not to be
confounded
with Inde-
pendancy.

Considered
with respect
to the Con-
stitution.

Exists princi-
pally in
England.

Considered
with respect
to Individu-
als.

The general Law of all Governments, at least temperate ones, and consequently just, is political Liberty; the full Enjoyment of which should be secured to each Individual: This Liberty is not the absurd Licence of doing whatever one pleases, but the Privilege of doing whatever is permitted or authorized by Law; it may be considered either as it relates to the Constitution or to the Individual. It is shewn, that in the Constitution of every State, there are two Powers, the Legislative and Executive, and that this latter has two Objects, the internal and external Policy; in the legal Distribution of those different Sorts of Power, consists the greatest Perfection of political Liberty, with respect to the Constitution; in Proof of which are explained the Constitution of the Republic of *Rome*, and that of *Great-Britain*: It is shewn, that the Principle of the latter is founded on the fundamental Law of the ancient *Germans*; namely, that Affairs of small Consequence were determined by the Chiefs, and those of Importance were referred to the General Assembly of the whole Nation, after being previously examined by the Chiefs. Political Liberty considered, with respect to Individuals, consists in the Security which the Law affords them, whereby one Individual is not in Dread of another. It is shewn, that it is principally by the Nature and Proportion of Punishments that this Liberty is established or destroyed: That Crimes against Religion

should be punished by the Privation of the Advantages which Religion procures; the Crimes against good Morals, by Infamy; Crimes against the public Tranquility, by Prison or Exile; Crimes against private Security, by corporal Punishments: That Writings are less criminal than Deeds; meer Thoughts are not punishable; Accusation without a regular Process, Spies, anonymous Letters; all those Engines of Tyranny, equally infamous with respect to the Instruments and the Employers, should be proscribed in every good Government, that no Accusations should be urged but in Face of the Law, which always punishes Guilt or Calumny: In every other Case, the Magistrate should say, *we should absolve from Suspicion, the Man who wants an Accuser, without wanting an Enemy.* That it is an excellent Institution to have public Officers appointed, who in the Name of the State may prosecute Criminals: This will produce all the Advantages of Informers, without their Inconveniencies and Infamy.

The Nature and Manner of imposing and collecting Taxes is afterwards explained: It is proved, that they should be proportioned to Liberty; consequently in Democracies they may be heavier than in other Governments, without being burthensome; because each Individual considers them as a Tribute he pays himself, and which secures the Tranquility and Fortune of each Member: Besides, in Democracies, the Misapplication of the public Revenues is more difficult, because it is more easily discovered and punished; each Individual having a Right to call the Treasurer to an Account. That in every Form of Government, those Taxes that are laid on Merchandizes are least burthensome, because the Consumer pays without perceiving it: That the excessive Number of Troops in Time of Peace, is only a Pretext to overcharge the People with Taxes; a Means of enervating the State, and an Instrument of Servitude. In fine, that the collecting of the Duties and Taxes by Officers appointed for this Purpose, whereby the whole Product enters the public Treasury, is by far less burthensome to the People, and consequently more advantageous than the farming out of the same Duties and Taxes, which always leaves in the Hands of a few private Persons, a Part of the Revenues of the State.

Liberty considered with respect to the levying of Taxes and the public Revenues.

The Augmentation of the Number of Troops enervates the State.

IV.

The Circumstances independant of the Nature of the Form of Government, which should modify the Laws, arise principally from the Nature of the different Regions of the Earth, and the different Characters of the People which inhabit them. Those arising from the Nature of the Regions of the Earth, are two-fold; some regard the Climate, others the Soil. No Body doubts but the Climate has an Influence on the habitual Disposition of Bodies, consequently on the Characters, the Laws should be therefore conformable to the Nature of the Climate in

Particular Circumstances which should modify the different Forms of Government.

The Climate produces the Difference in the Characters and Passions of Men.

indifferent Matters, and on the contrary check its vicious Effects; an exact Enumeration of which is made, and the Laws for correcting them explained, it is shewn, how in Countries where the Heat of the Climate inclines the People to Indolence, the Laws encourage them to Labour; where the Use of Spirituous Liquors is prejudicial, they are discouraged, &c.

Slavery is inconsistent with the Law of Nature and the civil Law.

The Use of Slaves being authoris'd in the hot Countries of *Asia* and *America*, and prohibited in the temperate Climates of *Europe*, the Lawfulness of civil Slavery is next enquired into; it is proved, that Men having no more Power over the Liberty than over the Lives of one another, Slavery in general is inconsistent with the Law of Nature; that there has never been perhaps but one just Law in Favour of Slavery, viz. the *Roman* Law, whereby the Debtor was rendered the Slave of the Creditor; the Limitation of this Servitude, both as to the Degree and as to the Time, is pointed out. That Slavery at the utmost can be tolerated in despotic States, where free Men, too weak against the Government, seek for their own Advantage, to become the Slaves of those who tyrannize over the State; or else in Climates where Heat so enervates the Body, and weakens the Spirits, that Men cannot be brought to undergo painful Duties only by the Fear of Punishment.

Countries where it may be tolerated.

Domestic Slavery depends on the Climate.

From thence we pass to the Consideration of the domestic Servitude of Women in certain Climates: It is shewn, that it should take Place in those Countries where they are in a State of cohabiting with Men before they are able to make Use of their Reason; marriagable by the Laws of the Climate, Infants by those of Nature. That this Subjection is still more necessary in those Countries where Poligamy is established, a Custom in some Degree founded on the Nature of the Climate and the Ratio of the Number of Women to that of Men; then the Nature of Repudiation and Divorce is examined, and it is proved, that if once allowed, it should be allowed in Favour of Women as well as of Men.

Political Slavery.

In fine, political Slavery is treated of; it is proved, that the Climate which has such Influence in producing domestic and civil Servitude, has not less in reducing one People under the Obedience of another; that the Northern People having more Strength and Courage than those of Southern Climates, the former are destined to preserve, the latter to lose their Liberty; in Confirmation of which, the various Revolutions which *Europe*, *Asia*, &c. have undergone, is unfolded; the Causes of the Rise and Fall of Empires is pointed out, particularly those of the *Roman* Empire; it is proved, that its Rise was principally owing to the Love of Liberty, of Industry, and of Country; Principles instilled into the Minds of the People from their earliest Infancy; to those intestine Dissentions, which kept all their Powers in Action, and which

It Reigns principally in hot Countries.

ceased at the Approach of an Enemy ; to their intrepid Constancy under Misfortunes, which made them never despair of the Republick ; to that Principle from which they never receded, of never concluding Peace until they were victorious ; to the Institution of Triumphs, which animated their Generals with a noble Emulation ; to the Protection they granted Rebels against their Sovereigns ; to their wise Policy of leaving to the Vanquished their Religion and their Customs ; in fine, to their Maxim of never engaging in War with two powerful Enemies at once, submitting to every Insult from one, until they had crushed the other. That its Fall was occasioned by the too great Extent of the Empire, which changed the popular Tumults into civil Wars ; by their Wars abroad, which forcing the Citizens to too long an Absence, made them lose insensibly the Republican Spirit ; by the Corruption which the Luxury of *Asia* introduced ; by the Proscriptions of *Sylla*, which debased the Spirit of the Nation, and prepared it for Slavery ; by the Necessity they were in of submitting to a Master, when their Liberty became burthensome to them ; by the Necessity they were in of changing their Maxims, in changing their Form of Government ; by that Succession of Monsters, who reigned almost without Interruption, from *Tiberius* to *Nerva*, and from *Comodus* to *Constantine* ; in fine, by the Translation and Division of the Empire, which was destroyed, first in the *West*, by the Power of the *Barbarians* ; and after having languished many Ages in the *East*, under weak or vicious Emperors, insensibly expired.

Enumeration of the Causes of the Rise and Fall of the Roman Empire.

The Laws relative to the Nature of the Soil is next explained ; it is shewn, that Democracies are better suited than Monarchies to barren and mountainous Countries, which require all the Industry of their Inhabitants ; that a People who till the Soil, require more Laws than a Nation of Shepherds, and those more than a People who live by Hunting ; those who know the Use of Coin, than those who are ignorant of it.

The Influence of the Nature of the Soil on the Laws.

The Laws relative to the Genius of the different People of the Earth at length is disclosed, and it is proved, that Vanity which magnifies Objects, is a good Resort of Government ; Pride, which depresses them, is a dangerous one ; that the Legislator, in some measure, should respect Prejudices, Passions, and Abuses ; as the Laws should not be the best, considered in themselves, but with respect the People for which they are made ; for Example, a People of a gay Character require easy Laws ; those of harsh Characters, more severe ones. The Manners and Customs are not to be changed by Laws, but by Recompences and Examples : In fine, what the different Religions have, conformable or contrary to the Genius and Situation of the People who profess them, is explained.

The Laws considered with respect to the Genius of the Inhabitants of the Earth.

The Relations of which the different Forms of Government are susceptible.

Virtues which Commerce introduces.

The Liberty of Trade not to be confounded with the Liberty of the Trader.

Should be interdicted to the Nobility in Monarchies.

Marriage to be encouraged.

Incestuous Marriages to be proscribed.

How Population is promoted.

The different States considered with respect to each other, may yield mutual Assistance, or cause mutual Injury. The Assistance they afford is principally derived from Commerce, its Laws are therefore to be unfolded; it is proved, that though the Spirit of Commerce naturally produces a Spirit of Interest, opposed to the Sublimity of moral Virtues, yet it renders a People naturally just, and banishes Idleness and Rapine. That free Nations, who live under moderate Governments, should apply themselves to it more than those who are enslaved; that one Nation should not exclude another from its Commerce without important Reasons; that the Liberty however of Commerce does not consist in allowing Merchants to act as they please; a Faculty which would be very often prejudicial to them, but in laying them under such Restraints only, as are necessary to promote Trade; that in Monarchies, the Nobility should not pursue it, much less the Prince: In fine, that there are Nations to whom Commerce is disadvantageous; it is not those who want for nothing, but those who are in want of every thing; as *Poland*, by whose Commerce the Peasants are deprived of their Subsistence, to satisfy the Luxury of their Lords: The Revolutions which Commerce has undergone, is next displayed, and the Cause of the Impoverishment of *Spain* by the Discovery of *America*, pointed out: In fine, Coin being the principal Instrument of Commerce, the Operations upon it are treated of, such as Exchange, Payment of public Debts, &c. whose Laws and Limits are settled.

Population and the Number of Inhabitants being immediately connected with Commerce, and Marriages having for their Object Population, every Thing relative thereto is accurately explained; it is shewn, that public Continnence is what promotes Propagation; that in Marriages, though the Consent of Parents is with Reason required, yet it should be subject to Restrictions, as the Law should be as favourable as possible to Marriage; that the Marriage of Mothers with their Sons, on account of the great Disparity of the Ages of the Contractors, could rarely have Propagation for Object, and considered even in this Light, should be prohibited; that the Marriage of the Father with the Daughter might have Propagation for Object, as the Virtue of engendering ceases a great deal later in Men, and has in consequence been authorised in some Countries, as in *Tartary*; that as Nature of herself inclines to Marriage, the Form of Government must be defective, where it stands in Need of being encouraged; that Liberty, Security, moderate Taxes, the Proscription of Luxury, are the true Principles and Support of Population; that Laws notwithstanding may be made with Success, for encouraging Marriages, when, in spite of Corruption, the People are attached to their Country; what Laws have been made to this Purpose, particularly

those of *Augustus*, are unfolded; that the Establishment of Hospitals may either favour or hurt Population, according to the Views in which they have been planned; that there should be Hospitals in a State where the greatest Part of the Citizens have no other Resource than their Industry; but that the Assistance which those Hospitals give should be temporary; unhappy the Country where the Multitude of Hospitals and Monasteries, which are only perpetual Hospitals, sets every Body at their Ease, except those who labour.

Hospitals necessary in rich States.

How they are to be conducted

To prevent the mutual Injuries which States may receive from each other, Defence and Attack are rendered necessary; it is shewn, that Republicks by their Nature being but small States, cannot defend themselves but by Alliances; but that it is with Republicks they should be formed. That the defensive Force of Monarchies consists principally in having their Frontiers fortified. That States as well as Men, have a Right to attack each other for their own Preservation, from whence is derived the Right of Conquest, the general Law of which is to do as little Hurt to the Vanquished as possible. That Republicks can make less considerable Conquests than Monarchies; that immense Conquests introduce and establish Despotism; that the great Principle of the Spirit of Conquest should be to render the Condition of the conquered People better, which is fulfilling at once the natural Law and the Maxim of State, how far the *Spaniards* receded from this Principle, in exterminating the *Americans*, whereby their Conquest was reduced to a vast Desert, and they were forced to depopulate their Country, and weaken themselves for ever, even by their Victory, is explained. That it may become necessary to change the Laws of a vanquished People, but never their Manners and Customs. That the most assured Means of preserving a Conquest, is to put the Vanquished and Victors on a Level if possible, by granting them the same Rights and Privileges; how the *Romans* conducted themselves in this Respect, is related; as also how *Cesar* with respect to the *Gauls*.

The Objects of Conquest is not Slavery but Conservation.

Means of preserving a Conquest.

VI.

After having treated in particular of the different Species of Laws, there remains no more to be done, but to compare them together, and to examine them, with respect to the Objects on which they are enacted. Men are governed by different Kinds of Laws, by the natural Law common to each Individual; by the divine Law, which is that of Religion; by the ecclesiastical Law, which is that of the Policy of Religion; by the civil Law, which is that of the Members of the same Community; by the political Law, which is that of the Government of the Community; by the Law of Nations, which is that of Communities considered with respect to each other; each of those have their distinct Objects, which are not to be confounded, nor

The Laws resulting from the Nature, Circumstances, and Relations, of the different Forms of Government.

what belongs to one be regulated by the other; it is necessary that the Principles which prescribe the Laws, reign also in the Manner of composing them; the Spirit of Moderation should as much as possible direct all the Dispositions: In fine, the Stile of the Laws, should be simple and grave, it may dispense with Motives, because the Motive is supposed to exist in the Mind of the Legislator; but when they are assigned, they should be founded on evident Principles.

VII

Conclusion.

Such is the Plan of the System of the Moral World, where the Inhabitants of this Earth are considered in their real State, and under all the Relations of which they are susceptible; the moral Philosopher without dwelling on mere speculative and abstract Truths, in pointing out the Duties of Man, and the Means of obliging him to discharge them, has less in View the metaphysical Perfection of the Laws, than what human Nature will admit of; the Laws that are existing, than those which should be established; and as a Citizen of the World confined to no Nation or Climate; he makes the Laws of a particular People less the Object of his Research, than those of all the People of the Universe.

P L A N of the Military Art, including the Instructions relative to Engineers, Gentlemen of the Artillery, and in general to all Land-Officers.

*Intenti expectant Signum, exultantiaque haurit
Corda pavor pulsans, Laudumque arrecta Cupido.*

I.

SINCE the Revolution which the Invention of Gunpowder has produced in *Europe*, but above all, since Philosophy born to console Mankind, and to make them happy, has been forced to lend its Light to teach Nations how to destroy one another, the Art of War forms a Science as vast as it is complicated, composed of the Assemblage of a great Number of Sciences united and connected together, lending each other mutual Assistance, and which the Youth of this Country who are intended for the the Military State, could never acquire but in a Military School, established by public Authority, and conducted by a Man of superior Talents and Abilities.

II.

Mathematics.

There the young Officers are first brought acquainted with Algebra and Geometry, elementary, transcendental and sublime, to teach them the general Properties of Magnitude and Extention; how to calculate the Relations of their different Parts; how to apply them for determining accessible and inaccessible Angles and Distances, tracing of Camps,

surveying of Land, drawing of Charts, cubing the Works of Fortifications, &c. and to infuse into them that Spirit of Combination, which is the Foundation of all Arts, where Imagination does not predominate, as necessary to the Military Gentleman as to the Astronomer, which has formed *Turenne* and *Coborn*, as *Archimedes* and *Newton*.

III.

These abstract Notions serve as an Introduction for attaining the Art which teacheth the Properties of Motion, to measure the Times and Spaces, to calculate the Velocities, and to determine the Laws of Gravity, to command the Elements by which we subsist, whose Forces it teaches to subdue, and learns how to employ all that is at our Reach in Nature, in the most advantageous Manner, either to assist us in our Enterprizes, by supplying our Weakness, or to satisfy our Wants, and procure us all Kind of Conveniencies.

Mecanicks
and Dyna-
micks.

IV.

They are taught the Application of this admirable Art, more particularly for regulating the Dimensions which suit the Linings of the Works of Fortification, that they may resist the Pressure of the Earth, which they are to sustain, by determining the Law according to which this Pressure acts. For estimating the Resistance that Counterforts are capable of, according to their Length, Thickness, and their Distances from one another, for calculating how the Efforts of Vaults act, in order to deduce general Rules for determining their Thickness, according to the Forms that are to be given them in the different Uses that are made of them in Fortification, either for Subterraneans, City-Gates, Magazines of Powder, &c. for assigning the Form of Bridges, relative to the spreading of the Arches, determining the Stress and Strength of Timber, the Proportions of the Parts of Works, that they may have an equal relative Strength with respect to the Models, according to which they are executed in large Dimensions.

Military Ar-
chitecture.

V.

Then is unfolded the Theory of the Force and Action of Gunpowder, as it serves to regulate the Proportions of Cannons, Mortars, Guns, &c. that of elastic Fluids, as it teacheth to determine the actual Degree of the Resistance of the Air to Shells and Bullets, and to assign the real Tract described by those military Projectiles.

Ballistic.

VI.

Then the Use that can be made of the Dilatation and Condensation of the Air, as of the Force that its Spring acquires by Heat, to move Machines, is explained, by shewing the Effects of Pumps, describing the Properties of all the Kinds that have hitherto been invented; pointing out their Defects and Advantages; to what Degree of Perfection they can be brought; determining the most advantageous Proportions and

Pnumatics.

Forms of their Parts, and of all the Machines contrived to make them move, either of those intended for the Use of private Persons, for extinguishing Fires, for supplying public Fountains, &c. unfolding the Construction of all those that have been hitherto executed in the different Parts of *Europe*, which are put in Motion either by Animals, by the Course of Rivers, by the Force of Fire, explaining how this Agent, the most powerful in Nature, has been managed with the greatest Art; afterwards is shewn how to calculate the Force of the Wind, the Advantages that can be drawn from it, for draining an aquatic or maracageous Land, or to water a dry Ground; exemplified by what has been practised in the different Parts of *Europe* in this Way.

VII.

Hydraulicks The Art of conducting, raising, and managing Water, is next disclosed; it is shewn how to raise Water above the Level of its Source by Means of its Gravity, without making Use of the Parts which enter into the ordinary Composition of Machines; how to discover by Calculation, if a Water of a given Source, or raised to a given Height, by any Machine, can attain to a given Place, either by Trenches, Aqueducts, or Pipes; how to construct Basons, Water-Houses, and Cisterns to preserve it; how to distribute it through the different Parts of a City, determining the most advantageous Dimensions and Dispositions of the Conduits, and describing the most useful and ingenious hitherto executed.

As nothing is more agreeable to the Sight than Water-Works, the Manner of laying them out, and the Construction of the Machines imagined to raise the Water into the Reservoirs, which are the Soul of all those Operations, are unfolded, in order that the Engineer may be able to point out to those who are willing to embellish their Gardens, what suits them as to the Expence they are willing to be at, or the Situation of the Place; and that the Officer may be able to judge of the Beauty of Objects of this Kind.

Water, being of all Agents, that from which the greatest Advantage can be drawn for animating Machines, it is shewn how to apply it to the Wheels of the different Kinds of Mills; what Velocity they should have relative to the Current which moves them, in order that the Machines may be capable of the greatest Effect; entering into the Detail of all their different Species; calculating the Force necessary to put them in Motion; the Effects they are capable of, by Calculations, comprehending the Friction of their Parts, and the other Accidents inseparable from Practice; determining when they act upon inclined Planes, the Angle they should form with the Horizon. In fine, comparing such Machines as are contrived for the same Purpose, in order to discover which are to be preferred, according to the local Circumstances and Conveniencies for their Execution.

VIII.

The Art of rendering Works capable of resisting the violent or slow Action of Water, presents itself next; the various Machines made use of in draining, and of sinking Piles, is described; then all that concerns the Construction of Sluices, as also the Manner of employing them, according to the different Uses to which they are applied, either in leveling the Canals of Navigation; draining of Marshes; rendering Rivers navigable; forming artificial Inundations; making of Harbours, &c.

Hydraulic
Architecture.

IX.

In order to render those Researches of real Use to the young Officers, Draughting. they are initiated in the Art of delineating Objects, as it teacheth how to represent all the Parts of Works already constructed, or that are intended to be constructed by Plans of them taken parallel to the Horizon, which shew the Distribution of all their Parts, their Dimensions, &c. by Profiles or Cuts of them taken perpendicular to the Horizon, which shew the Heights, Situations, &c. of all the Parts, by Plans of Elevation, or Cuts of the exterior Parts of the Work; in fine, by perspective Plans or Cuts, which represent the Object as seen at a certain Distance, which will enable them to judge of the Effect that all the Parts together produce.

X.

These Studies prepare the young Officers for attaining to a Proficiency in the Art of defending and attacking, which comprehend the Method of fortifying regular Poligons, according to the different Systems, shewing their Advantages with regard to the local Circumstances, and how far they have been followed with Success in the Fortifications of the most celebrated Towns in Europe; the Construction and Disposition of Batteries, the Management of Artillery, the pointing of Mortars and Cannon, the conducting of Trenches, the Manner of distributing the different Stages of Mines, the Form of their Excavation, the Rangement of the Chambers, the best contrived for the husbanding the Ground and the Annoyance of the Enemy, the Construction of Lines and the Mensuration of their Parts, the tracing of Camps, entrenched or not entrenched, in even or uneven Ground, the tracing of the Camps of Armies which besiege, included in Lines of Circumvallation and Contravallation, the Attack of a regular or irregular fortified Place, situated in an equal or an unequal Ground, exemplified by the Plans of the most celebrated Sieges, joining Theory to Practice, neglecting not one Detail that may be of Importance. All these Operations being made in large Dimensions, and a Front of Fortification being raised accompanied with the other detached Works to be attacked and defended as in a real Action.

Attack and
Defence.

XI.

Geography.

Geography, as an Introduction to History, is useful to all Persons, but the Profession for which Youth is intended should decide of the Manner more or less extensive, it is to be taught; the young Officers should have an exact Knowledge of the Countries which are commonly the Theatre of War, they are therefore instructed in Topography in the greatest Detail, employing the Method of refering to the different Places, the Passages in History which may render it remarkable, preferring military Facts to all others; by this Means their Notions are rendered more fixed, and their Memories though more burthened, will become stronger.

XII.

History.

The Life of Man is insufficient to study History in Detail, the Manner of teaching it should therefore be adapted to the State of Life for which Youth is intended: Those who are destined for the Law, should be taught it, as it serves to discover the Spirit and System of the Laws of which they will one Day be the Dispensers; those who are intended for the Church, as it relates to Religion and the ecclesiastical Discipline; the young Officers are taught it, as they may draw Instruction from the military Details, as it furnishes Examples of Virtue, Courage, Prudence, Greatness of Soul, Attachment to their Country and Sovereign; they are made to remark in Antient History that admirable Discipline, that Subordination which rendered a small Number of Men the Masters of the World; they are taught how to gather from the History of their own Country, so necessary and so neglected, the present State of Affairs, the Rights of their King and Country, the Interest of other Countries and Sovereigns, &c.

XIII.

Tactics.

The Theory and Practice of the different Parts of the Military Service being necessary to all Officers, they are instructed in what regards the Service of Camps, the Service of Towns, Reviews, Armaments, Equipments, &c. As to military Exercises, and Evolutions, all who are acquainted with the actual State of military Affairs, know how necessary it is to have a great Number of Officers sufficiently instructed in the Art of exercising Troops; it is manifest that a continual Practice is the surest Means to attain to a Proficiency in this Art; the young Officers therefore are taught the Management of Arms, and trained up to the different Evolutions, which one Day they will make others execute.

XIV.

Order of the
Studies.

The Order that is followed in the Employ of the Day is such, that the Variety and Succession of Objects may serve as a Recreation, which is the most infallible Means to hasten Instruction. The Lessons of Algebra, Geometry, Mechanicks, Hidrostaticks, Hydraulicks, Geo-

graphy, History, &c. are first given, and those on the various Branches of Drawing succeed.

XV.

As Youth is liable to take a Disgust against abstract Knowledge, when its Application is not rendered sensible, the Teachers of Mathematicks and Drawing frequently put in Practice in the Field, the Mathematical, Mechanical, &c. Operations which are susceptible, and which have been already delineated on Paper, Design at sight, Views, Landscapes, &c. this Method has the Advantage of procuring the Pupils an Amusement which instructs them, and rendering palpable the Truths that have been presented them, it inspires them at the same Time with a Desire of learning new ones, and making them execute after Nature agreeable Operations, it is a sure Means of forming their Taste.

Practical
Operations.

XVI.

As the Inequality of Ages and Genius, and even of the good and bad Dispositions of the Pupils, cause a great Difference, the State of the Examination is divided into three Classes. In the first are those who distinguish themselves the most by their Application; in the second are comprised those who do their best; the third comprehends those from whom little is expected. This State is laid before the Society, in order that it may have an exact Knowledge of the Progress of each.

Public Exa-
minations.

XVII.

Such are the Means, my brave Countrymen, which the DUBLIN SOCIETY have pursuant to their Resolution of the 4th of February, 1768, procured you, to enable you to study with Success, how to establish a Concert and an Harmony of Motion amongst those vast Bodies stiled Armies; how to combine all the Springs which ought to concur together; how to calculate the Activity of Forces, and the Time of Execution; how to take away from Fortune her Assendant, and to enchain her by Prudence; how to seize on Posts, and to defend them; how to profit of the Ground, and take away from the Enemy the Advantage of theirs; not to be dejected by Dangers, nor elated by Success; how to retire, change the Plan of Operation; how in the Glance of an Eye to Form the most decisive Resolutions; how to seize with Tranquility the rapid Instants which decide Victories, draw Advantages from the Faults of the Enemy; commit none, or what is greater, repair them, in which consists the ART OF WAR.

Conclusion:



PLAN of the Mercantile Arts, including the Instructions relative to those who are intended for Trade.

Docuit quæ maximus Atlas.

I.

Dignity of
the Trader.

WISE Regulations and well concerted Encouragements will contribute very little to promote Trade, unless they be rendered practicable, operative, and useful, by the Skill and Address of the judicious and industrious Trader; it is he who employs the Poor, rewards the Ingenious, encourages the Industrious, interchanges the Produce and Manufactures of one Country for those of another, binds and links together in one Chain of Interest, the Universality of the human Species and thus becomes a Blessing to Mankind, a Credit to his Country, a Source of Affluence to all around him, his Family, and himself. The Extent of Knowledge and Abilities notwithstanding, requisite to fit Youth for so great and valuable Purposes, have not been attended to in this Country, and those of the commercial Profession have laboured under the same Disadvantages in Point of Education, as the different Classes of Men we have already spoke of.

II.

The Disadvantages in
Point of
Education
those of the
commercial
Profession labour under.

A Number of Years are spent and frequently lost in drudging through the common Forms of a Grammer School, where Youth are obliged to learn what is dark and difficult, and what must afterwards cost them much Pains to unlearn, and if long pursued must in the End retard the quickest Parts, and go near to eclipse the brightest Genius: whilst on the contrary, if the Grammar School Studies were properly directed and carefully pursued, they would learn to pass a proper Judgment on what they read, with regard to Language, Thoughts, Reflections, Principles, and Facts, to admire and imitate the Solid more than the Bright, the True more than the Marvellous, the personal Merit and good Sense more than the external and adventitious; their Taste for Writing and Living might be in some Measure formed, their Judgment rectified, the first Principles of Honour and Equity instilled, the Love of Virtue and Abhorrence of Vice excited in their Minds: *quare ergo liberalibus Studiis Filios erudimus? non quia Virtutem dare possunt, sed quia Animum ad accipiendam Virtutem præparant, quemadmodum prima illa ut Antiqui vocabant, Literatura, per quam Pueris Elementa traduntur, non docet liberales Artes; sed mox percipiendis Locum parat, sic liberales Artes non perducunt Animum ad Virtutem, sed expediunt.*

III.

At a certain Age, not after certain Acquisitions, a Master of Mathematicks is looked out for, and in this Case great Pretensions, attested by his own Word, and low Prices, are sufficient Credentials to recommend him, although neither the Teacher nor the Student reap much Advan-

tage from it. When the Round of this Teacher's Form is once finished, the Student is then turned over to the Compting-House, where he is employed during the Time of his Apprenticeship, in copying Letters, going of Messages, and waiting on the Post-Office. The Master, though he hath Talents for communicating, hath not Time for attending to the Instruction of an Apprentice, who, on the other Hand, hath been so little accustomed to think, that this Improvement by Self-Application will be very inconsiderable, besides his Time of Life, and constant Habit of Indulgence, render him more susceptible of pleasurable Impressions, than of Improvement in Business, the more especially when he was not previously prepared to understand it; wherefore it is not at all surprising, if many, who having no Foundation in Knowledge to qualify them for the Compting-House, profit little from the Expence and Time of an Apprenticeship, and from seeing Business conducted with all the Skill and Address of the most accomplished Merchant: The Consequence must no Doubt, be fatal to Numbers, and the public Interest, as well as private, must suffer greatly by every Instance of this Nature. It is true, that there have been, and still are, Gentlemen, who, destitute of all previous mercantile Instruction, without Money, and without Friends, by the uncommon Strength of natural Abilities, supported only by their own indefatigable Industry and Application, and perhaps favoured with an extraordinary Series of fortunate Events, have acquired great Estates; but such Instances are rare, and rather to be admired than imitated; for we see many set out with large Capitals, who have shone in the commercial World while their Capitals lasted, as Meteors do in the natural, but like them, soon destroyed themselves, and involved in their Ruin all such who were so unhappy as to be within the Sphere of their Influence. *Novimus Novitios, qui cum se Mercaturæ vix dederunt, in magnis Mercimoniis se implicantes, Rem suam male gessisse; et profecto imperitos Mercatores, multis Captionibus suppositos, multisque insidiis expositos experientia videmus.*

IV.

Commerce is not a Game of Chance, but a Science, in which he who is most skilled bids fairest for Success, whereas the Man who shoots at Random, and leaves the Direction to Fortune, may go miserably wide of the Mark; of which the People of this Country at length made sensible, have come to the Resolution of no longer trusting the future Prospects of their Children in the World to a Foundation so weak and uncertain: but setting a proper Value on Education, are determined to be as careful in having the Minds of their Children adorned with Virtue and good Sense, as they are in setting off what relates to their Bodies. A School is erected in this Kingdom for training up Youth to Business, where every Master has a Salary proportioned to the Difficulty of his Department:

Establishment of a mercantile School.

The most intelligent Traders being appointed the Superintendants of this School, who take Care that none be admitted whose Parts are not previously enquired into, and whose Genius is not, in some Measure, turned to act with Dignity in the mercantile Profession; who enquire often into the Morals and Proficiency of the Students, converse frequently with the Masters on the Subject of Trade, and admit the Students according to their Seniority in Letters to such Conversations; and in short, take every other Method of encouraging Masters and Students to Industry and Attention, that they may go through the tedious and difficult Task with Alacrity and Spirit. That by these Means, our Youth may be long acquainted with the Arts of gaining, before they learn to spend Money; they may not be grown old in Debauchery and Riot before they are initiated into Business, and we may soon see a Spirit of Industry, Knowledge, Humanity, and good Sense, diffuse itself among all Ranks and Denominations, whilst Idleness and Folly, with all their mischievous Train, may be banished our Streets.

-v.

Order of
the Studies.

Mathema-
ticks and
Drawing.

In this School the young Merchant is first brought acquainted with Arithmetick, numeral and specious, which of all other Sciences is the most necessary to the mercantile Profession, the Teaching of which requires much Skill and Knowledge. Before it is applied to Computations in Business, every Rule is stated, exemplified, and illustrated, in an easy intelligible Manner, and the Examples so multiplied and diversified, that the Learner may be thoroughly grounded, and have Reason always ready for what he does: All the various Compendiums which serve to abreviate Operations are distinctly shewn and demonstrated: That Facility and Dispatch may be equally familiar, Theory then is reduced to Practice, in all the Cases which can occur to the Merchant, the Banker, the Custom-House, and Insurance-Office, to which every Observation relative to Insurance, Factorage, Exchange, and such other Branches of Business are joined, which serve to illustrate the Use of the different Examples. This not only forms the Mind of the young Merchant to Business, but when he comes to act for himself, will prevent many tedious and expensive Suits, which an Ignorance in the practical Arts of negotiating them is frequently apt to create.

He is then initiated in Geometry, elementary, transcendental, and sublime, which of all other Studies, contribute most to invigorate the Mind, to free it from Prejudices, Credulity, Superstition, and to accustom it to Attention and demonstrative Reasoning: The Theory is reduced to Practice in the Mensuration of Surfaces and Solids, Heights and Distances, and in constructing the Instruments he hath Occasion to use in laying down Plans and Maps of Countries, selling Land by Measure, ascertaining the Price of Labour, and determining the Quantity

of Liquors, for regulating of their Price and Duty, &c. some Proficiency in the different Branches of Drawing enables him to carry into Execution these practical Operations.

VI.

The young Merchant is instructed in the Use of the Globes, Maps, &c. and brought acquainted with the Situation, Extent, Produce, Manufactures, Commerce, Ports, Policies, and Regulations with respect to Trade of all the Nations in the World; how the several Parts of the World are connected together in their mutual Intercourse of Commerce, how the Redundancies of one Country supplies the Wants of another, in what Articles the Markets are scarce, and in what they are overstocked, to enable him at all Times to foresee when any Branch of Trade in which he is concerned is likely to be stagnated, and to take his Measures accordingly for preventing the bad Consequences; the national Commerce in general, the Trade of the Place where he lives, the Laws, Customs, and Usages relative to the Business of the Merchant, the Penalties to which he is liable, and the Privileges to which he is intitled; the Duties, Imposts, and other Charges, laid upon the Produce of those Islands in other Countries, with all the known Maxims that relate to the Prosperity of Trade, open also a wide Field for Improvement in Matters of real Use.

Geography
and History

VII.

Naval Architecture teaches the young Merchant the principal Parts of the Vessel and their Proportions, the Dimensions of her Bottom, the Curvature of her Flanks, the Sally of her Stem, &c. how to form the Plan of a Ship, how to discover by Calculation whether the Vessel constructed according to the Device, will have the Qualities requisite to her Destination, whether the Weight is proportioned to the Solidity of her Bottom, her Stability to the Quantity of Sail she is to carry, the Rapidity of her Motion to her Capacity: the Method of gauging Vessels, how to regulate the Duties on Anchorage, and other Duties of the same Species, &c.

Navigations

The young Merchant is also brought acquainted with the Use of the Sea Compass, the Construction and Use of Sea Charts, the Principles of Astronomy applied for finding the Latitude, the Variation of the Compass and Longitude at Sea; in fine the Manner of solving the different Problems relative to Navigation by graphical Operations and Calculation.

VIII.

It will be of little Consequence to have the Understanding improved, if the Heart be totally neglected; Man was made by Nature for Society, but the Merchant both by Nature and Practice: The young Merchant is therefore taught the Nature and Essence of Good, its Principles, Pow-

Moral Phi-
losophy.

ers, and Effects, how to blend Self-Love with Benevolence, to moderate his Passions, to subject all his Actions to the Test of Reason, and that it is his Duty and Interest to found all his Dealing on Inregrity and Honour, as he that accustoms himself to unfair Dealing will, by Degrees, be reconciled to every Species of Fraud, till Ruin and Infamy become the Consequence.

The Principle of Law and Government likewise constitute a Part of the mercantile Plan of Instruction, by which they learn to whom Obedience is due, for what it is paid, and in what Degree it may justly be required; and to give proper Instructions to their Representatives in the great Council of the Nation when they are deliberating on any Act which may be detrimental to the Interest of the Community with respect to Commerce, or any other Privilege whatsoever.

IX.

Composi-
tion,

The Study of Composition not only teaches but accustoms the young Merchant to range his Thoughts, Arguments, and Proofs, in a proper Order, and to cloath them in that Dress, which Circumstances render most natural; by this Means he is not only enabled to read the Works of the best Authors with Taste and Propriety, to observe the Elegance, Justness, Force, and Delicacy of the Turns and Expressions, and still more the Truth and Solidity of the Thoughts; hereby will the Connection, Disposition, Force, and Gradation of the different Proofs of a Discourse be obvious and familiar to him, while at the same Time he is led by Degrees to speak and write with Freedom and Elegance, which will infallibly raise the Opinion of the young Merchant in the Eye of his Correspondents, and of the Public.

X.

Book-Keep-
ing,

A Merchant ought to know upon all Occasions what is in his Power to do without embarrassing himself, and have such an Idea of his Dealings, and those with whom he deals, that his Speculations may be always within his Sphere, to effect which the Method of arranging and adjusting Merchants Transactions is, like other Sciences, communicated in a rational and demonstrative Manner, and not mechanically by Rules depending on the Memory alone. The Principles upon which the Science is founded is likewise reduced to Practice by proper Examples in foreign and domestic Transactions, such as Buying and Selling, Importing, Exporting, for proper Company, and Commission, Account, Drawing, and Remitting too, freighting and hiring Vessels for different Parts of the World, making Insurances and Under-writing, and the various other Articles that may be supposed to diversify the Business of the practical Compting-House. The Nature of all those Transactions, and the Manner of negotiating them, are particularly explained as they occur, the Forms of Invoices and Bills of Sales, together with the Nature of all

intermediate Accounts, which may be made use of to answer particular Purposes, are laid open; and the Form of all such Writs as may be supposed to have been connected with the Transactions in the Wastebook, are rendered so familiar, that the young Merchant may be able to make them out at once without the Assistance of Copies.

XI.

In order to accustom the young Merchants to think, write, and act like Men, before they come upon the real Stage of Action, an epistolary Correspondence is established among them, in order to accustom them to digest well whatever they read, and improve their Stile under the Correction of an accurate Master, to that clear, pointed, and concise Manner of Writing which ought, particularly, to distinguish a Merchant. Fictitious Differences among Merchants are likewise submitted to their Judgement, sometimes to two by the Way of Arbitration, and again to a Jury, whilst one assumes the Character of the Plaintiff, and another that of the Defendant, and each gives in such Memorials or Representations, according to the Nature of the Facts discussed, as he thinks most proper to support the Cause, the Patronage of which was assigned him.

Practical
Negotia-
tions.

XII.

Thus the Education of the young Merchant is conducted, that his Knowledge may be so particular, and his Morals so secured, that he may be Proof against the Arts of the Deceitful, the Snares of the Disingenuous, and the Temptations of the Wicked; that he may in a short Time be so expert in every Part of the Business of the practical Compting-House, that when he comes to act for himself, every Advantage in Trade will lie open to him, that his Knowledge, Skill, and Address, may carry him through all Obstacles to his Advancement, his Talents supply the Place of a large Capital, and when the beaten Track of Business becomes less advantageous, by being in too many Hands, he may strike out new Paths for himself, and thus bring a Balance of Wealth, not only to himself, but to the Community with which he is connected, by Branches of Trade unknown before.

Conclusion.

PLAN of the Naval Art, Including the Instructions relative to Ship-Builders, Sea-Officers, and in general to all those who are any Way concerned in the Business of the Sea.

*Qui dubiis ausus committere fluctibus Alnum,
Quas Natura negat, præbuit Arte Vias.*

CLAUD.

I.

AS nothing is executed in the Military Way, but by the Direction of Geometry and Mechanicks, no less indispensable is the Use of these Sciences in Naval Operations, viz. Ship-building, stowing, work-

ing, and conducting Vessels through the Sea. A Ship is so complicated a Machine, its various Parts have so close and so hidden a Dependance on one another, and the Qualities it ought to be endued with, are so many in Number, and so difficult to be reconciled, the Mechanism of its Motions depends upon so many Instruments, which have an essential Relation to each other, &c. that it is only by Experience, aided by the sublimest Geometry, it has been discovered, that all its Actions are subjected to invariable Laws, and that we can attain to certain Rules, which could enable the Master Ship-builders to give their Vessels the most advantageous Forms, relative to the Services for which they are destined, and instruct the Navigator how to draw from the Wind the greatest Force, to dispose of it at Pleasure, and to traverse the vastest Seas without Danger and without Fear.

Notwithstanding which, Mathematicks reduced by the Teachers of them in this Kingdom, to a few gross practical Rules, their Application to Sea Affairs, and to all other useful Enterprises, has not as yet been introduced; this Neglect has not only retarded the Progress that the Study of the Mathematicks otherwise would have made, by hindering it from being known that they are the Means the most proper to supply the Limitation of our natural Faculties, and that it is from them that all useful Arts are to receive their Perfection. But in the present Case, cannot but be attended with the most fatal Consequences, and the Disasters that happen but too often at Sea, are undoubtedly, in a great Measure, owing to it.

II.

Naval Architecture.

The constructing and repairing of Vessels is entirely abandoned to the Direction of Ship-Carpenters, whose Knowledge is confined to a few gross obscure Rules, which leave the Disposition of almost all the Work to Chance, or to the Caprice of Workmen; they rely in the most important Circumstances, on the blindest Practice, on that which is the most liable to Error; they change the upper Part of the Ship, they add a new Deck, or take one away, they alter totally the Form of her Bottom, &c. Making all those Changes, without knowing what Effects will ensue, even those that would manifest themselves in the Harbour, though they could determine them after the most infallible and precise Manner, in employing the least Knowledge of Geometry, and the simplest Operations of Arithmetick.

It was therefore necessary that a Marine School should be established, where the Youth who are intended for the Business of the Sea, should be taught the Nature of Fluids, and the Mechanism of floating Bodies, how to consider the Ship as a physical heterogeneous Body in all its different Situations, and relative to its different Uses; representing it to themselves not only when it is loaden, and at Anchor, but also when it sails, when it goes well, doubles a Cape, gets difficultly clear of a Coast,

So that Geometry and Mechanicks taking the Place that Chance and blind Practice had usurped, Master Shipbuilders may exercise their Employments with Discernment; substituting luminous and precise Rules in the Place of their imperfect practical ones; they may be no more exposed to the Trouble and Shame of attempting any thing rashly, but may be enabled to assign and foresee the Success of their Enterprises, and producing no Plans but what are supported by justifiable Calculations, in which each Quality the Ship ought to have, are discussed and estimated with Exactness; we can see, in verifying their Calculations, what Stress can be laid upon their Promises; we may have infallible Means of deciding in Favour of the different Plans proposed for the same Ship, and the Multitude of their Opinions, far from being hurtful, may on the contrary be profitable, since it will often furnish an Occasion of making a better Choice.

III.

The Ship being built, it is the Business of the Navigator to distribute the Loading in such a Manner, that she may sail without Danger, and at the same Time receive with the greatest Facility whatever Motions are to be given her, that is, he is to discover her most eligible Position in the Water, he is to dispose her Sails after a suitable Manner to oblige the Vessel to take the Route he intends to follow upon all Occasions, and to make her go well in spite of the Agitation of the Sea, and the Violence of the Wind, which often opposes; for this Effect, in a Glance of an Eye, he must be capable of rendering present to his Mind all the moveable Parts of the Ship, which he must look upon as a Body which he animates as he does his own, and that it is as it were an Extension of it; seize the actual State of Things in their continual Change, and form the most decisive Resolutions, which he must draw from no other Fund but his own Breast. This is without doubt, the most difficult Part of the Navigator's Art, but at the same Time, the most important for him to possess, as it furnishes him with the surest Resources in imminent Occasions, and renders him superior in Battle. It is surprising with what Readiness, the Ship well disposed, obeys, as it were, the Orders of the skilful Seaman; but on the contrary, if he does not know all the Nicety of this Part of his Art, his Ship, though excellent, is no more than a heavy Mass, which receives all its Motions from the Caprice of Winds and Waves, which in spite of his Courage and desperate Efforts, becomes but too surely a Prey to the Enemy, or ends very soon its Destiny by Shipwreck.

Mechanical
Navigation.

Notwithstanding which, no Attempt had been made in this Kingdom to lessen the Difficulties of attaining to a Proficiency in this Branch of the Naval Art, by instructing Sea-Officers in it after a methodical Manner. It was entirely abandoned to blind Practice, as if it could not be subjected to exact

Rules in the Employment of the physical Means which it makes use of to move the Vessel. When a Manœuvre is executed in the Presence of a young Sea-Officer, he does not know very often for what it is done, or how the Instruments that are made use of act; he is surrounded with Persons too busy to give him the least Eclaircissement; we may judge from thence how much Time he must lose to learn these gross Notions, which are to serve him instead of Theory: The imperfect Knowledge which the young Sea-Officer will attain to, will be (to the Disgrace of human Reason,) the Fruit of many Years unwearied Labour; and nevertheless, as it will favour of its defective Origin, it will not give him sufficient Insight, and will leave him without exact Rules, which he can absolutely rely upon; he will give, for Example, a certain Obliquity to the Sails; he will receive the Wind with a determined Incidence, but will he know whether there is nothing to be changed in one Sense or the other, in one or the other Disposition, his only Rule is servily to copy what he has seen practised perhaps erroneously by others on like Occasions; it was therefore necessary that the Youth intended for the Sea, should be methodically instructed in the useful Maxims of the Doctrine of the moveable Forces, applied to the Business of the Sea, so that rendering them familiar to themselves in taking Share in all the Manœuvres they will see executed, in order to apply them mechanically, without the painful Help of Reflection; they might see nothing for which they were not prepared beforehand, and of which they could give an Explication to themselves; and as they would not be obliged to execute any Manœuvre blindly, they might be sensible of the happy Effects that a reflected Exercise can produce, and the Quality of a good Practitioner would be less difficult to acquire.

IV.

The Art of
Piloting.

The Navigator not only ought to know how to produce the different Motions of his Ship, but he is to observe all the Particularities of its Route, esteem its daily Position, and the Course he is to steer, to arrive at the Harbour where he is to go: This is the only Branch of the Naval Art that is taught by Rule; but it is a general Complaint among Seamen, that very little of what is learned in Schools, is of real Use; which contributes very much to confirm them in the dangerous Error, that Theory is of little or no Service; this proceeds from the Generality of Teachers having not sufficient Skill to conform their Plans of Teaching to the Exigencies of Seamen, in shewing them how to modify their Rules of Navigation, according to the different Cases of Sailing; how to reduce to the smallest Compass, the Errors to which the Measures made use of for determining the Course and Distance, are liable to, and how to make proper Allowances for them, which would enable them, as often as the Reckoning would not agree with the Observation, to judge on which

Side lay the Error, and consequently how to correct them; all which supposes in the Teacher a profound Knowledge of the Theory of the Art, and a perfect Knowledge of all the Circumstances of the Ship's Motion, in all Cases of Wind and Weather.

Their not being sufficiently exercised in Astronomy, and astronomical Observations, make them neglect instructing Sea-Officers how to chuse the most favourable Circumstances for observing either by Night or Day. The only Observations practised by Sea-Officers, are the Sun's meridional Height, and its setting; they are entirely unacquainted with the Stars, though their Observations could be of great Use, particularly when the Sun does not serve, being observeable at all Hours of the Night, and the Incertitude to which the Reckoning is liable demands that the Sea-Officers should let no Occasion slip of taking Observations every Day; moreover the most reasonable Hopes of determining the Longitude at Sea, is founded on the Observation of the Distance of the Moon from a Star, or from the Sun; this Method gives actually the Longitude to half a Degree, and has the Advantage of being as easy put in Practice as that for determining the Latitude. If they had a little Skill in astronomical Observations, they could determine the Positions of so many Places, even of this Kingdom, which are placed in Charts after an uncertain Estimation; but on the contrary, they do not know even how to verify the Instruments that are in use at Sea, particularly their Compasses and Quadrants; for want of such a Knowledge, they are obliged to take them upon the bare Word of the Workman, who is interested to get them off his Hands at any Rate; and though they ought to be verified every Voyage, on Account of the Accidents that might arise to them, it is not done. This Particular, however minute, nevertheless is worthy of Attention, since nothing should be neglected in the present Case, seeing, in spite of all the Care that can be taken, the Errors that are committed being but too sensible, and as great ones may be occasioned in the Reckoning by the Imperfection of the Instruments, as in Deductions deduced from Calculation.

We may conclude from these Considerations, that the Ship-builders and Navigators of this Kingdom were no way apprised of the important Resources they could draw from Geometry and Mechanicks, though in no Profession so eminent as in theirs, and that they could never be sufficiently skilled in their respective Arts, until a Marine School was established, conducted by a Person exercised sufficiently in the sublime Mathematicks, as to be able to understand the different mathematical Tracts that have been published in great Number of late Years, upon the different Branches of the Naval Art, such as Ship-building, Stowing, working Vessels at Sea, &c. by the most eminent Mathematicians of Europe, who should make it his Business to communicate to them after

Establishment of a
Marine
School.

a methodical Manner, all the Improvements their respective Arts have received, and receive daily from Mathematicks.

V.

Draughting. He is aided in this important Employment by Drawing-masters, as the Ship-builders cannot finish properly their Plans, without a Tincture of this Art, and some Proficiency in it, may enable the Navigator to take Views of Lands, draw such Coasts, and plan such Harbours, as the Ship should touch at, which will contribute very much to render the Geography of our Globe more correct, and lessen the Dangers of Navigation; but what is perhaps of more Consequence, it will make them acquire the Habit of observing Objects with Distinctness, and recollect exactly every Part of them, and recall all the Circumstances of their Appearances. In one Word, as the Science, which is entirely occupied in weighing, measuring and comparing Magnitudes, is necessary in all Stations and Occurrences of Life, so the Art which teaches how to represent them to the Eye is indispensable.

AN EXTRACT from the Plan of the School of Mechanic Arts, where Architects, Painters, Sculptors, and in general all Artists and Manufacturers receive the Instructions in Geometry, Perspective, Staticks, Dynamicks, Physicks, &c. which suit their respective Professions, and may contribute to improve their Taste and their Talents.*

Rem quam ago, non opinionem sed opus esse, eamque non Sedes alicujus aut placiti, sed utilitatis esse et amplitudinis immensæ fundamenta.

BACON.

I.

In the mechanic Arts is to be considered the Theory and Practice.

HOWEVER vigorous, indefatigable, or supple is the naked Hand of Man, it is capable of producing but a small Number of Effects. He can perform great Matters but by the Help of Instruments and Rules, which are as Muscles superadded to his Arms. The different Systems of Instruments and Rules conspiring to the same End, hitherto invented to impress certain Forms on the Productions of Nature, either to supply our Wants, our Pleasures, our Amusements, our Curiosity, &c. constitute the mechanic Arts.

Every Art has its Theory and Practice; its Theory is grounded on Geometry, Perspective, Staticks, Dynamicks, whose Precepts corrected by those of Physicks, as it procures the Knowledge of the Materials, their Qualities, Elasticity, Inflexibility, Friction, the Effects of the Air, Water, Cold, Heat, Aridity, &c. produce the Rules and Instruments of the Art. Practice is the habitual Use of those Instruments and Rules.

* This Plan being too extensive is omitted for the present

It is scarce possible to improve the Practice without Theory, and reciprocally to be Master of the Theory without Practice, as there is in every Art a great Number of Circumstances relative to the Materials, to the Instruments, and to the Operation which can be learned only by Use. It is the Business of Practice to point out the Difficulties, and to furnish the Phenomena. It is the Business of the Theory to explain the Phenomena, to remove Difficulties and to open the Road to further Improvement; from whence it follows, that only such Artists who have a competent Knowledge of the Theory, can become eminent in their Profession.

The Knowledge of the Theory absolutely necessary to every Artist.

But unfortunately such is the Influence of Prejudice in this Country, that Artists, Mechanicks, &c. are considered as incapable of acquiring any Knowledge in the Principles of their respective Professions, and our Youth destined to receive a liberal Education, are taught to think it beneath them to give a constant Application to Experiments and particular sensible Objects, for to practice or even to study the mechanic Arts, is to stoop to Things whose Research is laborious, the Meditation ignoble, the Exposition difficult, the Exercise dishonourable, the Number endless, and the Value inconsiderable. Prejudice which has debased an useful and estimable Class of Men, and peopled our Towns with arrogant Reasoners, useless Contemplators, and the Country with idle and haughty Landlords.

The Judicious, sensible of the Injustice and of the fatal Consequences attending this Contempt for the mechanic Arts, the Industry of the People and Establishment of Manufactures being the most assured Riches of this Country, have come to the Resolution that the Justice which is due to the Arts and Manufactures, shall be rendered them; that the mechanick Arts shall be raised from that State of Meaness, which Prejudice has hitherto kept them; that the Protection of the Noblemen and Gentlemen of Fortune shall secure the Artists and Mechanicks from that Indigence in which they languish, who have thought themselves contemptible because they have been despised; that they shall be taught to have a better Opinion of themselves, as being the only Means of obtaining from them more perfect Productions.

A School of mechanic Arts is established, where all the Phenomena of the Arts are collected, to determine the Artists to study, teach the Men of Genius to think usefully, and the Opulent to make a proper Use of their Authority and their Rewards. There the Artists receive the Instructions they stand in need of, they are delivered from a Number of Prejudices, particularly that from which scarce any are free, of imagining that their Art has acquired the last Degrees of Perfection; their narrow Views exposing them often to attribute, to the Nature of Things, Defects which arise wholly from themselves; Difficulties appearing to

The Establishment of a School of mechanic Arts.

them unsurmountable, when they are ignorant of the Means of removing them. They are rendered capable of reflecting and combining, and of discovering, in short, the only Means of excelling; the Means of saving the Matter, and the Time, of aiding Industry, either by a new Machine, or by a more commodious Method of Working. There Experiments are made, to advance whose Success, every one contributes, the Ingenious direct, the Artist executes, and the Man of Fortune defrays the Expence of the Materials, Labour and Time. There Inspectors are appointed who take Care that good Stuff is employed in our Manufactures, and that they are properly supplied with Hands; that each Operation employs a different Man, and that each Workman shall do, during his Life, but one Thing only; from whence it will result, that each will be well and expeditiously executed, and the best Work will be also the cheapest. Thus, in a short Time, our Arts and Manufactures will be brought to as great a Degree of Perfection, as in any other Part of *Europe*.

GENERAL CONCLUSION.

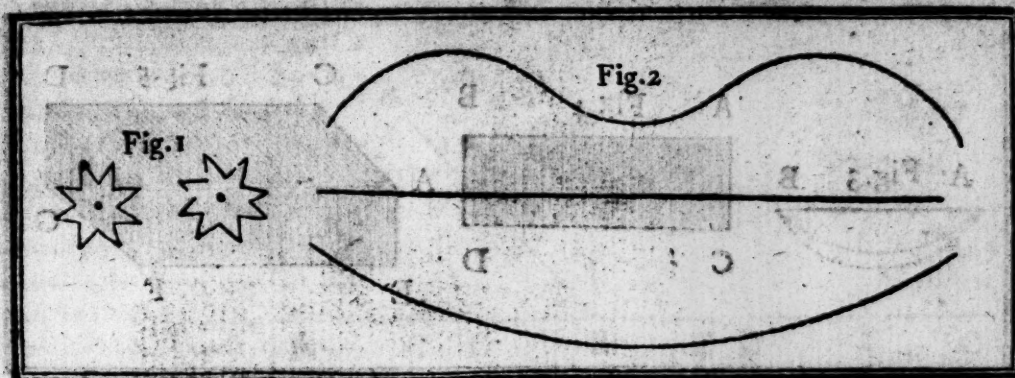
Such is the Plan of the new Scene of useful and agreeable Knowledge calculated for all Stations in Life, which the Nobility and Gentry of the Kingdom of *Ireland*, pursuant to their Resolution of the 4th of February 1768, have opened to Youth, in the Drawing-School established under their immediate Inspection. Encouraging Men of Genius and Education, from all Parts, to appear as Teachers, inviting the Artists and Connoisseurs to devote their Attention to excite the Emulation of the Pupils by adjudging and distributing the Premiums granted to engage them to advance more and more their Studies to the Point of Perfection, and taking under their Patronage such young Citizens favoured by Nature more than by Fortune, who discover happy Dispositions and superior Talents for the Service of their Country.

ERRATA.

Page LXIII Line 15, for the Centrifugal Force diminishes the Centrifugal Force, read the Centrifugal Force diminishes the Centripetal Force.

Page LXXI Line 14, for $\frac{400}{49\frac{1}{2}}$ read $\frac{400}{94\frac{1}{2}}$

Page LXXXV Line 41, for this Expression 69 for (a), 70 for (b), read, this Expression, for (a) 70, for (b) 69.



DEFINITIONS.

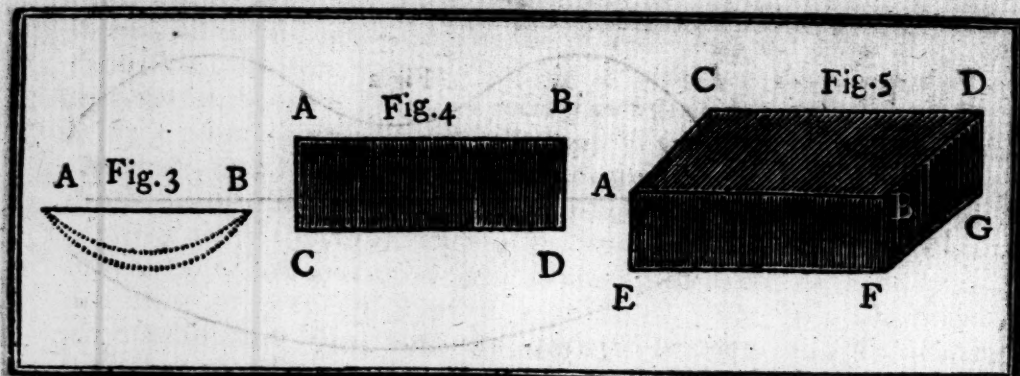
I.

A Point, is that which has no parts, or which hath no magnitude: Fig. 1.

IN this definition, as well as in the second and fifth, Euclid simply explains the manner of conceiving the first objects of Geometry, a Point, a Line, and a Superficies; he does not demonstrate that there are such objects in the class of real beings. These notions, though very useful in geometry, are only abstractions which are not to be realised, by being represented as existing independent of the mind, where they took their rise. There are no mathematical points in nature, (at least what Euclid says does not prove it); but there exist things which have extension, which may be treated as simple marks without magnitude, as often as they are considered not as composed of parts, but merely as the limits of some other magnitude. Thus, when it is required to measure the distance of two stars, the Astronomer proceeds, as if those stars were indivisible points: and he is in the right; since he does not propose to determine their magnitude, but the distance that separates them, of which they are looked upon as the terms. The same is to be understood with respect of the other notions of this kind. We represent under the form of a line, or of a length without breadth, every magnitude whose length alone is the object of our consideration, whatever may be its breadth, its depth, or its other qualities. The imagination, always disposed to transform into realities what has none, forms of those abstractions a class of beings which seem to exist independent of the mind. The Geometer has a right to adopt those beings, as they may serve to render his speculations on magnitude, considered in different points of view, more intelligible; but it is by no means allowed to him, to form wrong notions as to their origin and their real use.

II.

A Line is Length without breadth. Fig. 2.



DEFINITIONS.

III.

THE *Extremities of a Line*, are points (A, B). *Fig. 3.*

IV.

A *straight Line*, is that which lies evenly between its extreme points (A, B). *Fig. 3.*

This definition is imperfect, since it presents no essential character of a straight line; for which reason, Euclid could make no use of it: it is no more quoted in the body of the work. He is obliged to have recourse to other principles (for example, to the 12th axiom) as often as he has occasion of employing truths, which depend on a perfect definition of a straight line.

V.

A *Superficies*, is that which hath only length and breadth. *Fig. 4.*

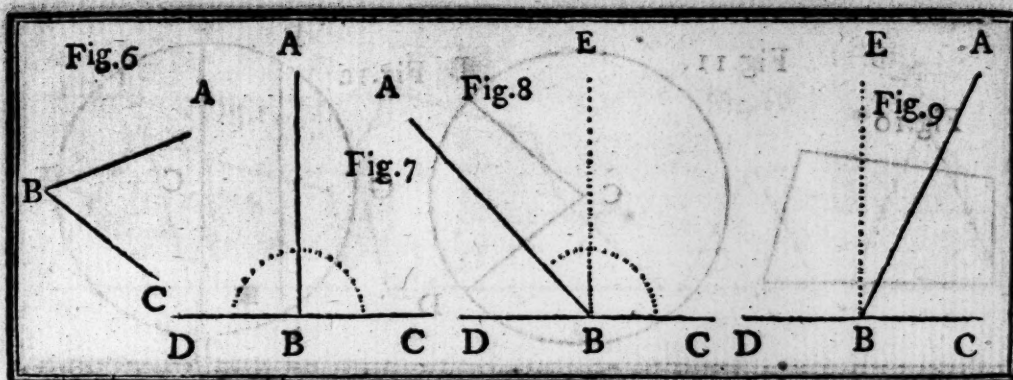
VI.

The *Extremities of a Superficies*, are lines (AB, CD, AC, BD,). *Fig. 4.*

VII.

A *Plane Superfices*, or simply a *Plane*, (AD) is that which lies evenly between its extremities (AB, CD, AC, BD,). *Fig. 5.*

This definition is liable to the same exceptions as the fourth.



DEFINITIONS.

VIII.

A *Plane Angle*, is the inclination of two lines (AB, BC,) to one another, which meet together, and which are situated in the same plane. Fig. 6.

IX.

A *Plane Rectilineal Angle*, is the inclination of two straight lines to one another. Fig. 6.

N. B. When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line.

X.

When a straight line (AB) standing on another straight line (CD) makes the adjacent angles (ABD, ABC,) equal to one another, each of the angles is called a *right angle*; and the straight line (AB) which stands on the other (CD) is called a *perpendicular*. Fig. 7.

XI.

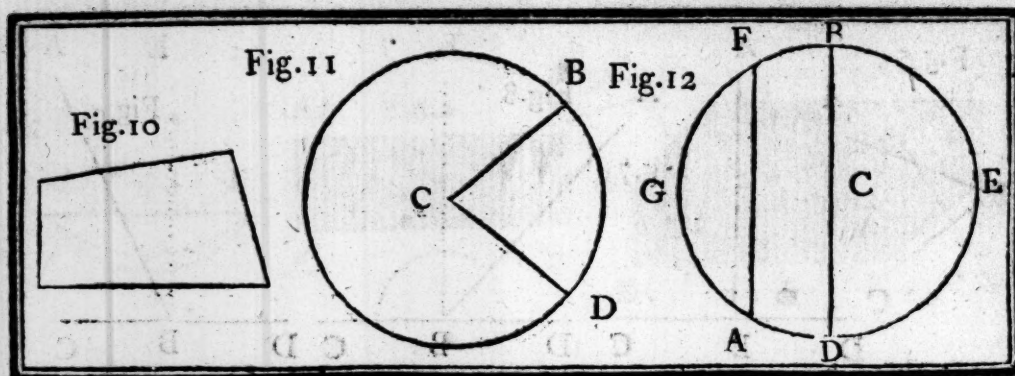
An Obtuse Angle, (ABC) is that which is greater than a right angle (EBC). Fig. 8.

XII.

An Acute Angle, (ABC) is that which is less than a right angle (EBC). Fig. 9.

XIII.

A Term or Boundary, is the extremity of any magnitude.



DEFINITIONS.

XIV.

A Figure, is that which is inclosed by one or more boundaries. *Fig. 10.*

XV.

A *Circle*, is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines (CB, CD,) drawn from a certain point (C) within the figure to the *circumference*, are equal to one another. *Fig. 11.*

XVI.

This point (C) is called the *center* of the circle, and the straight lines (CB, CD,) drawn from the center to the circumference, are called the *Rays*. *Fig. 11.*

XVII.

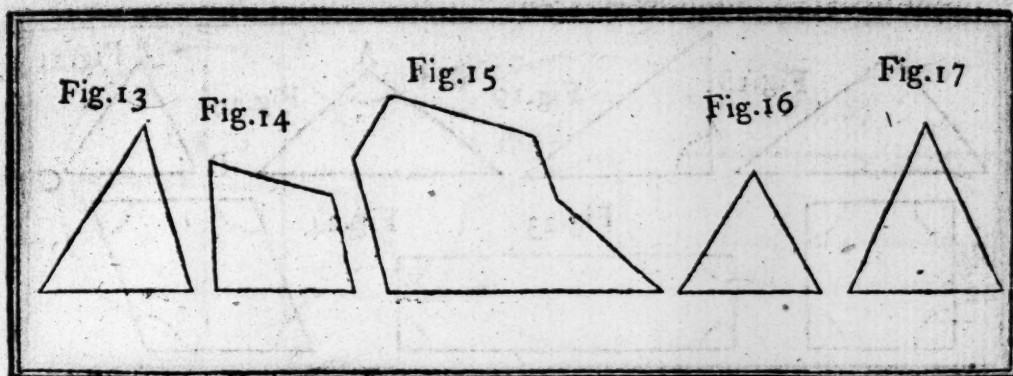
A *Diameter of a Circle*, is a straight line (DB) drawn thro' the center, and terminated both ways by the circumference. *Fig. 12.*

XVIII.

A *Semicircle*, is the plane figure (DEB) contained by a diameter (BD) and the part of the circumference (DEB) cut off by the diameter (DB). *Fig. 12.*

XIX.

A *Segment of a Circle*, is a figure contained by a straight line (AF) called a *Chord*, and the part of the circumference it cuts off (AGF, or AEF) called an *Arc*. *Fig. 12.*



DEFINITIONS.

XX.

Rectilineal Figures, are those which are contained by straight lines. Fig. 13, 14, 15, 16, 17.

XXI.

Trilateral Figures, or triangles, are those which are contained by three straight lines. Fig. 13, 16, 17.

XXII.

Quadrilateral Figures, are those which are contained by four straight lines. Fig. 14.

XXIII.

Multilateral Figures, or polygons, are those which are contained by more than four straight lines. Fig. 15.

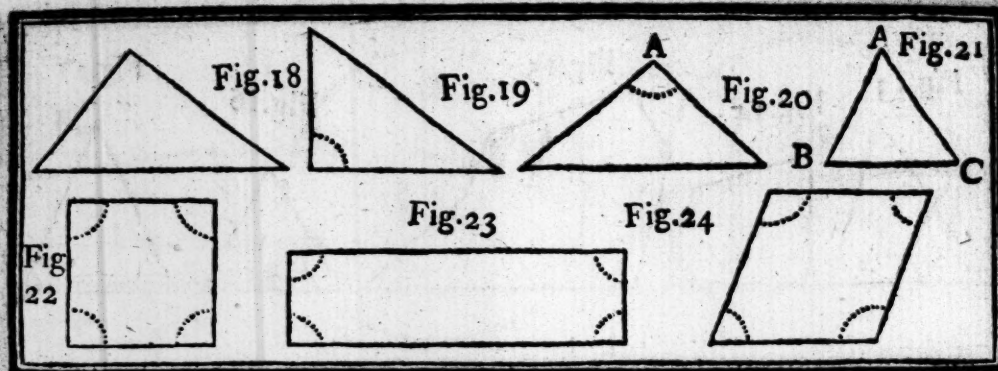
XIV.

As to three sided figures in particular:

An *Equilateral Triangle*, is that which has three equal sides. Fig. 16.

XXV.

An *Isofceles Triangle*, is that which has only two sides equal. Fig. 17.



DEFINITIONS.

XXVI.

A *Scalene Triangle*, is that which has three unequal sides. Fig. 18.

XXVII.

Likewise, among those same trilateral figures :

A Right angled Triangle, is that which has a right angle. Fig. 19.

XXVIII.

An Obtuse angled Triangle, is that which has an obtuse angle, (A). Fig. 20.

XXIX.

An Acute angled Triangle, is that which has three acute angles, (A, B, C). Fig. 21.

XXX.

After the same manner in the species of four sided figures :

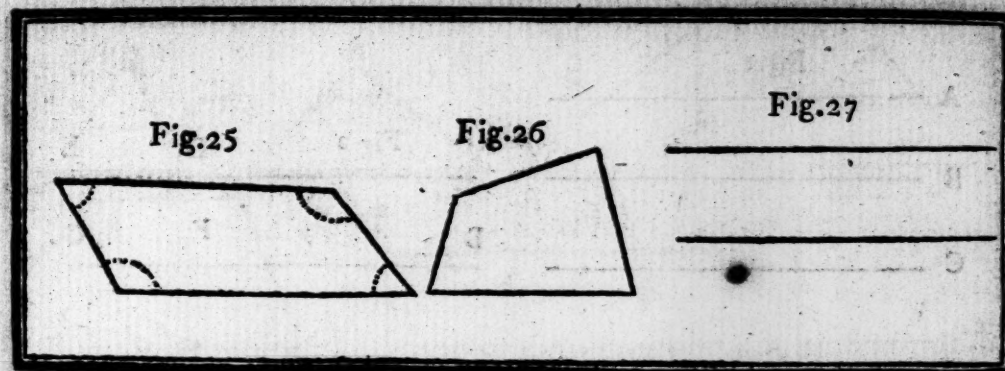
A Square, is that which has all its sides equal, and all its angles right angles. Fig. 22.

XXXI.

An Oblong, is that which has all its angles right angles, but has not all its sides equal. Fig. 23.

XXXII.

A Rhombus, is that which has all its sides equal, but its angles are not right angles. Fig. 24.



DEFINITIONS.

XXXIII.

A Rhomboid, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles. *Fig. 25.*

XXXIV.

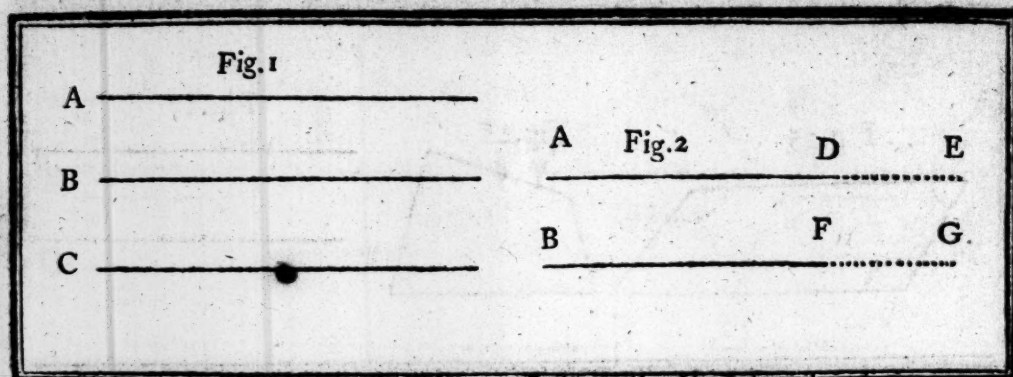
All other four sided figures besides these, are called *Trapeziums*. *Fig. 26.*

XXXV.

Parallel straight Lines, are such as are in the same plane, and which being produced ever so far both ways, do not meet. *Fig. 27.*

It is for this reason that every quadrilateral figure whose opposite sides are parallel, is called a Parallelogram. *Fig. 25.*





POSTULATES.

I.

LET it be granted, that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any center, at any distance from that center.

A X I O M S;

O R,

COMMON NOTIONS.

I.

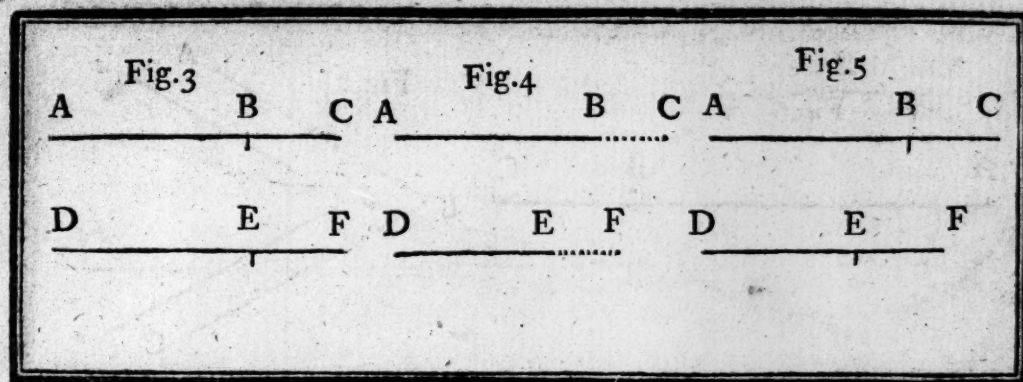
TWO magnitudes, which are equal to the same third, are equal to one another.

If the line A is equal to the line B, and the line C equal to the same line B, the line A will be equal to the line C. Fig. 1.

II.

If to equal magnitudes be added equal magnitudes, the wholes will be equal.

If to the line AD be added the part DE, and to the line BF, which is equal to the line AD, be added the part FG, equal to the part DE, the wholes AE, BG, will be equal to one another.



A X I O M S.

III.

If equals be taken from equals, the remainders are equal.

If from the whole line AC, be taken the part BC, and from the whole line DF, equal to AC, be taken the part EF, equal to BC; the remainders AB, DE, will be equal. Fig. 3.

IV.

If equals be added to unequals, the wholes are unequal.

If to the line AB, be added the part BC, and to the line DE, less than AB, be added the part EF, equal to the part BC; the wholes AC, DF, will be unequal. Fig. 4.

V.

If equals be taken from unequals, the remainders are unequal.

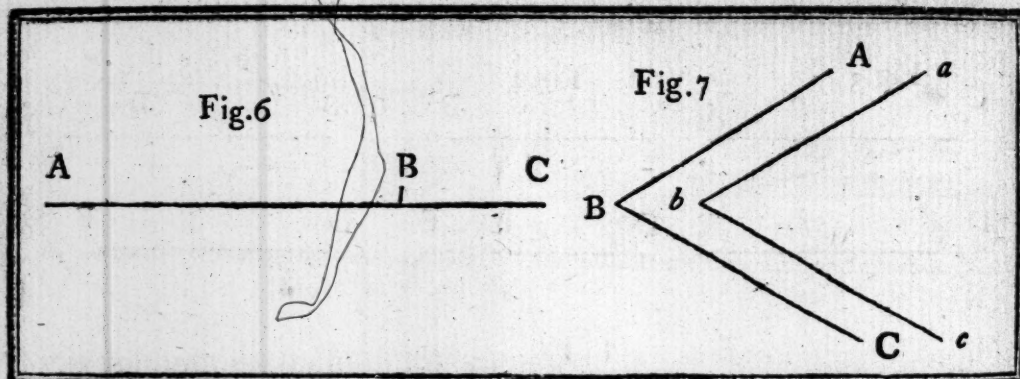
If from the line AC, be taken the part BC, and from the line DF, less than AC, be taken the part EF equal to BC; the remainders AB, DE, are unequal. Fig. 5.

VI.

Magnitudes which are double, or equimultiples of the same magnitude, are equal to one another.

VII.

Magnitudes which are halves, or equisubmultiples of the same magnitude, are equal to one another.



A X I O M S.

VIII.

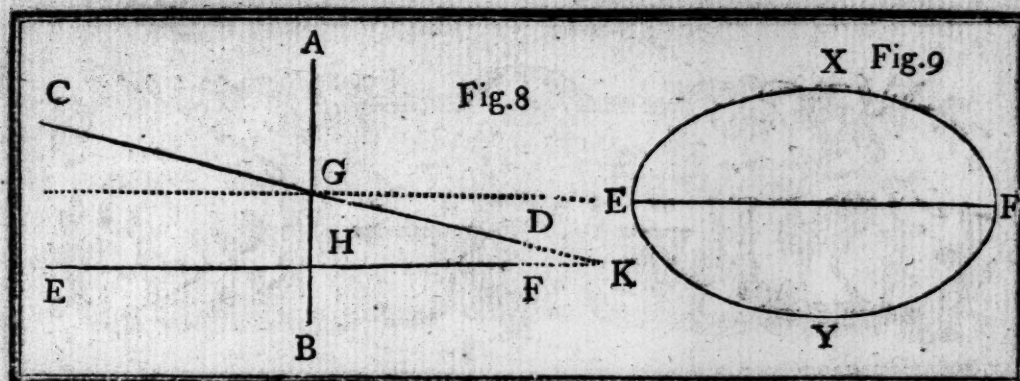
THE whole is greater than its part.

The whole line AC, is greater than its part BC. Fig 6.

IX

Magnitudes, which coincide with one another, are equal.

This axiom is called the principle of congruency; the notion of congruency, includes the notion of terms, and the notion of the possibility of their coincidence. Two magnitudes coincide, when their terms perfectly agree; or when they may be contained within the same bounds. Euclid regards the principle of congruency as a common notion: he is authorised from the universal practice of determining the equality of magnitudes, by applying one to the other, as in the mensuration of magnitudes by the foot, cubit, pearch, &c. or by including them within the same bounds, as in the measure of liquids, of grain, and the like, by pints, gallons, pecks, bushels, &c. So that we judge by the eye, or hand, how one agrees with the other, and accordingly determine their equality. It would be wrong to suppose, that such a principle could only conduct to a practice purely mechanical, incompatible with geometrical precision. Euclid has found the means of converting this maxim, into a very scientific principle. On congruency he lays down but a few obvious truths, from which he rigourously demonstrates the more complex ones which depend on this principle. Those obvious truths are as follow.



A X I O M S.

1. **A**LL points coincide.
2. Straight lines, which are equal to one another coincide; and reciprocally, straight lines whose extremities coincide are equal.
3. If in two equal angles ($ABC, abc,$) the vertexs (B & b) coincide, and one of the sides (BA) with one of the sides (ba) the other side (BC) will coincide also with the other side (bc). Likewise, all angles whose sides coincide are equal. *Fig. 7.*

Euclid has not separately enounced, those particular axioms subordinate to the general one; he nevertheless makes use of them, as will easily appear in analyzing several of his demonstrations.

X.

All right angles are equal to one another.

XI.

If a straight line (AB) cuts two other straight lines ($CD, EF,$) situated in the same plane, so as to make the two interior angles ($DGH, FHG,$) on the same side of it, taken together, less than two right angles; these two lines ($CD, EF,$) continually produced, will at length meet upon the side (K) on which are the angles which are less than two right angles. *Fig. 8.*

This truth is not simple enough, to be placed among the axioms; it is a consequence of the XXVII proposition of the first book; it is only there, that it can be properly established.

XII.

Two straight lines cannot inclose a space.

If the two straight lines EF and EXF inclose a space; those two lines cannot be both straight lines; one of them at least as EXF must be a curve line. Fig. 9.



EXPLICATION of the SIGNS.

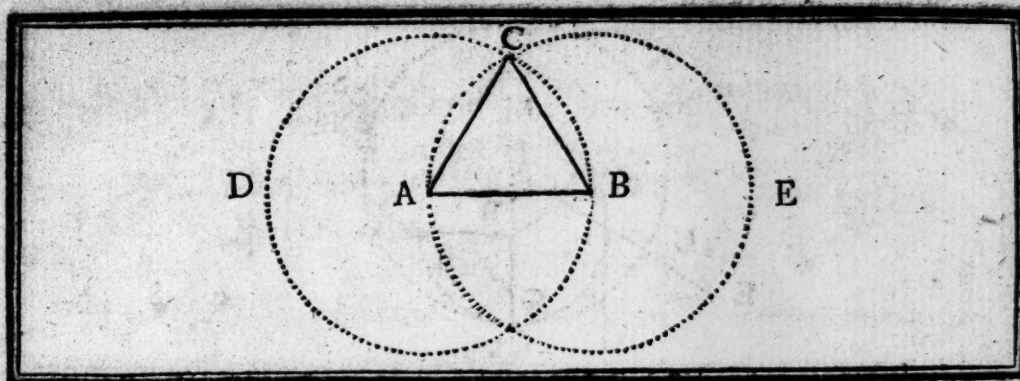
$\perp$	- - -	Perpendicular.		$\angle$	- - -	Right Angle.
$<$	- - -	Greater than		$\triangle$	- - -	Triangle.
$>$	- - -	Less than		$=$	- - -	Equal.
$+$	- - -	More.		$\square$	- - -	Square.
$-$	- - -	Less.		$\odot$	- - -	Circle.
$\sphericalangle$	- - -	Angle.		$\bigcirc$	- - -	Circumference.



ABBREVIATIONS.

Plle.	- - -	Parallel.
Pgr.	- - -	Parallelogram.
Rgle.	- - -	Rectangle.





PROPOSITION I. PROBLEM I.

UPON a given finite straight line (AB); to construct an equilateral triangle (ABC).

Given
the straight line AB.

Sought
the construction of an equilateral Δ
upon the finite straight line AB.

Resolution.

1. From the center A, at the distance AB, describe $\odot$ BCD. Pos. 3.
2. From the center B, at the distance BA, describe $\odot$ ACE. Pos. 3.
3. Mark the point of intersection C.
4. From the point A to the point C, draw the straight line AC. Pos. 1.
5. From the point B to the point C, draw the straight line BC. Pos. 1.

DEMONSTRATION.

BECAUSE the point A is the center of $\odot$ BCD (Ref. 1.), and the lines AB, AC, are drawn from the center A to the $\odot$ BCD (Ref. 4.).

1. Those two lines AB, AC, are rays of the same $\odot$.

D. 16. B. 1.

2. Consequently, the line AC is = to the line AB.

D. 15. B. 1.

Likewise, because the point B is the center of $\odot$ ACE (Ref. 2.), and the lines BA, BC, are drawn from the center B to the $\odot$ ACE (Ref. 5.).

3. Those two lines are rays of the same circle ACE.

D. 16. B. 1.

4. Consequently, the line BC is also = to the same line AB.

D. 15. B. 1.

5. Therefore, AC, BC, are each of them = to AB (Arg. 2. and 4.).

But if two magnitudes are equal to a same third, they are equal to one another.

Ax. 1.

6. The line AC is therefore = to the line BC.

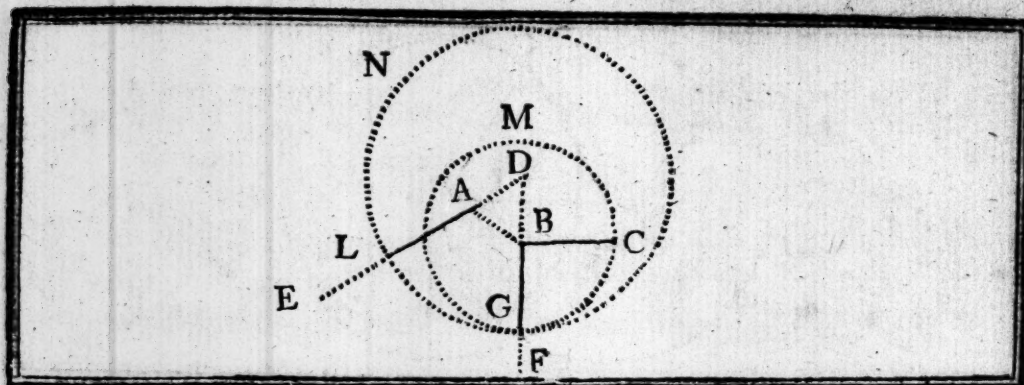
But each of those two lines = to one another (Arg. 6.), is also = to the line AB (Arg. 5.).

7. Wherefore, the three lines AB, BC, AC, which form the three sides of Δ ABC, are = to one another.

8. Consequently, the Δ ABC constructed upon the given finite straight line AB, is an equilateral triangle.

D. 24. B. 1.

Which was required to be done.



PROPOSITION II. PROBLEM II.

FROM a given point (A), to draw a straight line (AL), equal to a given straight line (BC).

Given

1. The point A.
2. The straight line BC.

Sought

$AL = BC$.

Resolution.

1. From the point A to the point B, draw the straight line AB. *Pof. 1.*
2. Upon this straight line AB construct the equilateral $\triangle ADB$. *P. 1. B. 1.*
3. Produce indefinitely the sides DA and DB of this $\triangle$. *Pof. 2.*
4. From the center B, at the distance BC, describe $\odot CGM$. *Pof. 3.*
5. And from the center D, at the distance DG, describe $\odot GLN$; *Pof. 3.*
which cuts the straight line DA produced, somewhere in L.

DEMONSTRATION.

BECAUSE the lines BC and BG, are drawn from the center B to the $\odot CGM$ (*Ref. 4.*).

1. Those two lines are rays of the same $\odot CGM$.

D. 16. B. 1.

2. Consequently, $BC = BG$.

D. 15. B. 1.

And because the lines DG and DL, are drawn from the center D to the $\odot GLN$ (*Ref. 5.*).

3. Those lines, are also rays of the same $\odot GLN$.

D. 16. B. 1.

4. Consequently, $DG = DL$.

D. 15. B. 1.

But the lines DA & DB, being the sides of an equilateral $\triangle ADB$ (*Ref. 2.*).

5. The line DA, is = to the line DB.

D. 24. B. 1.

Cutting off therefore from the equal lines DG, DL, (*Arg. 4.*); their equal parts DB, DA, (*Arg. 5.*).

6. The remainder AE is = to the remainder BG.

Ax. 3.

Since therefore the line AL is = to the line BG (*Arg. 6.*), and the line BC is also = to the same line BG (*Arg. 2.*).

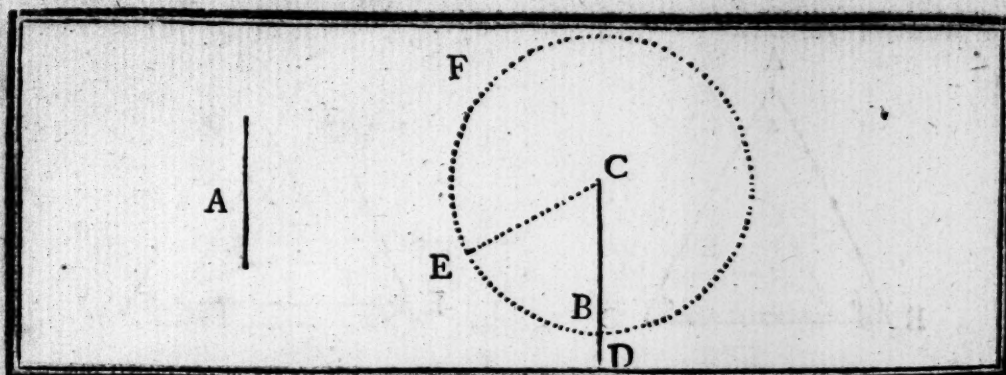
7. The line AL is = to the line BC.

Ax. 1.

But it is manifest that this line AL, is a line drawn from the given point A (*Ref. 3.*).

8. Wherefore from the given point A, a straight line AL, equal to the given straight line BC, has been drawn.

Which was to be done.



PROPOSITION III. PROBLEM III.

TWO unequal straight lines (A & CD) being given; to cut off from the greater (CD) a part (CB) equal to the less A .

Given
the line $CD >$ line A .

Sought
from CD to cut off $CB = A$.

Resolution.

1. From the point C draw the straight line $CE =$ to the given one A . *P. 2. B. 1.*
2. From the center C and at the distance CE , describe $\odot CEB$; *Pos. 1.* which cuts the greater CD in B .

DEMONSTRATION.

THE straight lines CB , CE , being drawn from the center C to the $\odot BEF$ (*Ref. 2.*).

1. They are rays of the same $\odot BEF$. *D. 16. B. 1.*
2. Consequently, $CB = CE$. *D. 15. B. 1.*

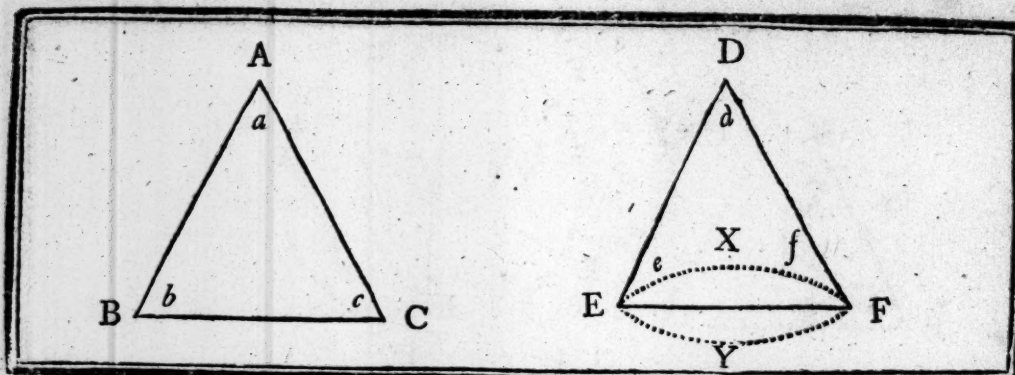
But the straight line A being $=$ to the straight line CE (*Ref. 1.*); and the straight line CB being likewise $=$ to CE (*Arg. 2.*).

3. The straight line A is $=$ to the straight line CB . *Ax. 1.*
And since CB is a part of CD .

4. From CD the greater of two straight lines, a part CB has been cut off $=$ to A the less.

Which was to be done.





PROPOSITION IV. THEOREM I.

IF two triangles (BAC, EDF), have two sides of the one, equal to two sides of the other, (*i. e.* $AB = DE$, & $AC = DF$), & have likewise the angle contained (a) equal to the angle contained (d): they will also have the base (BC), equal to the base (EF); & the two other angles (b & c) equal to the two other angles (e & f) each to each, *viz.* those to which the equal sides are opposite; and the whole triangle (BAC) will be equal to the whole triangle (EDF).

Hypothesis.

- I. $AB = DE$.
- II. $AC = DF$.
- III. $\angle a = \angle d$.

Thesis.

- I. $BC = EF$.
- II. $\angle b = \angle e$ & $\angle c = \angle f$.
- III. $\triangle BAC = \triangle EDF$.

Preparation.

Suppose the $\triangle BAC$ to be laid upon the $\triangle EDF$, in such a manner that

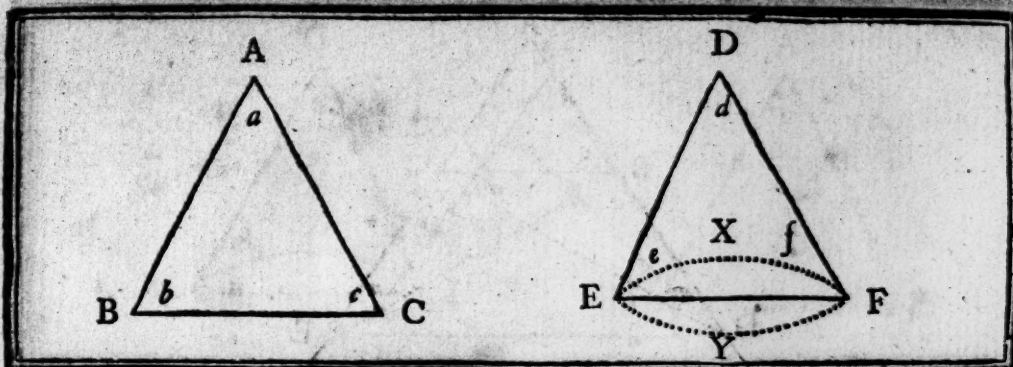
1. The point A falls upon the point D.
2. And the side AB falls upon the side DE.

DEMONSTRATION.

SINCE the line AB is $=$ to the line DE (*Hyp.* 1.), & the point A falls upon the point D (*Prep.* 1.), & the line AB upon the line DE (*Prep.* 2.).

1. The point B will fall necessarily upon the point E. Ax. 9.
Because the $\angle a = \angle d$ (*Hyp.* 3.), & the point A falls upon the point D (*Prep.* 1.), & the side AB upon the side DE (*Prep.* 2.).
2. The side AC will fall necessarily upon the side DF. Ax. 9.
Moreover, since this side AC is $=$ to the side DF.
3. The point C must fall also upon the point F. Ax. 9.
4. Wherefore, the extremities B and C of the base BC, coincide with the extremities E and F of the base EF.
5. And consequently, the whole base BC coincides with the whole base EF; for if the base BC did not coincide with the base EF, though the extremities B and C of the base BC, coincide with the extremities E and F of the base EF; two straight lines would inclose a space (EXF or EYF); which is impossible. Ax. 12.

Since therefore, the base BC coincides with the base EF (*Arg.* 5.).



6. This base BC will be = to the base EF.

Ax. 9.

Which was to be demonstrated. I.

Again, the base BC coinciding with the base EF (*Arg. 5.*), & the two other sides AB, AC, of $\triangle BAC$ coinciding with the two other sides DE, DF, of $\triangle EDF$ (*Prop. 2, Arg. 2.*).

7. Those two $\triangle BAC$, $\triangle EDF$, are necessarily equal to one another.

Ax. 9.

Which was to be demonstrated. III.

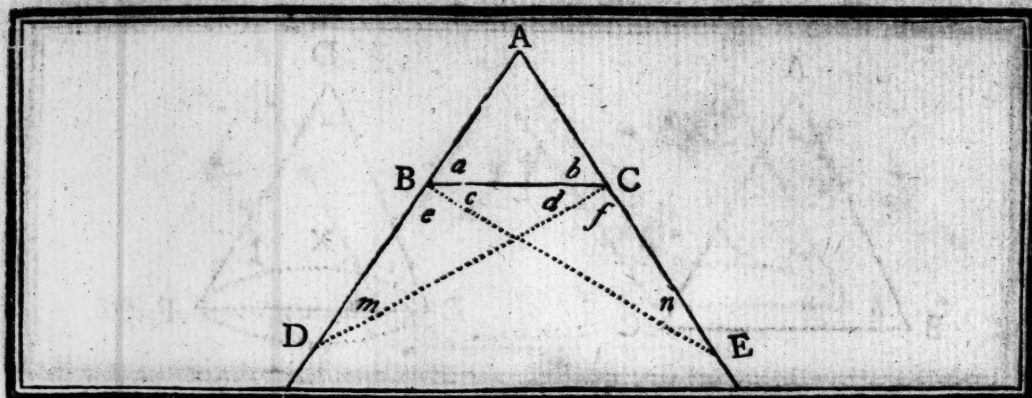
In fine, since the $\angle b$ & $\angle e$ to which the equal sides AC, DF are opposite (*Hyp. 2.*); likewise, the $\angle c$ & $\angle f$ to which the equal sides AB, DE, are opposite (*Hyp. 1.*), coincide both as to their vertices and their sides (*Arg. 1, 2, 3, 5.*).

8. It follows, that the $\angle b$ & $\angle e$, as also the $\angle c$ & $\angle f$, to which the equal sides are opposite, are equal to one another.

Ax. 9.

Which was to be demonstrated. II.





PROPOSITION V. THEOREM II.

IN every isosceles triangle (BAC): the angles (a & b) at the base (BC) are equal, & if the equal sides (AB, AC,) be produced: the angles ($c + e$ & $d + f$) under the base (BC) will be also equal.

Hypothesis.

Thesis.

I. The $\triangle BAC$ is an isosceles $\triangle$.I. $\forall a$ & $\forall b$ are equal.

II. AB & AC are produced indefinitely.

II. $\forall c + e$ & $\forall d + f$ are also equal.

Preparation.

1. In the side AB produced take any point D.

2. Make $AE = AD$.

P. 3. B. 1.

3. Through the points B & E, as also C & D, draw BE, CD. Pos. 1.

DEMONSTRATION.

BECAUSE in the $\triangle DAC$ the two sides AD, AC, are equal to the two sides AE, AB of $\triangle EAB$, each to each (Prep. 2. Hyp. 1.); and the $\angle A$ contained by those equal sides is common to the two $\triangle$.

1. The base DC is $=$ to the base BE; & the two remaining $\angle m$ & $b + d$ of $\triangle DAC$, are equal to the two remaining $\angle n$ & $a + c$ of $\triangle EAB$, each to each of those to which the equal sides are opposite.

P. 4. B. 1.

And because the whole line AD is $=$ to the whole line AE (Prep. 2.), and the part AB $=$ to the part AC (Hyp. 1.); cutting off &c.

2. The remainder BD will be $=$ to the remainder CE.

Ax. 3.

Again, since in the $\triangle DBC$ the sides DB, DC, are equal to the sides CE, EB, of $\triangle ECB$, each to each (Arg. 2. and 1.), & likewise $\angle m$ is equal to $\angle n$ (Arg. 1.).

3. The two remaining $\angle$ of the one, are $=$ to the two remaining $\angle$ of the other, each to each, viz. $\angle c + e = \angle d + f$ & $\angle d = \angle c$.

P. 4. B. 1.

The whole $\angle a + c$ & $b + d$ being therefore $=$ to one another, as also their parts $\angle c$ & $\angle d$ (Arg. 1. & 3.); cutting off &c.

4. The remaining $\angle a$ & b are likewise $=$ to one another.

Ax. 3.

But those $\angle$ are the two $\angle$ at the base BC.

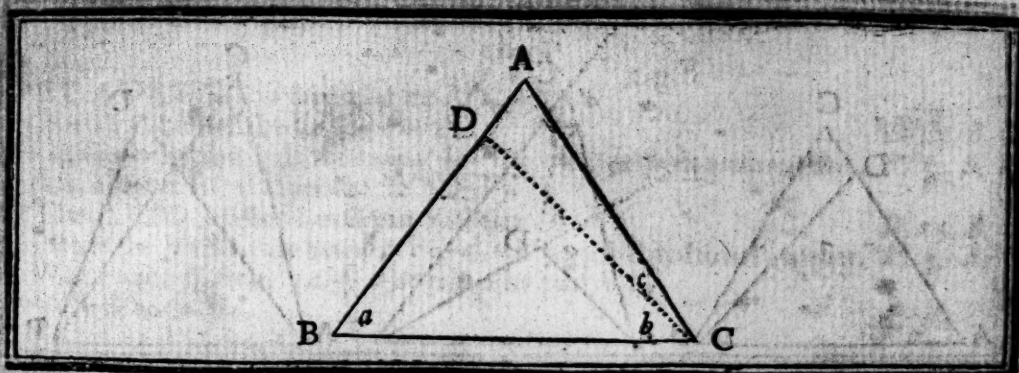
5. Therefore $\angle a$ & $\angle b$ at the base BC are $=$ to one another.

Which was to be demonstrated. I.

Moreover, since $\angle e + c = \angle d + f$ (Arg. 3.) are the $\angle$ under the base.

6. It is evident that the $\angle e + c$ & $\angle d + f$ under the base, are also $=$ to one another.

Which was to be demonstrated. II.



PROPOSITION VI. THEOREM III.

IF a triangle (ACB) has two angles (a & $b + c$) equal to one another; the sides which are opposite to those equal angles, will be also equal to one another.

Hypothesis.

In the $\triangle ACB$, $\forall a = \forall b + c$.

Thesis.

The side $CA =$ to the side BA .

DEMONSTRATION.

If not,

1. The sides CA , BA , will be necessarily unequal. C. N.
2. Consequently one of them, as BA , will be $>$ the other CA . C. N.

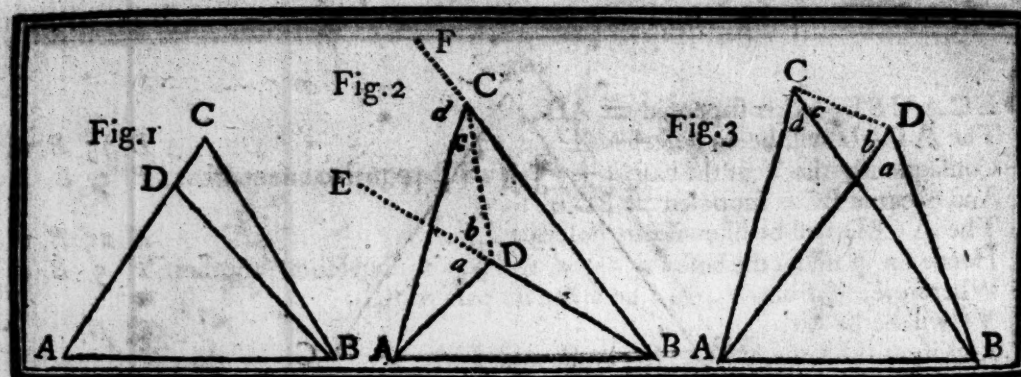
Preparation.

1. Cut off therefore from the $>$ side BA , a part $=$ to the $<$ side CA . P. 3. B. 1.
2. Draw from the point C to the point D , the straight line CD . Pos. 1.

IN the $\triangle ACB$, DBC , the side $BD =$ to the side CA (*Prep. 1.*), the side BC is common to the two $\triangle$, & $\forall$ contained $a = \forall$ contained $b + c$ (*Hyp. 1.*).

1. Consequently, the two $\triangle ACB$, DBC , have two sides of the one equal to two sides of the other, each to each, & $\forall$ contained $a = \forall$ contained $b + c$.
2. Wherefore the $\triangle ACB$ is $=$ to $\triangle DBC$. P. 4. B. 1.
But the $\triangle ACB$ being the whole, & the $\triangle DBC$ its part.
3. It follows, that the whole would be $=$ to its part.
4. Which is impossible. Ax. 8.
5. Therefore as the sides CA , BA , which are opposite to the equal $\forall a$ & $b + c$, cannot be unequal.
5. Those sides are equal to one another, or $CA = BA$. C. N.

Which was to be demonstrated.



PROPOSITION VII. THEOREM IV.

FROM the extremities (A & B) of a straight line (AB), from which have been drawn to the same point (C), two straight lines (AC, BC): there cannot be drawn to any other point (D) situated on the same side of this line, two other straight lines (AD, BD,) equal to the two first each to each.

Hypothesis.

1. AC, BC, also AD, BD, are straight lines;
2. Drawn from the same points A & B;
3. To two different points D & C, situated on the same side of the line AB.

Thesis.

It is impossible that $AC = AD$,
& $BC = BD$.

DEMONSTRATION.

If not,

There is on the same side of the line AB a point D so situated, that $AC = AD$, & $BC = BD$. Consequently this point will be placed,

- CASE 1. Either in the side AC, or BC. Fig. 1.
- CASE 2. Or within the $\triangle ACB$. Fig. 2.
- CASE 3. Or lastly without the $\triangle ACB$. Fig. 3.

CASE I. If the point D be supposed to be in one of the sides, as in AC. Fig. 1.

BECAUSE the point D is supposed to be a point in AC different from the point C.

1. The line AD is either $>$ or $<$ the line AC. }
2. Consequently it is impossible that $AD = AC$. }

C. N.

Which was to be demonstrated.

CASE II. If the point D be supposed to be situated within the $\triangle ACB$. Fig. 2.

Preparation.

1. From the point D to the point C, draw the straight line DC. Pos. 1.
2. Produce at will BD to E & BC to F. Pos. 2.

BECAUSE AC is supposed = AD.

1. The $\triangle CAD$ will be an isosceles $\triangle$. D. 25. B. 1.
2. Consequently the $\angle$ at the base $a + b$ & c will be equal to one another. P. 5. B. 1.
And because BC is supposed = BD.
3. The $\triangle CBD$ will be likewise an isosceles $\triangle$. D. 25. B. 1.
4. Hence the $\angle$ under the base b & $c + d$, will be also equal to one another. P. 5. B. 1.
Wherefore, if from $\angle c + d$ be taken its part $\angle d$.
5. $\angle b$ will be $> \angle c$. C. N.
And if to the same $\angle b$ be afterwards added $\angle a$.
6. Much more then will the whole $\angle a + b$ be $> \angle c$. C. N.
7. Consequently $\angle a + b$ & $\angle c$ are not equal. C. N.
But it has been demonstrated that in consequence of the supposition of this case, $\angle a + b$ & $\angle c$ should be equal (*Arg. 2.*).
8. From whence it follows that this supposition cannot subsist, unless those angles at the same time be equal and unequal.
9. Which is impossible. C. N.
10. Therefore the supposition which makes $AC = AD$ & $BC = BD$, is in itself impossible.

Which was to be demonstrated.

CASE III. If the point D be supposed to be without the $\triangle ACB$. *Fig. 3.*

Preparation.

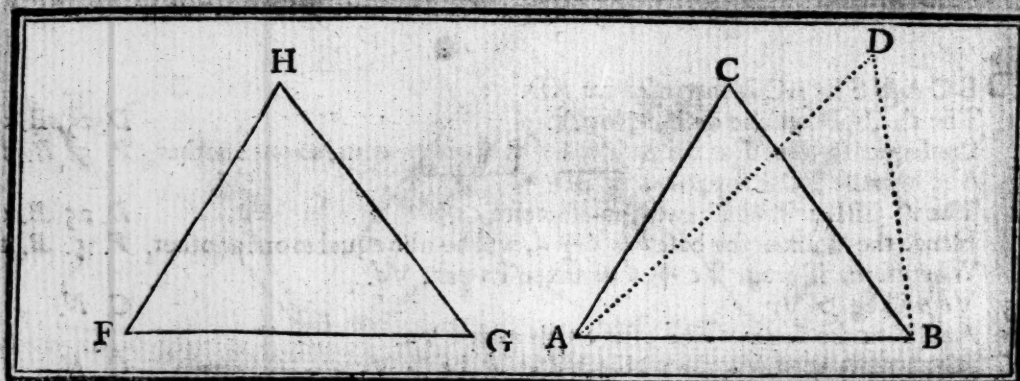
From the point D to the point C let there be drawn the straight line DC.

Pos. 1.

BECAUSE AC is supposed = AD.

1. The $\triangle CAD$ will be an isosceles $\triangle$. D. 25. B. 1.
2. Consequently $\angle b$ & $d + c$ at the base are equal to one another. P. 5. B. 1.
Again, because BC is likewise supposed = BD;
3. The $\triangle CBD$ will be an isosceles $\triangle$. D. 25. B. 1.
4. Hence $\angle c$ & $\angle b + a$ at the base will be equal to one another. P. 5. B. 1.
If therefore we take from $\angle b + a$ its part $\angle a$.
5. The $\angle c$ will be $>$ the remaining $\angle b$. C. N.
And if to this same $\angle c$ be added $\angle d$.
6. Much more then will the whole $\angle c + d$ be $> \angle b$. C. N.
7. Wherefore $\angle c + d$ & $\angle b$ are not equal to one another. C. N.
But it has been proved that in consequence of the supposition of this case, $\angle c + d$ & $\angle b$ are equal to one another. (*Arg. 2.*).
8. From whence it follows that this supposition cannot subsist, unless those angles be at the same time equal and unequal.
9. Which is impossible. C. N.
10. Therefore, the supposition which makes $AC = AD$ & $BC = BD$ is impossible.

Which was to be demonstrated.



PROPOSITION VIII. THEOREM V.

IF two triangles (FHG, ACB,) have the three sides (FH, HG, GF,) of the one equal to the three sides (AC, CB, BA,) of the other, each to each, they are equal to one another, & the angles contained by the equal sides are likewise equal, each to each.

Hypothesis.

- I. $FH = AC$.
- II. $HG = CB$.
- III. $GF = BA$.

Thesis.

$$\triangle FHG = \triangle ACB, \text{ and } \begin{cases} \angle F = \angle A. \\ \angle G = \angle B. \\ \angle H = \angle C. \end{cases}$$

Preparation.

Let the $\triangle FHG$ be applied to the $\triangle ACB$, so that;

1. The point F may coincide with the point A.
2. And the base FG with the base AB.

DEMONSTRATION.

BECAUSE the point F coincides with the point A (*Prep. 1.*), & the line FG with the line AB (*Prep. 2.*), & those lines are equal (*Hyp. 3.*).

1. The point G must coincide with the point B.

Ax. 9.

The extreme points F & G of the side FG, coinciding therefore with the extreme points A & B of the side AB (*Prep. 1. Arg. 1.*); & the straight lines FH, GH, being equal to the straight line AC, BC, each to each.

2. The straight lines FH, GH, will necessarily coincide with the straight lines AC, BC, each with each.

If not; then from the extremities A & B of a line AB, there may be drawn to two different points C & D, on the same side of AB, two straight lines AC, BC, equal to two other straight lines AD, BD, each to each. Which is impossible.

P. 7. B. 1.

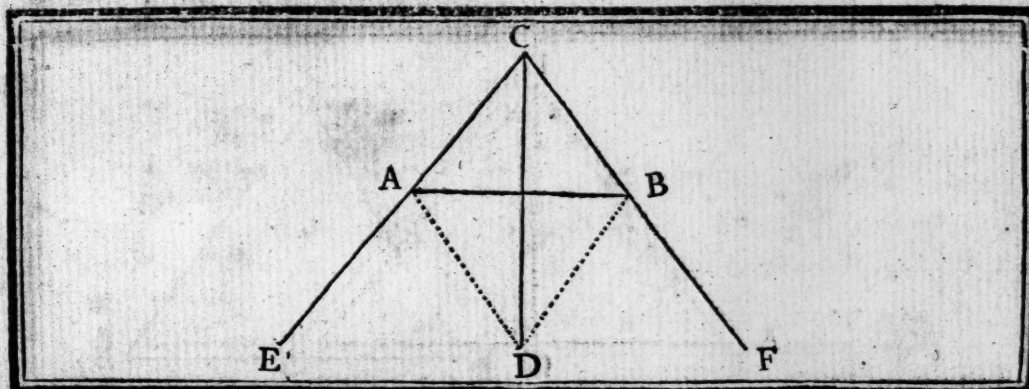
3. Those sides therefore coincide.

4. But the base FG coinciding with the base AB (*Prep. 2.*), the side FH with the side AC, & the side GH with the side BC, (*Arg. 2.*).

5. It follows, that the $\triangle ACB$, $\triangle FGH$, are equal to one another; as likewise their $\angle$ contained by the equal sides, each to each.

Ax. 9.

Which was to be demonstrated,



PROPOSITION IX. PROBLEM. IV.
TO divide a given rectilinear angle (ECF), into two equal angles (ECD, FCD,).

Given
 A rectilinear $\angle$ ECF.

Sought
 $\angle$ ECD = $\angle$ FCD.

Resolution.

1. Take CA of any length. P. 3. B. 1.
2. Make CB = CA. Pos. 1.
3. From the point A to the point B, draw the straight line AB. P. 1. B. 1.
4. Upon the straight line AB, construct the equilateral $\triangle$ ADB. Pos. 1.
5. From the point C to the point D, draw the straight line CD.

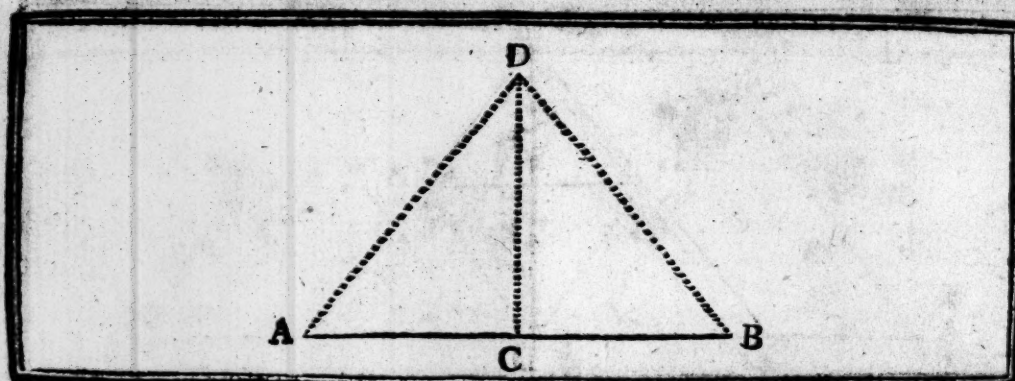
DEMONSTRATION.

BECAUSE AC = BC (*Ref. 2.*), DA = DB (*Ref. 4.*), and the side DC common to the two $\triangle$ CAD, CBD.

1. Those two $\triangle$ have the three sides of the one equal to the three sides of the other, each to each. P. 8. B. 1.
2. Consequently the $\angle$ FCD, ECD, contained by the equal sides CA, CD ; & CB, CD, are equal to one another.

Which was to be done.





PROPOSITION X. PROBLEM V.
TO divide a given finite straight line (AB) into two equal parts (AC, BC).

Given
 A finite straight line AB.

Sought
 $AC = BC$.

Resolution.

1. Upon the straight line AB construct the equilateral $\triangle ADB$. *P. 1. B. 1.*
2. Divide into two equal parts $\angle ADB$ by the straight line DC. *P. 9. B. 1.*

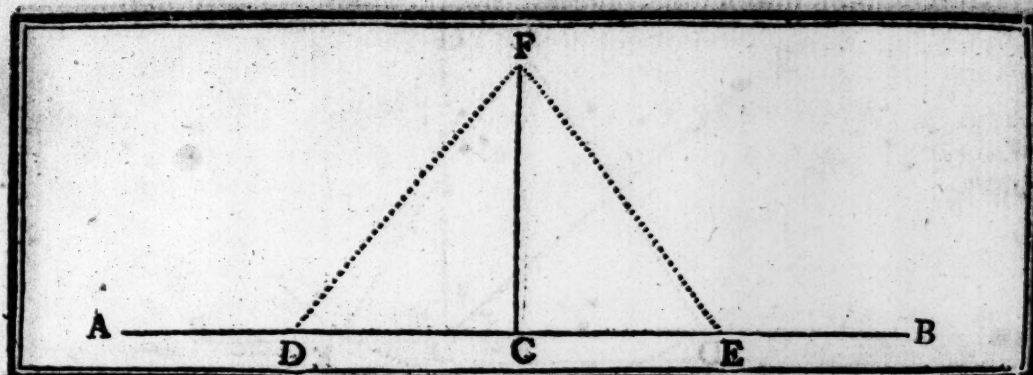
DEMONSTRATION.

BECAUSE $AD = BD$ (*Ref. 1.*), & the side DC is common to the two $\triangle ADC$, BDC , & $\angle$ contained $ADC = \angle$ contained BDC (*Ref. 2.*).

1. Those two $\triangle ADC$, BDC , have two sides in the one equal to two sides in the other, each to each, & $\angle$ contained $ADC = \angle$ contained BDC (*Ref. 2.*).
2. Consequently, the base $AC =$ to the base BC . *P. 4. B. 1.*

Which was to be done.





PROPOSITION XI. PROBLEM VI.
FROM a given point (C), in an indefinite straight line (AB), to raise a perpendicular (CF) to this line.

Given

The indefinite straight line AB, & the point C in this straight line.

Sought

The straight line CF raised from the point C $\perp$ upon AB.

Resolution.

1. On both sides of the point C take CD, CE, equal to one another. P. 3. B. 1.
2. Upon the straight line DE, construct the equilateral $\triangle DFE$. P. 1. B. 1.
3. From the point F to the point C, draw the straight line FC. P. 1.

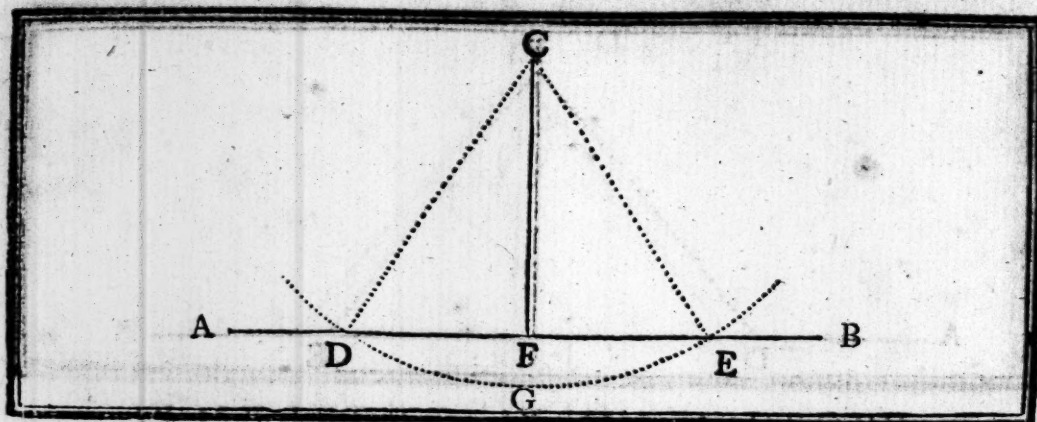
DEMONSTRATION.

BECAUSE CD is $=$ to CE (*Ref. 1.*), FD $=$ FE (*Ref. 2.*), & the side CF is common to the two $\triangle DFC$, $\triangle EFC$.

1. It is evident that those two $\triangle$ have the three sides of the one, equal to the three sides of the other, each to each.
2. Consequently, the adjacent $\angle FCD$, $\angle FCE$, (contained by the equal sides FC, CD, and FC, CE,) are equal to one another. P. 8. B. 1.
 But it is the straight line FC, which falling upon AB, forms those adjacent $\angle =$ to one another.
3. Wherefore, the straight line FC is $\perp$ upon AB. D. 10. B. 1.

Which was to be done.





PROPOSITION XII. PROBLEM VII.
FROM a given point (C), without a given indefinite straight line (AB);
 to let fall a perpendicular (CF) to this line.

Given
*The indefinite straight line AB, &
 the point C without this line.*

Sought
*The straight line CF, let fall from
 the point C $\perp$ upon AB.*

Resolution.

1. Take any point G, upon the other side of the straight line AB,
 with respect to the point C.
2. From the center C, at the distance CG, describe an arc
 of $\odot$ DGE cutting the indefinite line AB in two points D & E. *Pos. 3.*
3. Divide the line DE into two equal parts in the point F. *P. 10. B. 1.*
4. From the point C to the point F, draw the straight line CF. *Pos. 1.*

Preparation.

From the point C to the points D & E, draw the straight
 lines CD & CE.

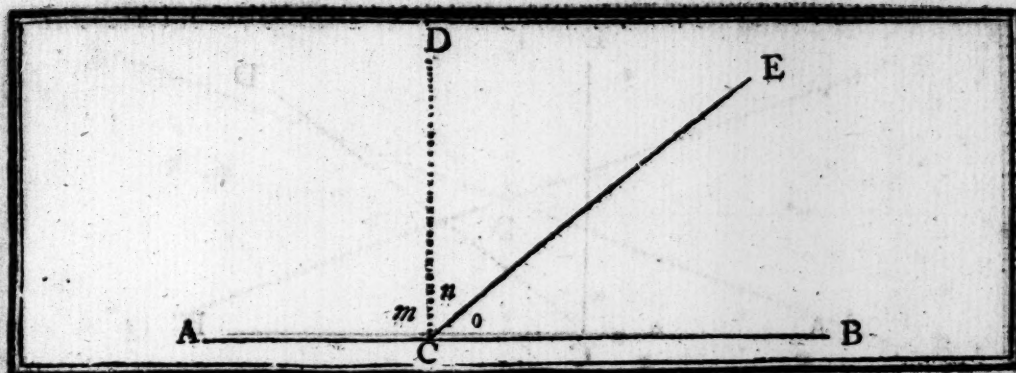
Pos. 1.

DEMONSTRATION.

BECAUSE the lines CD, CE, are drawn from the center C to
 the $\odot$ DGE (*Ref. 2. and Prep.*).

1. Those lines are rays of the same $\odot$. *D. 16. B. 1.*
2. Consequently, the line CD is = to the line CE. *D. 15. B. 1.*
 Since therefore CD is = to CE (*Arg. 2.*), $DE = FE$ (*Ref. 3.*), &
 the side CF is common to the two $\triangle$ DCF, ECF.
3. Those two $\triangle$ have the three sides of the one equal to the three sides
 of the other, each to each.
4. Wherefore the $\angle$ CFD, CFE, contained by the equal sides FC, FD,
 and FC, FE, are = to one another. *P. 8. B. 1.*
 But those two $\angle$ CFD, CFE, = to one another (*Arg. 4.*), are the
 adjacent angles formed by the line CF which falls upon the line AB.
5. Therefore, each of those two $\angle$ CFD, CFE, is a $\angle$, and the
 line CF is $\perp$ upon the line AB. *D. 10. B. 1.*

Which was to be demonstrated.



PROPOSITION XIII. THEOREM VI.

THE angles which one straight line EC makes with another AB upon one side of it, are either two right angles, or are together equal to two right angles.

Hypothesis.

EC is a straight line meeting AB in the point C.

Thesis.

I. Either each of $\angle ACE$, $\angle ECB$, is a $\angle$.

II. Or their sum is $=$ to two $\angle$.

SUP. I. If $\angle ACE$ is $=$ to $\angle ECB$.

DEMONSTRATION.

BECAUSE the adjacent angles ACE, ECB, formed by the straight lines CE & AB, are equal to one another (*Sup.*).

I. It follows, that each of them is a $\angle$.

D. 10. B. 1.

Which was to be demonstrated.

SUP. II. If $\angle ACE$ is not $=$ to $\angle ECB$.

Preparation.

From the point of concurrence C, raise upon AB the $\perp$ CD.

P. 11. B. 1.

DEMONSTRATION.

BECAUSE DC is $\perp$ upon AB (*Prep.*).

1. The two $\angle DCA$ & $\angle DCB$ are $\angle$. D. 10. B. 1.

But as $\angle DCB$ is $=$ to the two $\angle n + o$; if the $\angle DCA$ or $\angle m$, be added to each.

2. The two $\angle DCA + \angle DCB$, are $=$ to the three $\angle m + n + o$. Ax. 2.

Again, because $\angle ECA$ is $=$ to the two $\angle m + n$; if the $\angle ECB$ or $\angle o$ be added to each.

3. The two $\angle ECA$, $\angle ECB$, are also $=$ to those same three $\angle m + n + o$. Ax. 2.

4. Consequently, the two $\angle ECA$ & $\angle ECB$ are $=$ to the two $\angle DCA$ & $\angle DCB$. Ax. 1.

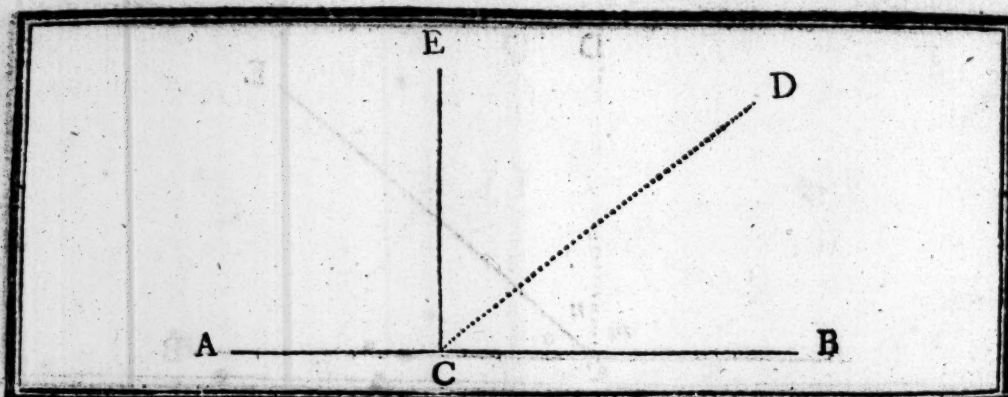
But the two $\angle DCA$ & $\angle DCB$, being two $\angle$ (*Arg. 1.*).

5. It is evident that the sum of the two $\angle ECA$ & $\angle ECB$, is also $=$ to two $\angle$.

Ax. 1.

Which was to be demonstrated.

D 2



PROPOSITION XIV. THEOREM VII.

IF two straight lines (AC, BC,) meet at the opposite sides of a straight line (EC), in a point C, making with this straight line (EC) the sum of the two adjacent angles (ACE, ECB,) equal to two right angles; those two straight lines (AC, BC,) will be in one and the same straight line.

Hypothesis.

- I. The two lines AC, BC, meet in the point C.
- II. The adjacent $\angle ACE + ECB$ are $=$ to two $\angle$.

Thesis.

The lines AC, BC, are in one & the same straight line AB.

DEMONSTRATION.

If not,

AC may be produced from C to D, so that DC & AC may make but one and the same straight line ACD.

Pos. 2.

Preparation.

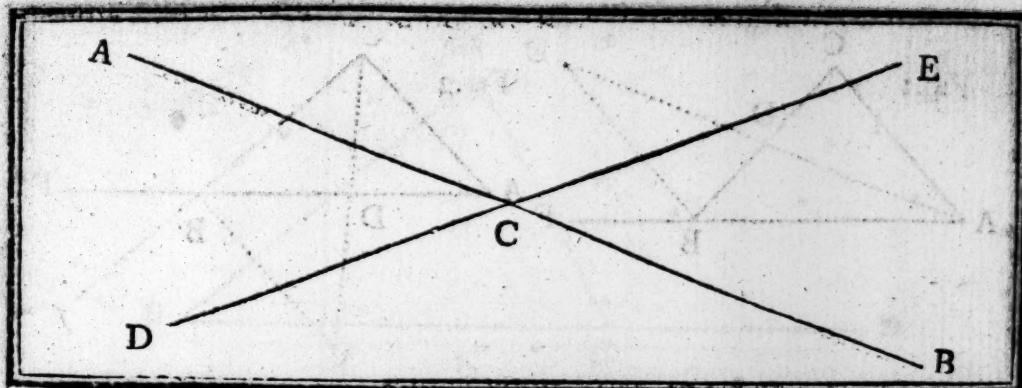
Produce then AC from C to D.

Pos. 2.

BECAUSE ACD is a straight line upon which falls the line EC.

1. It follows, that the sum of the adjacent $\angle ACE + ECD$ is $=$ to two $\angle$. P. 13. B. 1.
But the $\angle ACE + ECB$ being also $=$ to two $\angle$ (Hyp. 2.).
2. The two $\angle ACE + ECB$ are therefore $=$ to the two $\angle ACE + ECD$. Ax. 1.
Taking away therefore from each the common $\angle ACE$.
3. The remaining $\angle ECB$, $\angle ECD$, will be equal to one another. Ax. 3.
But $\angle ECB$ being the whole & $\angle ECD$ its part.
4. It follows, that the whole is equal to its part. Ax. 1.
5. Which is impossible. Ax. 8.
6. Consequently, the lines AC & BC, are in one & the same straight line.

Which was to be demonstrated.



PROPOSITION XV. THEOREM VIII.

IF two straight lines (AB, DE,) cut one another in (C), the vertical or opposite angles (ECA, DCB, & ACD, BCE,) are equal.

Hypothesis.
AB, DE, are straight lines which
cut one another in the point C.

Thesis.
I. $\angle ECA = \angle DCB$.
II. $\angle ACD = \angle BCE$.

DEMONSTRATION.

BECAUSE the straight line AC falls upon the straight line DE (*Hyp.*).

1. The sum of the two adjacent $\angle ECA + \angle ACD$ is $=$ to two $\angle$. *P. 13. B. 1.*

Again, since the straight line DC falls upon the straight line AB (*Hyp.*).

2. The sum of the adjacent $\angle ACD + \angle DCB$ is also $=$ to two $\angle$. *P. 13. B. 1.*

3. Consequently, the $\angle ECA + \angle ACD$ are $=$ to $\angle ACD + \angle DCB$. *Ax. 1.*

Taking away therefore from those equal sums (*Arg. 3.*) the common $\angle ACD$.

4. The remaining $\angle ECA$, $\angle DCB$, which are vertically opposite, are equal. *Ax. 3.*

Which was to be demonstrated. I.

In the same manner it will be proved:

5. That $\angle ACD$ is $=$ to $\angle BCE$, which is vertically opposite to it.

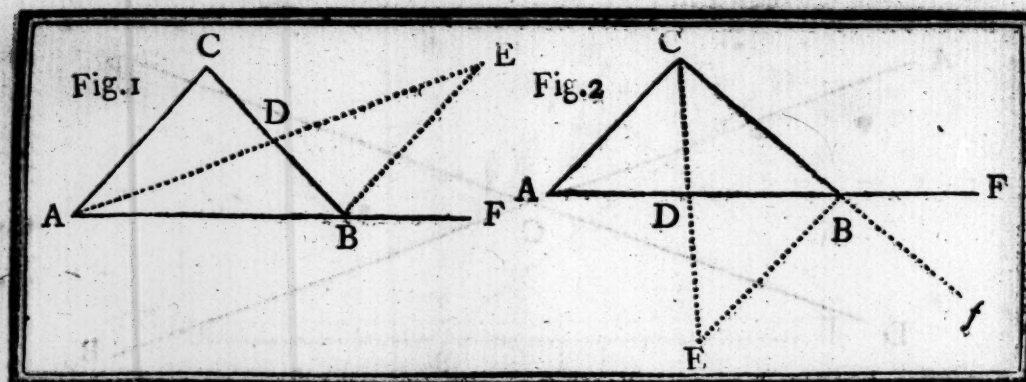
Which was to be demonstrated. II.

COROLLARY I.

FROM this it is manifest, that if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

COROLLARY II.

AND consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.



PROPOSITION XVI. THEOREM IX.

IF one side as (AB) of a triangle (ACB) be produced, the exterior angle (CBF) is greater than either of the interior opposite angles (ACB, CAB).

Hypothesis.

Thesis.

- I. ACB is a Δ .
 II. CBF is an exterior $\angle$ formed by the side AB produced.
 III. $\angle$ ACB & CAB are the interior opposite ones.

The exterior $\angle$ CBF > the interior opposite $\angle$ ACB or CAB.

Préparation.

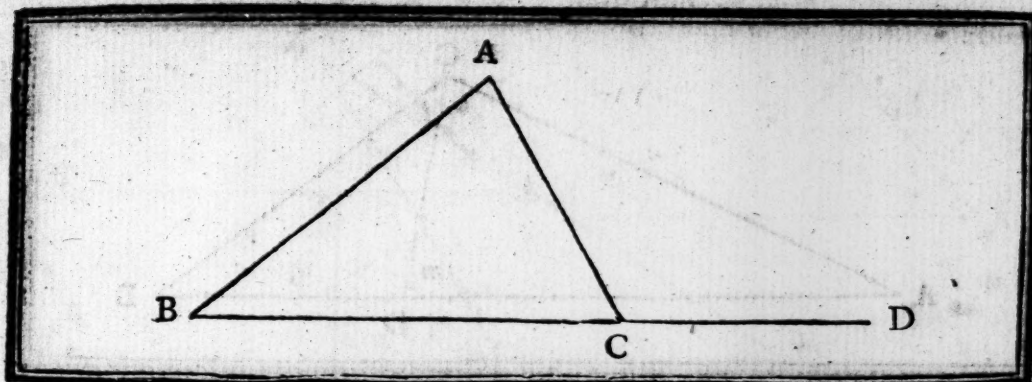
1. Divide CB into two equal parts at the point D. (Fig. 1.) P. 10. B. 1.
2. From the point A to the point D, draw the line AD, & produce it indefinitely to E. Pos. 1.
3. Make DE = DA. P. 3. B. 1.
4. From the point B to the point E, draw the straight line BE. Pos. 1.

DEMONSTRATION.

THE straight lines AE, BC, (Fig. 1.) intersect each other at the point D. (Prep. 2.).

1. Consequently, the opposite vertical $\angle$ CDA, BDE, are = to one another. P. 15. B. 1.
 Wherefore since in the Δ ACD, DEB, the side CD is = to the side DB (Prep. 1.), AD = DE (Prep. 3.), & $\angle$ contained CDA is = to $\angle$ contained BDE (Arg. 1.).
2. It follows, that the remaining $\angle$ of the one are equal to the remaining $\angle$ of the other, each to each of those to which the equal sides are opposite. P. 4. B. 1.
 But the $\angle$ ACD, DBE, are opposite to the equal sides AD, DE, (Prep. 3.).
3. Therefore $\angle$ ACD is = to $\angle$ DBE.
 But $\angle$ CBF being the whole, & $\angle$ DBE its part.
4. It follows, that $\angle$ CBF > $\angle$ DBE. Ax. 8.
5. Wherefore the exterior $\angle$ CBF is also > the interior $\angle$ ACB. C. N.
 In the same manner, dividing the side AB into two equal parts in the point D (Fig. 2.) it will be proved.
6. That the exterior $\angle$ ABf is > the interior $\angle$ CAB.
 But this $\angle$ ABf is vertically opposite to $\angle$ CBF.
7. Wherefore $\angle$ ABf = $\angle$ CBF. P. 15. B. 1.
8. Consequently, the exterior $\angle$ CBF is > the interior $\angle$ CAB. C. N.

Which was to be demonstrated.



PROPOSITION XVII. THEOREM X.

ANY two angles as $(\angle ABC, \angle ACB)$ of a triangle $(\triangle BAC)$, are less than two right angles.

Hypothesis.
ABC is a $\triangle$.

Thesis.
The $\angle ABC + \angle ACB$ are $<$ two $\angle$.

Preparation.

Produce the side BC (upon which the two $\angle ABC, \angle ACB$, are placed) to D.

Pos. 2.

DEMONSTRATION.

BECAUSE $\angle ACD$ is an exterior $\angle$ of the $\triangle BAC$.

1. It is $>$ its interior opposite one $\angle ABC$.

Since therefore $\angle ACD$ is $>$ $\angle ABC$; if the $\angle ACB$ be added to each. *P. 16. B. 1.*

2. The $\angle ACD + \angle ACB$ will be $>$ the $\angle ABC + \angle ACB$.

Ax. 4.

But the $\angle ACD + \angle ACB$ are the adjacent $\angle$ s, formed by the straight line AC, which falls upon BD (*Prep.*).

3. Consequently, those $\angle ACD + \angle ACB$ are $=$ to two $\angle$ s.

P. 13. B. 1.

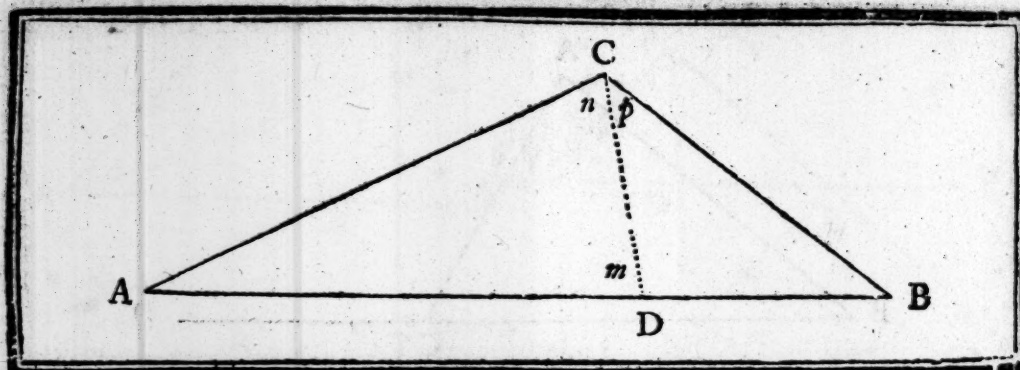
But the $\angle ACD + \angle ACB$ being $=$ to two $\angle$ s (*Arg. 3.*) & those same $\angle$ s being $>$ the $\angle ABC + \angle ACB$ (*Arg. 2.*).

4. It follows, that the $\angle ABC + \angle ACB$ are $<$ two $\angle$ s.

C. N.

Which was to be demonstrated.





PROPOSITION XVIII. THEOREM. XI.

IN every triangle (ACB); the greater side is opposite to the greater angle.

Hypothesis.
ACB is a Δ , whose side AB is $>$ AC.

Thesis.
 $\angle$ ACB, opposite to $>$ side AB, is greater than $\angle$ ABC opposite to the lesser side AC.

Preparation.

Because the side AB is $>$ AC (*Hyp.*).

1. Make $AD = AC$.

2. From the point C to the point D, draw the straight line CD.

P. 3. B. 1.

Pos. 1.

DEMONSTRATION.

BECAUSE the side AD is $=$ to the side AC (*Prep. 1.*).

1. The Δ ACD is an isosceles Δ .

2. Consequently, the $\angle$ m & n at the base CD are $=$ to one another. But $\angle$ m being an exterior $\angle$ of Δ DCB.

D. 25. B. 1.

P. 5. B. 1.

3. It follows, that it is $>$ the interior opposite $\angle$ DBC.

P. 16. B. 1.

But $\angle$ m is $=$ to $\angle$ n (*Arg. 2.*)

4. Therefore $\angle$ n is also $>$ $\angle$ DBC.

C. N.

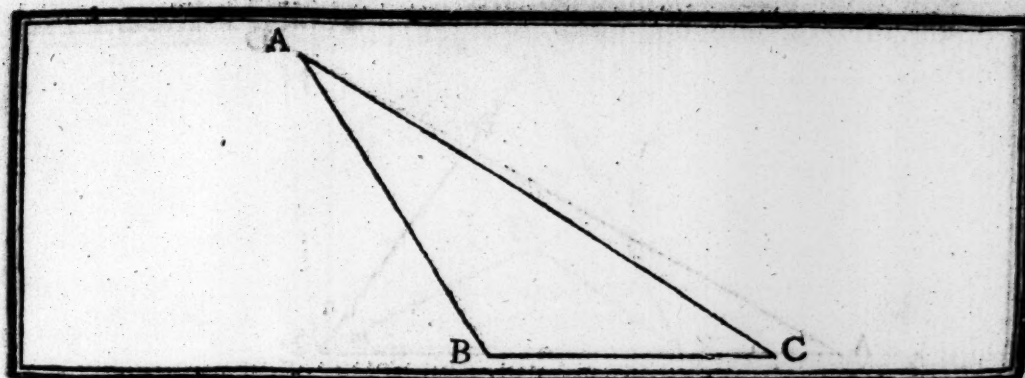
And if to $\angle$ n be added $\angle$ p .

5. Much more will $\angle$ $n + p$ or $\angle$ ACB, opposite to the greater side AB, be $>$ $\angle$ DBC, or ABC, opposite to the lesser side AC.

C. N.

Which was to be demonstrated.





PROPOSITION XIX. THEOREM XII.

IN every triangle (BAC), the greater angle, has the greater side opposite to it.

Hypothesis.
In the $\triangle BAC$, $\angle C$ is $>$ $\angle A$.

Thesis.
The side AB opposite to $\angle C$ is $>$ the side CB opposite to $\angle A$.

DEMONSTRATION.

If not,

The side AB is either equal, or less than the side CB.

C. N.

CASE I. Suppose AB to be $=$ to CB.

BECAUSE the side AB is $=$ to the side CB (*Sup. 1.*).

1. The $\triangle BAC$ is an isosceles $\triangle$.
2. Consequently, the $\angle C$ & $\angle A$ at the base, are $=$ to one another.
But those $\angle C$ & $\angle A$ are not $=$ to one another (*Hyp.*).
3. Therefore neither are the sides AB, CB $=$ to one another.

D. 25. B. 1.

P. 5. B. 1.

CASE II. Suppose AB to be $<$ CB.

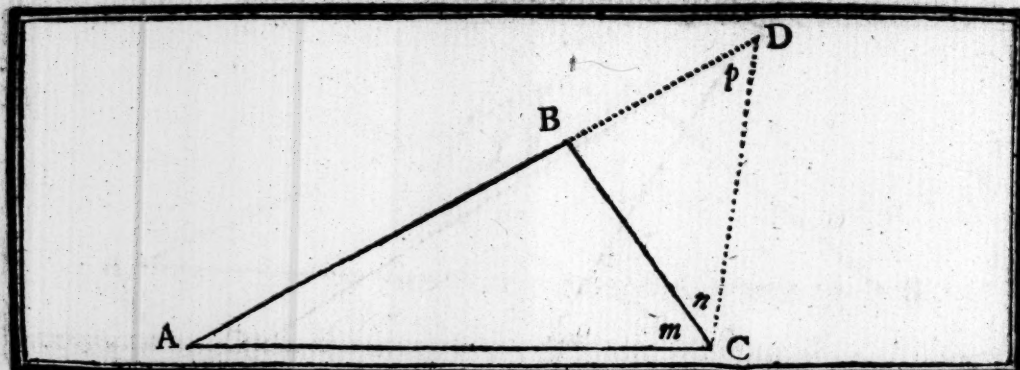
BECAUSE the side AB is $<$ the side CB (*Sup. 2.*).

1. It follows, that $\angle C$ opposite to the lesser side AB, is $<$ $\angle A$ opposite to the greater side CB.
But $\angle C$ is not $<$ $\angle A$ (*Hyp.*).
2. Consequently, the side AB cannot be $<$ the side CB.
The side AB being therefore neither $=$ to the side CB (*Case 1.*); nor $<$ the side CB (*Case 2.*).
3. It follows, that this side AB is $>$ the side CB.

P. 18. B. 1.

C. N.

Which was to be demonstrated.



PROPOSITION XX. THEOREM XIII.

ANY two sides (AB, BC,) of a triangle (ABC) are together greater than the third side (AC).

Hypothesis.
ABC is a $\triangle$.

Thesis.
Any two sides, as AB + BC,
are $>$ the third AC.

Preparation.

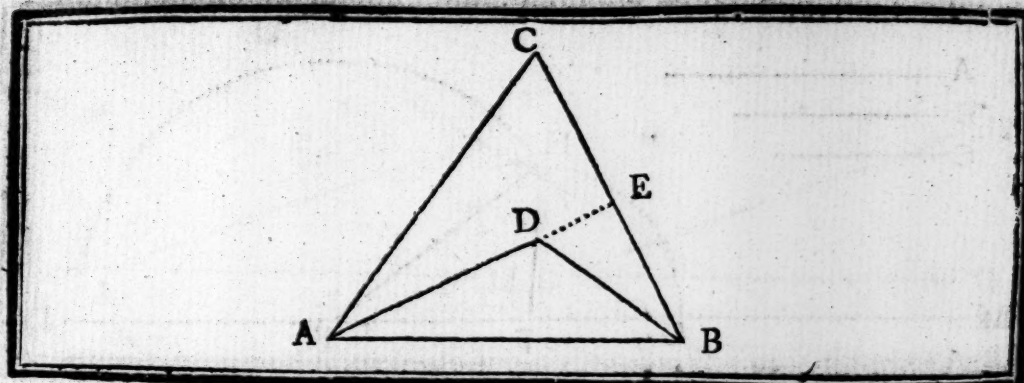
1. Produce one of the two sides, as AB, towards D indefinitely. *Pof. 2.*
2. Make BD = to BC. *P. 3. B. 1.*
3. From the point C to the point D, draw the straight line CD. *Pof. 1.*

DEMONSTRATION.

BECAUSE in the $\triangle BDC$ the side BD is = to the side BC (*Prep. 2.*).

1. This $\triangle$ is an isosceles $\triangle$. *D. 25. B. 1.*
2. Consequently, the $\angle$ at the base n & p are = to one another. *P. 5. B. 1.*
But $\angle m + n$ being the whole, & $\angle n$ its part.
3. It follows, that $\angle m + n$ is $>$ $\angle n$. *Ax. 8.*
But $\angle m + n$ being $>$ $\angle n$ (*Arg. 3.*), & this $\angle n$ being = to $\angle p$, (*Arg. 2.*).
4. It is evident that $\angle m + n$ is $>$ $\angle p$. *C. N.*
Since therefore in the $\triangle ADC$, $\angle m + n$ is $>$ $\angle p$ (*Arg. 4.*).
5. The side AD opposite to the greater $\angle m + n$ is also $>$ the side AC opposite to the lesser $\angle p$. *P. 19. B. 1.*
But because the straight line BD is = to the straight line BC (*Prep. 2.*), if the side AB be added to both.
6. It follows, that AB + BD or AD is = to the sum of the two sides AB + BC. *Ax. 2.*
But AD is $>$ the side AC (*Arg. 5.*).
7. Wherefore, the sum of the two sides AB + BC is also $>$ the third side AC. *C. N.*

Which was to be demonstrated.



PROPOSITION XXI. THEOREM XIV.

IF from the ends (A & B) of the side (AB) of any triangle (ACB) there be drawn to a point (D) within the triangle, two straight lines (DA, DB); these straight lines will be less than the other two sides (CA, CB,) of the triangle; but will contain a greater angle (ADB).

Hypothesis.

DA, DB, are two straight lines drawn from the points A & B to the point D, within the $\triangle$ ACB.

Thesis.

- I. $DA + DB < CA + CB$.
II. $\angle ADB > \angle C$.

Preparation.

Produce the straight line DA, until it meets the side CB in E. *Pos. 2.*

DEMONSTRATION.

BECAUSE the figure ACE is a $\triangle$ (D. 21. B. 1.).

1. The two sides $CA + CE$ are $>$ the third AE. *P. 20. B. 1.*

If the line EB be added to each of these.

2. The sides $CA + CB$ (that is $CA + CE + EB$) are $>$ the lines $AE + EB$. *Ax. 4.*

Again, the figure DEB being also a $\triangle$ (D. 21. B. 1.).

3. The two sides $EB + ED$ are $>$ the third DB. *P. 20. B. 1.*

If we add to each of these the line DA.

4. The lines $AE + EB$ (that is $DA + ED + EB$) are $>$ the lines $DA + DB$. *Ax. 4.*

But it has been proved that the sides $CA + CB$ are $>$ the lines $AE + EB$ (Arg. 2.).

5. Much more then will the sides $CA + CB$ be $>$ the lines $DA + DB$. *C. N.*

Which was to be demonstrated. I.

AGAIN, because $\angle ADB$ is an exterior $\angle$ of $\triangle DEB$ (Prep.), & the $\angle DEB$ is its interior opposite one.

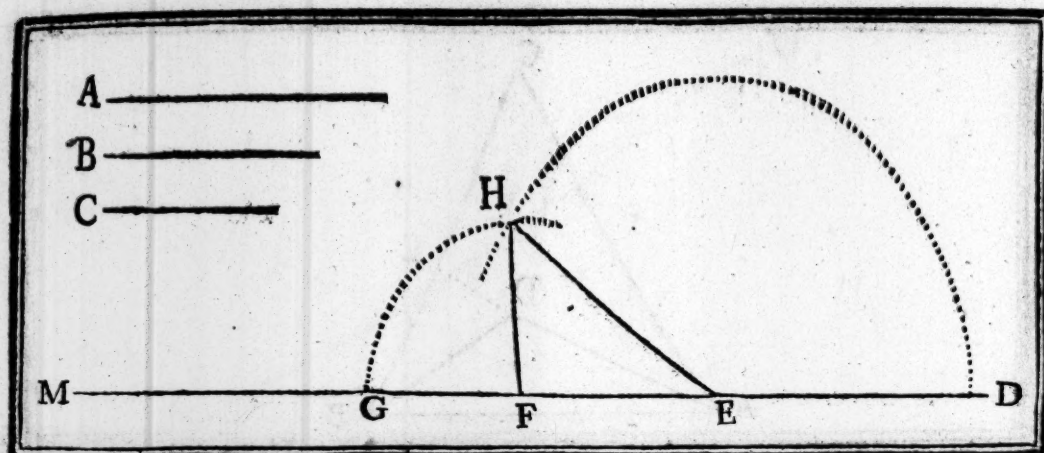
1. It follows, that $\angle ADB$ is $>$ $\angle DEB$. *P. 16. B. 1.*

2. For the same reason; $\angle DEB$ is $>$ $\angle C$.

But since $\angle ADB >$ $\angle DEB$ (Arg. 1.), & $\angle DEB >$ $\angle C$ (Arg. 2.).

3. It is evident, that $\angle ADB$ is much $>$ $\angle C$. *C. N.*

Which was to be demonstrated. II.



PROPOSITION XXII. PROBLEM VIII.

TO make a triangle (FHE) of which the sides shall be equal to three given straight lines (A, B, C,) ; supposing any two whatever of these given straight lines to be greater than the third.

Given

The straight lines A, B, C, such that
 $A + B > C$, $A + C > B$, $C + B > A$.

Sought

The construction of a $\triangle FHE$ such, that
 EH may be $= A$, $FE = B$, & $FH = C$.

Resolution.

1. Draw the indefinite straight line DM. Pos. 1.
2. Make $ED =$ to the given A, $FE =$ to the given B, & $FG =$ to the given C. P. 3. B. 1.
3. From the center E at the distance ED, describe the $\odot DH$. Pos. 3.
4. From the center F at the distance FG, describe the $\odot GH$. Pos. 2.
5. From the points E & F, to the point of intersection H, draw the straight lines EH, FH.

DEMONSTRATION.

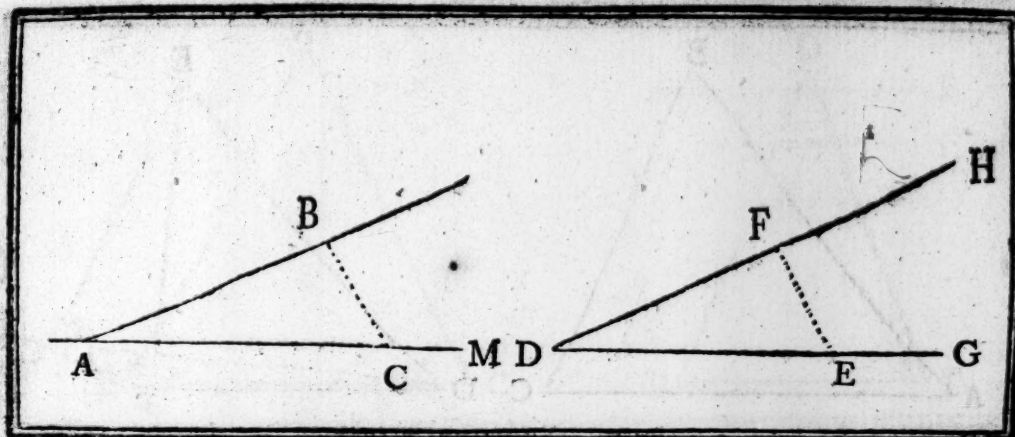
THE straight lines ED, EH, being drawn from the center E to the $\odot DH$ (Ref. 3 & 5.).

1. Those two straight lines ED, EH, are rays of the same $\odot DH$. D. 16. B. 1.
2. Consequently, the straight line ED is $=$ to the straight line EH. D. 15. B. 1.
 Since therefore ED is $=$ to EH (Arg. 2.), & the given straight line A is also $=$ to the same line ED (Ref. 2.).
3. It follows, that EH is $=$ to the given A. Ax. 1.
 After the same manner it will be proved, that
4. The line FH is $=$ to the given C.
 But the side EH being $=$ to the given A (Arg. 3.), the side FH $=$ to the given C (Arg. 4.), & in fine the side FE $=$ to the given B. (Ref. 2.).
5. It is evident, that the three sides EH, FE, FH, of $\triangle FHE$, are $=$ to the three given straight lines A, B, C.

Which was to be done.

REMARK.

THE condition added, that any two of the given lines should be greater than the third, is essential, in consequence of the XX prop. of the I. Book ; without this restriction the circles described from the centers E & F would not cut one another ; a defect which would render the construction impossible.



PROPOSITION XXIII. PROBLEM IX.

AT a given point (A) in a given straight line (AM) to make a rectilineal angle (BAC) equal to another given rectilineal angle (HDG).

Given

- I. An indefinite straight line AM.
- II. The point A in the straight line AM.
- III. The rectilineal angle HDG.

Sought

An angle BAC made on AM, at the point A = to $\angle$ HDG.

Resolution.

1. In the sides DG, DH, of the given $\angle$ HDG, take any two points E & F.
2. From the point E to the point F, draw the straight line EF. *Pos. 1.*
3. Upon the indefinite straight line AM & at the point A, construct a $\triangle$ ABC whose three sides shall be = to the three sides of $\triangle$ DFE.

P. 22. B. 1.

DEMONSTRATION.

BECAUSE the three sides AB, AC, BC, of $\triangle$ ABC are = to the three sides DF, DE, FE, of $\triangle$ DFE, each to each (*Ref. 3.*).

1. It follows, that the $\angle$ BAC & HDG, opposite to the equal sides BC, FE, are = to one another.

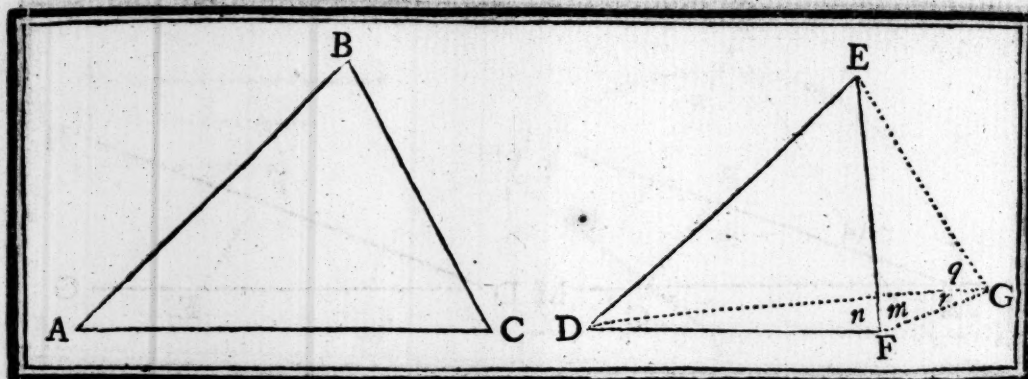
P. 8. B. 1.

But $\angle$ BAC being = to the given $\angle$ HDG; as also made on the the given straight line AM, at the given point A (*Ref. 3.*).

2. It follows, that at the given point A, in the given straight line AM, the rectilineal $\angle$ BAC is made = to the given rectilineal $\angle$ HDG.

Which was to be done.





PROPOSITION XXIV THEOREM. XV.

IF two triangles (ABC , DEF ,) have two sides (BA , BC ,) of the one equal to two sides (ED , EF ,) of the other, each to each; but the angle contained (B) greater than the angle contained (DEF); the base (AC) opposite to the greater angle, will be also greater than the base (DF) opposite to the lesser angle.

Hypothesis.

- I. $BA = ED$.
- II. $BC = EF$.
- III. $\angle B > \angle DEF$.

Thesis

The base AC is $>$ the base DF .

Preparation.

1. At the point E , in the line DE , make $\angle DEG =$ to the given $\angle B$.
2. Make $EG =$ to BC or to EF .
4. From the points D & F to the point G , draw the straight lines DG , FG .

P. 23. B. 1.

P. 3. B. 1.

Pos. 1.

DEMONSTRATION.

BECAUSE in the $\triangle ABC$ the sides BA , BC , are $=$ to the sides ED , EG , of $\triangle DEG$ (*Hyp. 1, Prep. 2.*), & $\angle$ contained $B =$ to $\angle$ contained DEG (*Prep. 1.*).

1. It follows, that the base AC is $=$ to the base DG .
Again, because EG is $=$ to the side EF (*Prep. 2, Hyp. 2.*).
2. The $\triangle FEG$ is an isosceles $\triangle$.
3. Consequently, $\angle m = \angle r + q$.
Since therefore $\angle m = \angle r + q$ (*Arg. 3.*); if from the last be taken its part q .
4. The $\angle m$ will be $> \angle r$.
And if to $\angle m$ be added $\angle n$.
5. Much more will the whole $\angle m + n$ be $> \angle r$.
6. Consequently, the side DG opposite to the greater $\angle m + n$, is $>$ the side DF opposite to the lesser $\angle r$.
But the straight line DG being $> DF$ (*Arg. 6.*), & this same straight line DG being $=$ to the base AC (*Arg. 1.*).
7. It is evident that the base AC is also $>$ the base DF .

P. 4. B. 1.

D. 25. B. 1.

P. 5. B. 1.

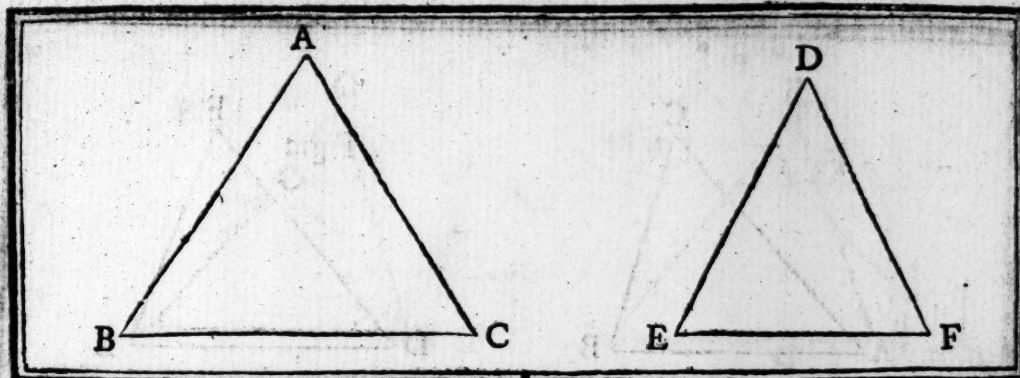
C. N.

C. N.

P. 19. B. 1.

C. N.

Which was to be demonstrated.



PROPOSITION XXV. THEOREM XVI.

IF two triangles (BAC, EDF,) have two sides of the one equal to two sides of the other, each to each, but the base (BC) of the one greater than the base (EF) of the other; the angle (BAC) opposite to the greater base (BC), will be also greater than the angle (D) opposite to the lesser base (EF).

Hypothesis,

- I. $AB = DE$.
- II. $AC = DF$.
- III. $BC > EF$.

Thesis,

The angle A opposite to the greater base BC, is $> \angle D$ opposite to the lesser base EF.

DEMONSTRATION.

If not,

The angle A is either equal or less than the angle D.

C. N.

CASE I. Suppose $\angle A$ to be $=$ to $\angle D$.

BECAUSE $\angle A$ is $=$ to $\angle D$ (Sup. 1.), & the sides AB, AC, & DE, DF, which contain those $\angle$ s, are equal each to each, (Hyp. 1 & 2.).

1. The base BC is $=$ to the base EF.

P. 4. B. 1.

But the base BC is not $=$ to the base EF (Hyp. 3.).

2. Therefore $\angle A$ cannot be $=$ to $\angle D$.

CASE II. Suppose $\angle A$ to be $<$ $\angle D$.

BECAUSE $\angle A$ is $<$ $\angle D$ (Sup. 2.), & the sides AB, AC, & DE, DF, which contain those $\angle$ s are equal, each to each, (Hyp. 1 & 2.).

1. The base BC is $<$ the base EF.

P. 24. B. 1.

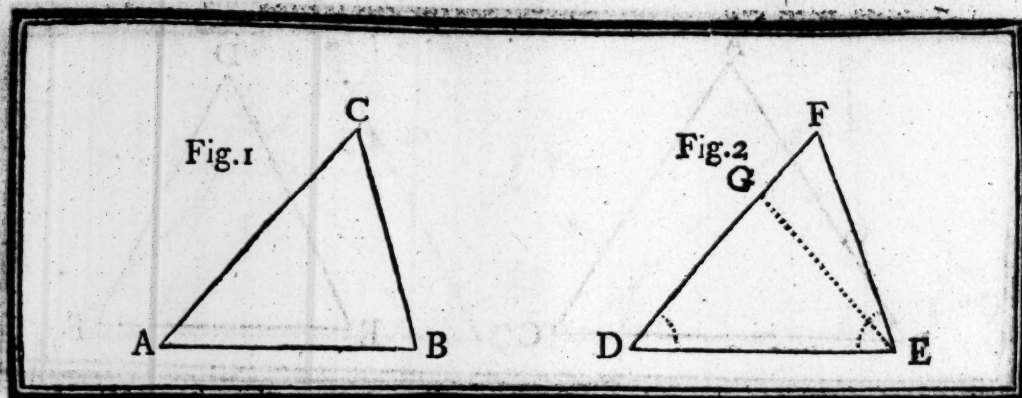
But the base BC is not $<$ the base EF (Hyp. 3.).

2. Therefore $\angle A$ is not $<$ $\angle D$.

But it has been shewn that neither is it equal to it (Case. 1.).

3. Consequently, $\angle A$, which is opposite to the greater base BC, is $>$ $\angle D$, which is opposite to the lesser base EF.

Which was to be demonstrated.



PROPOSITION XXVI. THEOREM XVII.

IF two triangles (ACB, DFE,) have two angles (A & B) of one, equal to two angles (D & FED) of the other, each to each, & one side equal to one side, viz. either the sides, as (AB & DE) adjacent to the equal angles; or the sides, as (AC & DF) opposite to equal angles in each: then shall the two other sides (AC, BC, or AB, BC,) be equal to the two other sides (DF, EF, or DE, EF,) each to each, & the third angle (C) equal to the third angle (F).

Hypothesis.

- I. $\angle A = \angle D$.
- II. $\angle B = \angle FED$.
- III. $AB = DE$.

CASE I.

When the equal sides AB, DE, are adjacent to the equal angles A & D, B & FED (Fig. 1 & 2.).

Thesis.

- I. $AC = DF$.
- II. $BC = EF$.
- III. $\angle C = \angle F$.

DEMONSTRATION.

If not,

The sides are unequal, & one, as DF will be $>$ the other AC.

Preparation.

1. Cut off from the greater side DF a part DG = to AC. P. 3. B. 1.
2. From the point G to the point E, draw the straight line GE. Pof. 1.

BECAUSE in the $\triangle ACB$, DGE, the side AC is = to the side DG, (Prep. 1.), $AB = DE$ (Hyp. 3.), & $\angle A$ is = to $\angle D$. (Hyp. 1.).

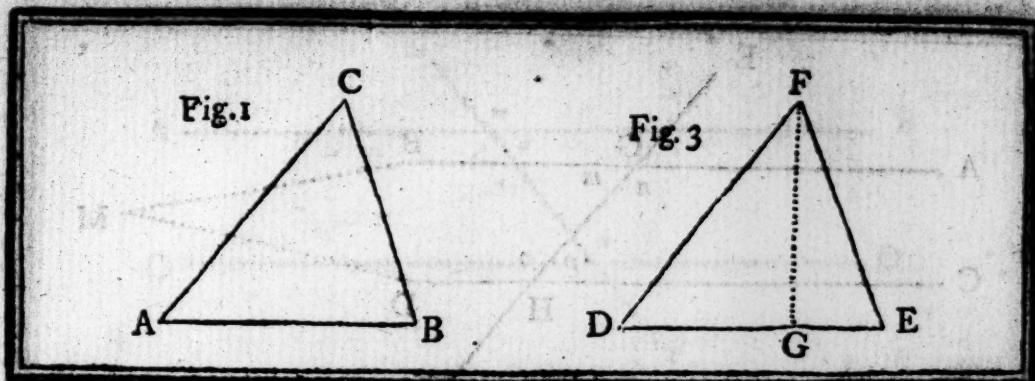
1. The $\angle B$ & GED opposite to the equal sides AC & DG are equal. P. 4. B. 1.
But $\angle B$ being = to $\angle GED$ (Arg. 1.), & this same $\angle B$ being also = to $\angle FED$ (Hyp. 2.).
2. It follows, that $\angle GED$ is = to $\angle FED$. Ax. 1.
But $\angle FED$ being the whole & $\angle GED$ its part :
3. The whole would be = to its part.
4. Which is impossible. Ax. 8.
5. The sides AC, DF, are therefore not unequal.
6. Consequently, they are equal, or $AC = DF$. C. N.

Which was to be demonstrated. I.

Since then in the $\triangle ACB$, DFE, the side AC is = to the side DF, (Arg. 6.), $AB = DE$ (Hyp. 3.), & $\angle A$ is = to $\angle D$ (Hyp. 1.).

7. The third side BC is also = to the third side EF, & the $\angle C$ & F, opposite to the equal sides AB, DE, are also = to one another. P. 4. B. 1.

Which was to be demonstrated, II & III.



CASE II.

Hypothesis.

- I. $\angle A = \angle D$.
- II. $\angle B = \angle E$.
- III. $AC = DF$.

When the equal sides AC, DF,
are opposite to the equal angles
B & E. (Fig. 1. & 3.)

Thesis.

- I. $AB = DE$.
- II. $BC = EF$.
- III. $\angle C = \angle F$.

DEMONSTRATION.

If not,

The sides AB, DE, are unequal; and one, as DE, will be $>$ the other AB.

Preparation.

1. Cut off from the greater side DE, a part DG = to AB.

P. 3. B. 1.

2. From the point G to the point F, draw the straight line GF.

Pos. 1.

BECAUSE then in the $\triangle ACB$, DFG , the side AC is = to the side DF (Hyp. 3.), $AB = DG$ (Prep. 1.), & $\angle A$ is = to $\angle D$, (Hyp. 1.).

1. The other $\angle B$ & $\angle DGF$, to which the equal sides AC, DF, are opposite, are = to one another.

P. 4. B. 1.

The angle B being therefore = $\angle DGF$ (Arg. 1.), & this same $\angle B$ being also = to $\angle E$ (Hyp. 2.).

2. It follows, that $\angle E$ is = to $\angle DGF$.

Ax. 1.

But $\angle DGF$ is an exterior $\angle$ of $\triangle GFE$, & $\angle E$, is its interior opposite one.

3. Therefore the exterior $\angle$ will be equal to its interior opposite one.

P. 16. B. 1.

4. Which is impossible.

5. Consequently, the sides AB, DE, are not unequal.

C. N.

6. They are therefore equal, or $AB = DE$.

Which was to be demonstrated. I.

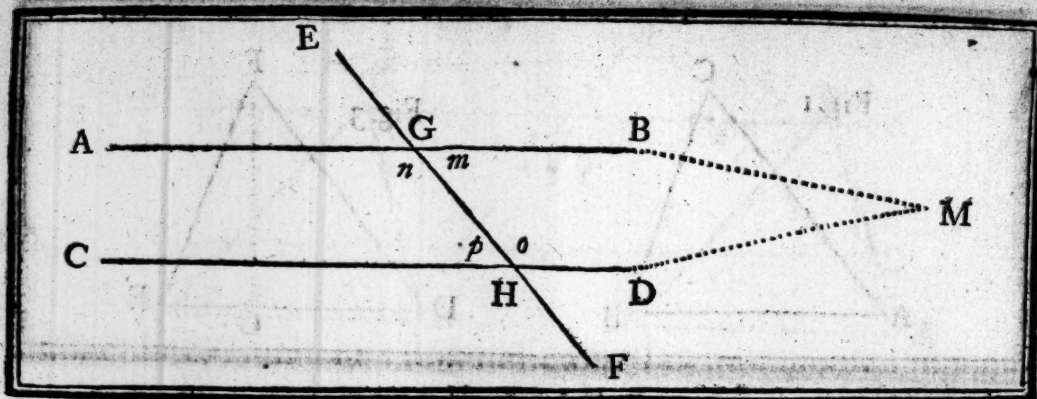
Since then in the $\triangle ACB$, DFE , the side AC is = to the side DF, (Hyp. 3.), $AB = DE$ (Arg. 6.), & $\angle A$ is = to $\angle D$ (Hyp. 1.).

7. It is evident, that the third side BC is = to the third side EF, & the $\angle C$ & $\angle F$, to which the equal sides AB, DE, are opposite, are equal to one another.

P. 4. B. 1.

Which was to be demonstrated. II. & III.

F



PROPOSITION XXVII. THEOREM XVIII.

IF a straight line (EF), falling upon two other straight lines (AB, CD,) situated in the same plane, makes the alternate angles (m & p , or n & o ;) equal to one another: these two straight lines (AB, CD,) shall be parallel.

Hypothesis.

Thesis.

- I. AB, CD, are two straight lines in the same plane.
 II. The line EF cuts them so that $\angle m = \angle p$, or $\angle n = \angle o$.

The lines AB, CD,
 are plle.

DEMONSTRATION.

If not,

The straight lines AB, CD, produced will meet either towards BD or towards AC.

D. 35. B. 1.

Preparation.

Let them be produced & meet towards BD in the point M.

Pos. 2.

BECAUSE the $\angle n$ is an exterior angle of $\triangle GMH$, & $\angle o$ its interior opposite one.

1. The $\angle n$ is $> \angle o$.

P. 16. B. 1.

But $\angle n$ is $=$ to $\angle o$. (Hyp. 2.)

2. This $\angle n$ is therefore not $> \angle o$.

C. N.

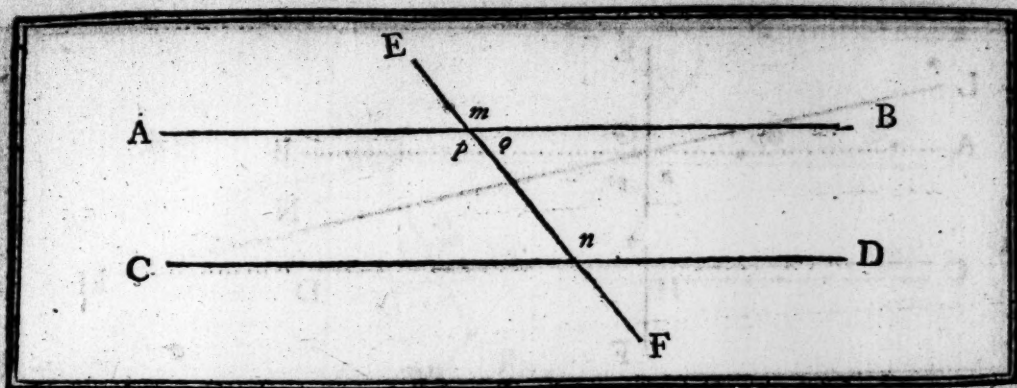
3. Consequently, it is impossible that the straight lines AB, CD, should meet in a point as M.

4. From whence it follows that they are plle straight lines.

D. 35. B. 1.

Which was to be demonstrated.





PROPOSITION XXVIII. THEOREM XIX.

IF a straight line (EF) falling upon two other straight lines (AB, CD,) situated in the same plane, makes the exterior angle (*m*) equal to the interior & opposite (*n*) upon the same side, or makes the interior angles (*o* + *n*) upon the same side equal to two right angles; those two straight lines AB, CD, shall be parallel to one another.

CASE I.

Hypothesis.
 $\angle m = \angle n$.

Thesis.
AB, CD, are p^lle lines.

DEMONSTRATION.

BECAUSE the $\angle m$ & p are vertical or opposite $\angle$.

1. They are $\equiv$ to one another.

The $\angle p$ being therefore $\equiv$ to $\angle m$ (*Arg. 1.*), & $\angle n$ being $\equiv$ to the same $\angle m$ (*Hyp.*).

P. 15. B. 1.

2. It is evident that $\angle p$ is also $\equiv$ to $\angle n$.

Ax. 1.

But the equal $\angle p$ & n (*Arg. 2.*), are also alternate $\angle$.

3. Consequently, the straight lines AB, CD, are p^lle.

P. 27. B. 1.

CASE II.

Hypothesis.
The $\angle o + n$ are $\equiv$ to 2 $\angle$.

Thesis.
AB, CD, are p^lle. lines.

DEMONSTRATION.

BECAUSE the straight line EF falling upon the straight line AB, forms with it the adjacent $\angle o$ & p .

1. Those $\angle o + p$ are $\equiv$ to two $\angle$.

P. 13. B. 1.

The $\angle o + p$ being therefore $\equiv$ to two $\angle$. (*Arg. 1.*), & the $\angle o + n$ being also $\equiv$ to two $\angle$ (*Hyp.*).

2. It follows, that the $\angle o + p$ are $\equiv$ to $\angle o + n$.

Ax. 1.

And if the common angle *o* be taken away from both sides.

3. The remaining $\angle p$ & n will be equal to one another.

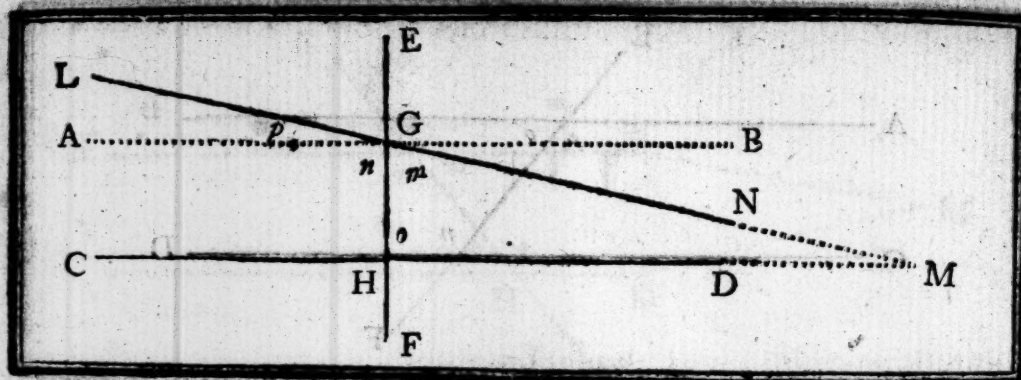
Ax. 3.

But those equal $\angle p$ & n (*Arg. 3.*), are at the same time alternate $\angle$.

4. Consequently, the straight lines AB, CD, are p^lle.

P. 27. B. 1.

Which was to be demonstrated.



L E M M A.

IF a straight line (EF), meeting two straight lines (LN, CD,) situated in the same plane, makes the alternate angles ($p + n$ & o) unequal; those two straight lines (LN & CD,) being continually produced, will at length meet in (M), upon that side on which is the lesser of the alternate angles (o).

Preparation.

For since $\angle p + n$ is supposed $> \angle o$.

1. There may be made in the greater $\angle p + n$, on the straight line EF, at the point G, an angle $n = \angle o$.
2. And AG may be produced at will to B.

P. 23. B. 1.
Pos. 2.

DEMONSTRATION.

BECAUSE the two lines AB, CD, are cut by a third EF, so that the alternate $\angle n$ & o are $=$ to one another (*Prep. 1.*).

1. Those two lines AB, CD, are p^lle.

P. 27. B. 1.

But the line LN cuts one of the two p^lles, viz. AB in G.

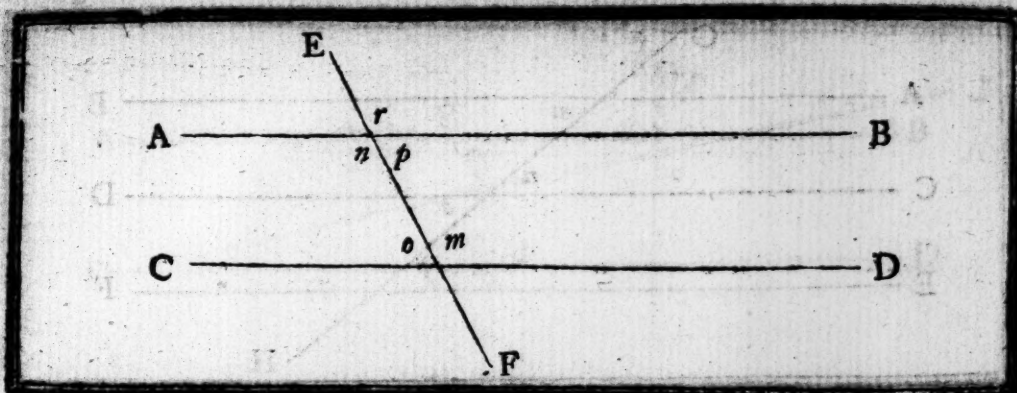
2. Therefore, if produced sufficiently, it will cut also the other CD somewhere in M, upon that side on which is the lesser of the alternate $\angle o^*$. C. N.

Which was to be demonstrated.

C O R O L L A R Y.

WHEN $\angle o < \angle p + n$, the two interior angles $o + m$ are necessarily $<$ two $\angle$; since the two angles $p + n$ & m are equal to two $\angle$. P. 13. B. 1. Consequently, when the two interior $\angle$ s are $<$ two $\angle$; the lines LN, CD, which form those angles with EF, will meet somewhere on the side of the line EF, where those angles are situated, provided they are produced sufficiently.

* Euclid regards as a self evident principle that, a straight line (EF), which cuts one of two parallels as (AB) will necessarily cut the other (CD), provided this cutting line (EF) be sufficiently produced. See the prep. of propositions XXX, XXXVII, and of several others.



PROPOSITION XXIX. THEOREM XX.

IF a straight line (EF), falls upon two parallel straight lines (AB, CD), it makes the alternate angles (n & m) equal to one another; and the exterior angle (r) equal to the interior & opposite upon the same side (m); and likewise the two interior angles upon the same sides ($p + m$) equal to two right angles.

Hypothesis.

AB, CD, are two pple lines, cut by
the same straight line EF.

Thesis.

- I. $\angle n = \angle m$.
II. $\angle r = \angle m$.
III. $\angle p + m = 2\angle$.

DEMONSTRATION.

If not,

The $\angle m$ & n are unequal,

And one of them as $\angle m$ will be $<$ the other $\angle n$.

C. N.

BECAUSE the $\angle m$ is $<$ $\angle n$; if the $\angle p$ be added to both.

1. The $\angle m + p$ will be $<$ the $\angle n + p$.

Ax. 4.

But since the $\angle n$ & $\angle p$ are adjacent $\angle$ s, formed by the straight line EF which falls upon AB.

2. These $\angle n + p$ are $=$ to two $\angle$ s.

P. 13. B. 1.

3. Consequently, the $\angle m + p$ (less than the $\angle n + p$) are also $<$ two $\angle$ s.

C. N.

4. From whence it follows, that the lines AB, CD, are not pple.

Cor. of lem.

But the straight lines AB, CD, are pple. (Hyp.).

5. Consequently, the $\angle m$ & n are not unequal.

P. 27. B. 1.

6. They are therefore equal, or $\angle n = \angle m$.

C. N.

Which was to be demonstrated. I.

Moreover, $\angle r$ & $\angle n$ being vertically opposite.

7. These angles are $=$ to one another.

P. 15. B. 1.

But $\angle m$ being $=$ to $\angle n$ (Arg. 6.), & $\angle r$ being $=$ to the same $\angle n$, (Arg. 7.).

8. It follows, that $\angle r$ is $=$ to $\angle m$.

Ax. 1.

Which was to be demonstrated. II.

Likewise, $\angle n$ being $=$ to $\angle m$ (Arg. 6.); if $\angle p$ be added to both.

9. The $\angle n + p$ will be $=$ to $\angle m + p$.

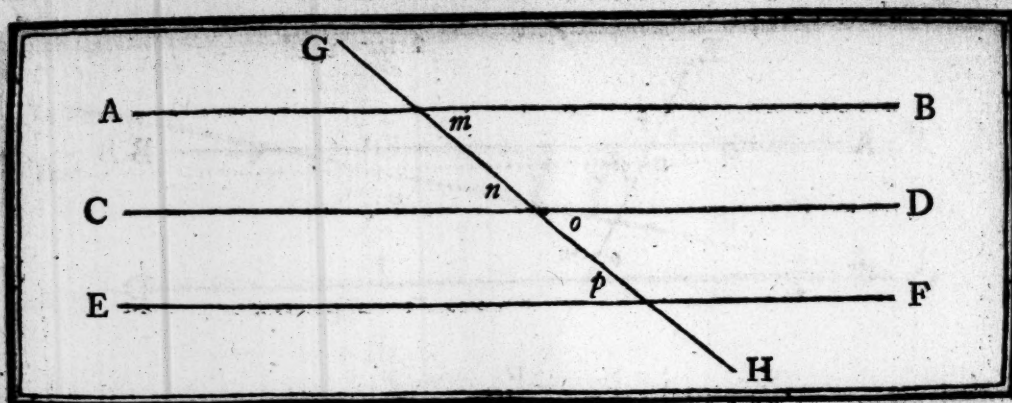
Ax. 2.

But the $\angle n + p$ are $=$ to two $\angle$ s (Arg. 2.).

10. From whence it follows that the $\angle m + p$ are also $=$ to two $\angle$ s.

Ax. 1.

Which was to be demonstrated. III.



PROPOSITION XXX. THEOREM XXI.
THE straight lines (AB, EF), which are parallel to the same straight line (CD), are parallel to one another.

Hypothesis.

AB, EF, are straight lines, p^lle to CD.

Thesis.

The straight lines AB, EF are p^lle to one another.

Preparation.

Draw the straight line GH, cutting the three lines AB, CD, EF.

DEMONSTRATION.

BECAUSE the straight lines AB, CD, are two p^lles, (*Hyp.*) cut by the same straight line GH. (*Prep.*)

1. The alternate $\angle m$ & n are = to one another.

P. 29. B. 1.

Likewise since the straight lines CD, EF are two p^lles. (*Hyp.*) cut by the same straight line GH. (*Prep.*)

2. The exterior angle n is = to its interior opposite one on the same side p . *P. 29. B. 1.*

3. But $\angle n$ being = to $\angle m$ (*Arg. 1.*) & the same $\angle n$ being also = to $\angle p$ (*Arg. 2.*)

3. The $\angle m$ & p will be = to one another.

Ax. 1.

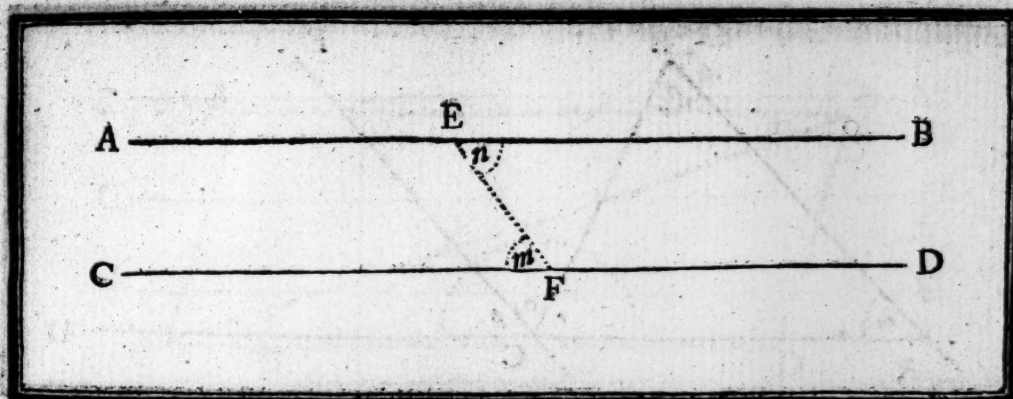
But these $\angle m$ & p (*Arg. 3.*) are alternate $\angle$ s, formed by the two straight lines AB, EF, which are cut by the straight line GH.

4. Consequently, these straight lines AB, EF are p^lle.

P. 27. B. 1.

Which was to be demonstrated





PROPOSITION XXXI. PROBLEM X.

TO draw a straight line (AB), thro' a given point (E), parallel to a given straight line (CD).

Given

The straight line CD and the point E.

Sought

*The straight line AB, p^{lle} to CD,
& passing thro' the point E.*

Resolution.

1. In the given straight line CD take any point F.
2. From the point F to the point E, draw the straight line FE. *Pos. 1.*
3. At the point E in the straight line FE, make $\angle n = \angle m$. *P. 23. B. 1.*
4. And produce the side EB to A. *Pos. 2.*

DEMONSTRATION.

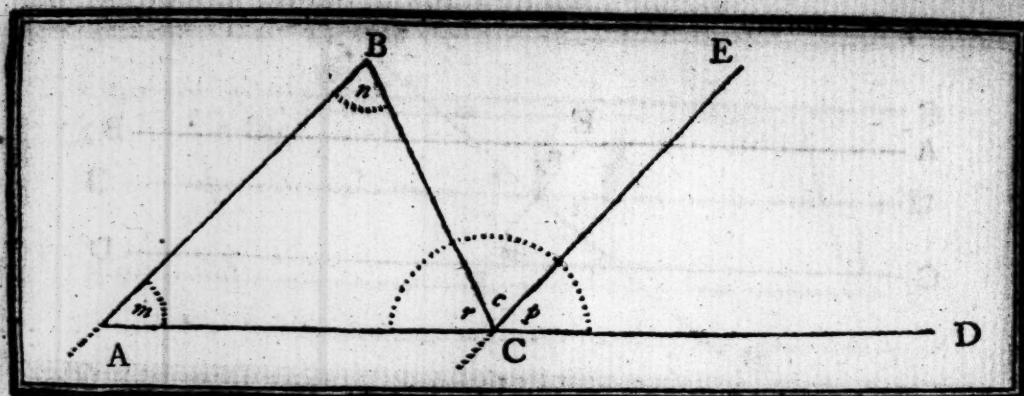
BECAUSE the alternate $\angle m$ & n , formed by the straight line EF, which cuts the two lines AB, CD, are $=$ to one another (*Ref. 3.*).

1. The straight lines AB, CD, are p^{lle}.

P. 27. B. 1.

Which was to be demonstrated.





PROPOSITION XXXII. THEOREM XXII.

IF a side as (AC) of any triangle (ABC) be produced, the exterior angle ($c + p$) is equal to the sum of the two interior and opposite angles ($n + m$); and the three interior angles ($n + m + r$) are equal to two right angles.

Hypothesis.

ABC is a Δ , one of whose sides AC, is produced indefinitely to D.

Thesis.

I. $\forall c + p$ is $\equiv$ to $\forall m + n$.

II. the $\forall n + m + r$ are $\equiv$ to $2\angle$.

Preparation.

Thro' the point C, draw the straight line CE, pple to the straight line AB.

P. 31. B. 1.

DEMONSTRATION.

BECAUSE the straight lines AB, CE, are two plles (*Prep.*) cut by the same straight line BC.

1. The alternate $\forall n$ & c are $\equiv$ to one another.

P. 29. B. 1.

Likewise because the straight line AB, CE, are two plles (*Prep.*) cut by the same straight line AD.

2. The exterior angle p is $\equiv$ to its interior opposite one m , on the same side.

P. 29. B. 1.

The $\forall c$ being therefore $\equiv$ to $\forall n$ (*Arg. 1.*), & $\forall p = \forall m$, (*Arg. 2.*).

3. The $\forall c + p$ is $\equiv$ to the $\forall n$ & m taken together.

Ax. 2.

Which was to be demonstrated. I.

Since then the $\forall c + p$ is $\equiv$ to $\forall n + m$ (*Arg. 3.*); if the $\forall r$ be added to both sides.

4. The $\forall c + p + r$ will be $\equiv$ to the three $\forall n + m + r$ of the Δ ABC. Ax. 2.

But these $\forall c + p + r$ are the adjacent $\forall$, formed by the line BC, which meets AD at the same point C.

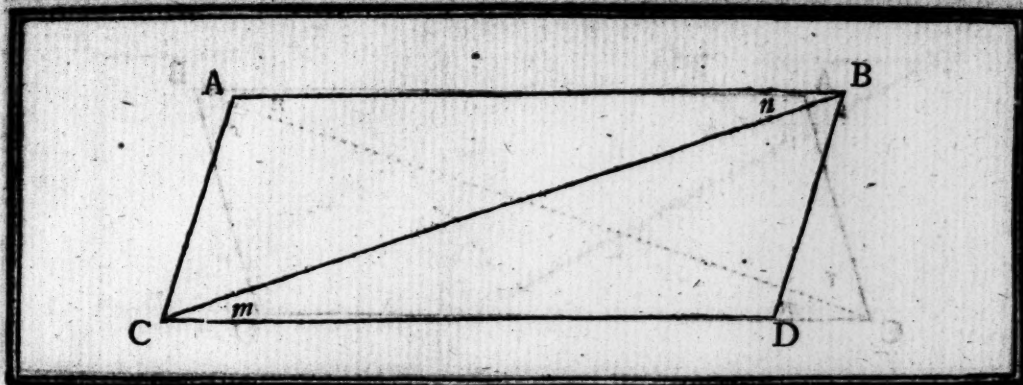
5. Consequently, the $\forall c + p + r$ are $\equiv$ to two $\angle$.

P. 13. B. 1.

Wherefore, the three $\forall n + m + r$, which are $\equiv$ to $\forall c + p + r$, (*Arg. 4.*) are also $\equiv$ to two $\angle$.

Ax. 1.

Which was to be demonstrated. II.



PROPOSITION XXXIII. THEOREM XXIII.

THE straight lines (AC, BD,) which join the extremities (A, C, & B, D,) of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

Hypothesis.

AC, BD, are two straight lines, which join towards the same parts, the extremities of two = & p^{lle} straight lines AB, CD.

Thesis.

I. The straight lines AC, BD, are equal.
II. And those straight lines AC, BD, are p^{lle}.

Preparation.

From the point B to the point C, draw the straight line BC.

DEMONSTRATION.

BECAUSE the straight lines AB, CD, are two p^{lle}s (*Hyp.*) cut by the same straight line BC (*Prep.*).

1. The alternate $\angle n$ & m are = to one another.
Since therefore in the two $\triangle CAB$, BDC , the side CD is = to the side AB (*Hyp.*), the side BC is common to the two $\triangle$, & the $\angle m$ is = to the $\angle n$ (*Arg. 1.*).

P. 29. B. 1.

2. It follows, that the base AC is = to the base BD.

Which was to be demonstrated. I.

3. Likewise that the $\angle ACB$, DBC , to which the equal sides AB, CD, are opposite, are also = to one another.

P. 4. B. 1.

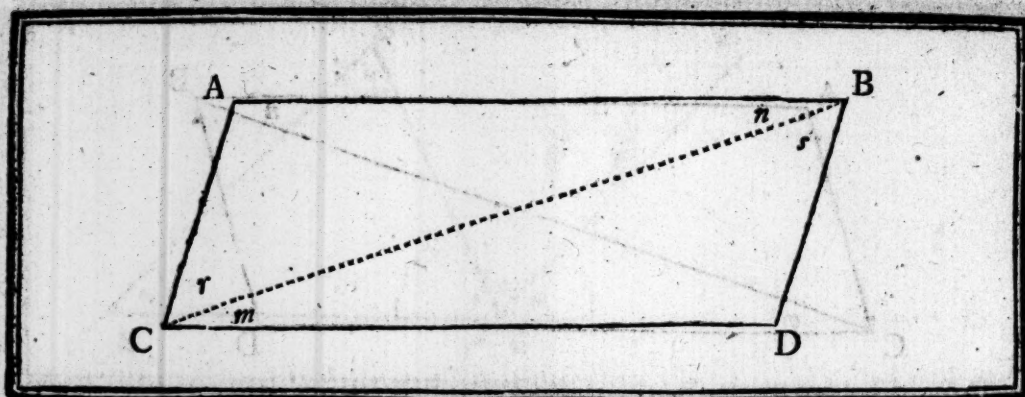
But those equal $\angle ACB$, DBC , (*Arg. 3.*) are alternate $\angle$ formed by the straight lines AC, BD, cut by the straight line BC.

4. Consequently, the straight lines AC, BD, are p^{lle}.

P. 27. B. 1.

Which was to be demonstrated. II.

G



PROPOSITION XXXIV. THEOREM XXIV.

THE opposite sides (AC, BD, & CD, AB,) and the opposite angles (A, D, & $m + r$, $n + s$) of a parallelogram (AD) are equal to one another, & the diagonal (BC) divides it into two equal parts.

Hypothesis.

- I. AD is a Pgr.
- II. BC is the diagonal of this Pgr.

Thesis.

- I. The sides AC, BD, & CD, AB, are = to one another, & $\forall A = D$.
- II. $\forall m + r = \forall n + s$.
- III. The $\triangle CAB$, BDC , formed by the diagonal, are = to one another.

DEMONSTRATION.

BECAUSE the straight lines AB, CD, are two plles (Hyp. 1.) cut by the same straight line CB (Hyp. 2.).

1. The alternate $\forall m$ & n are = to one another. P. 29. B. 1.
Again, because the straight lines AC, BD, are two plles (Hyp. 1.) cut by the same straight line CB (Hyp. 2.).
2. The alternate $\forall r$ & s are = to one another. P. 29. B. 1.
But the $\triangle BDC$, CAB , have two $\forall m$ & s = to two $\forall n$ & r , (Arg. 1 & 2.), & the side BC adjacent to those equal $\forall$ is common to the two $\triangle$.
3. Consequently, the sides AC & BD, opposite to the equal $\forall n$ & m , also the sides CD, AB, opposite to the equal $\forall s$ & r , are = to one another, & the third $\forall A$ is = to the third $\forall D$. P. 26. B. 1.

Which was to be demonstrated. I.

But $\forall m$ being = to $\forall n$ (Arg. 1.), & $\forall r = \forall s$ (Arg. 2.).

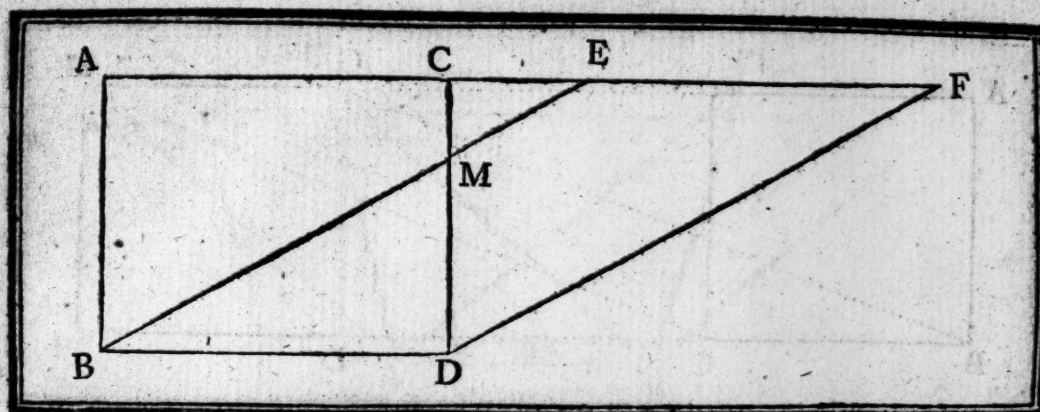
4. The whole $\forall m + r$ is = to the whole $\forall n + s$. Ax. 2.

Which was to be demonstrated. II.

In fine, because in the $\triangle CAB$, BDC , the side CD is = to the side AB, (Arg. 3.), the side BC is common to the two $\triangle$; and $\forall m$ is = to $\forall n$ (Arg. 1.).

5. Those two $\triangle CAB$, BDC , formed by the diagonal BC, are = to one another. P. 4. B. 1.

Which was to be demonstrated. III.



PROPOSITION XXXV. THEOREM XXV.

PARALLELOGRAMS (AD, ED,) upon the same base (BD) & between the same parallels (AF, BD,); are equal to one another.

Hypothesis.

- I. AD & ED are two Pgrs.
 II. And those two Pgrs, are upon the same base BD, & between the same plles AF, BD.

Thesis.

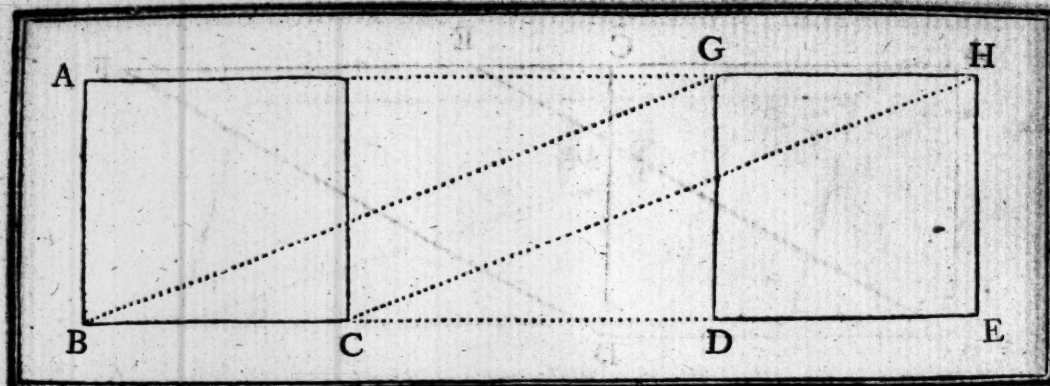
The Pgr AD is = to the Pgr ED.

DEMONSTRATION.

BECAUSE the figure AD is a Pgr (Hyp. 1.).

1. The opposite sides AC, BD, & AB, CD, are = to one another. P. 34. B. 1.
 Likewise, because the figure ED is a Pgr (Hyp. 1.).
2. The opposite sides EF, BD, & BE, DF, are = to one another. P. 34. B. 1.
 But the straight line AC being = to the straight line BD (Arg. 1.), & the straight line EF being also = to the same straight line BD (Arg. 2.).
3. It follows, that the straight line AC, is = to the straight line EF. Ax. 1.
 Since therefore AC is = to EF (Arg. 3.); if CE be added to both.
4. The straight line AE is necessarily = to the straight line CF. Ax. 2.
 Therefore in the $\triangle ABE$, CDF, the side AB is = to the side CD, (Arg. 1.), the side BE is = to the side DF (Arg. 2.), & the base AE is = to the base CF (Arg. 4.).
5. Consequently, the $\triangle ABE$ is = to the $\triangle CDF$. P. 8. B. 1.
 Taking away therefore from those equal $\triangle ABE$, CDF, (Arg. 5.) their common part CME.
6. The remaining trapeziums ABMC, MD FE, are = to one another. Ax. 3.
 Adding in fine to those equal trapeziums ABMC, MD FE, (Arg. 6.) the common part MBD.
7. The Pgrs AD & ED will be = to one another. Ax. 2.

Which was to be demonstrated.



PROPOSITION XXXVI. THEOREM XXVI.

PARALLELOGRAMS (AC, GE,) upon equal bases (BC, DE,) & between the same parallels (AH, BE,) are equal to one another.

Hypothesis.

- I. AC, GE, are two Pgrs.
- II. And those two pgrs are upon equal bases BC, DE, & between the same plles AH, BE.

Thesis.

The Pgr AC is = to the Pgr GE.

Preparation.

1. From the point B to the point G, draw the straight line BG. } *Pos. 1.*
2. From the point C to the point H, draw the straight line CH. }

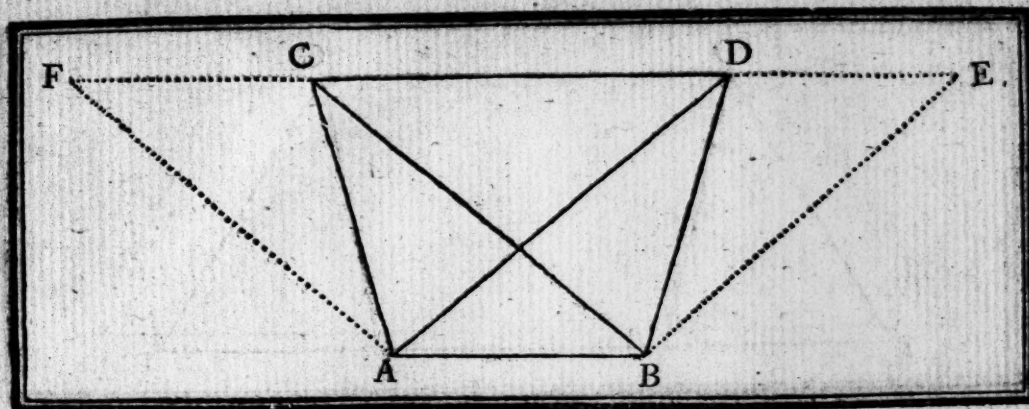
DEMONSTRATION.

BECAUSE the figure GE is a Pgr (*Hyp. 1.*).

1. The opposite sides DE, GH, are = to one another. *P. 34. B. 1.*
But the straight line BC is = to DE (*Hyp. 2.*), & GH is = to the same straight line DE (*Arg. 1.*).
2. Therefore BC is = to GH. *Ax. 1.*
But since BC is = to GH (*Arg. 2.*); & they are plles (*Hyp. 2.*) whose extremities are joined by the straight lines GB, HC, (*Prep. 1 & 2.*).
3. It is evident that those straight lines GB, HC, are = & plle. *P. 33. B. 1.*
4. Consequently, the figure GC is a Pgr. *D. 35. B. 1.*
Moreover, the Pgrs AC, GC, being upon the same base BC, & between the same plles AH, BE, (*Hyp. 2.*).
5. Those Pgrs AC, GC, are = to one another. *P. 35. B. 1.*
It will be proved after the same manner.
6. That the Pgr GC is = to the Pgr GE.
Since therefore the pgr AC is = to the pgr GC (*Arg. 5.*), & the Pgr GE is = to the same Pgr GC (*Arg. 6.*).
7. It follows, that the Pgr AC is = to the Pgr GE.

Ax. 1.

Which was to be demonstrated.



PROPOSITION XXXVII. THEOREM XXVII.

TRIANGLES (ACB, ADB,) upon the same base (AB) & between the same parallels (AB, CD,) are equal to one another.

Hypothesis.

- I. ACB, ABD, are two Δ .
- II. And those two Δ are upon the same AB, & between the same plles AB, CD.

Thesis.

The Δ ACB is = to the Δ ADB,

Preparation.

1. Produce the straight line CD both ways to E & F. Pas. 2.
2. Thro' the points A & B, draw the straight lines AF, BE, pple to the sides BC, AD; which will meet the produced CD P. 31. B. 1. somewhere in F & in E.

DEMONSTRATION.

BECAUSE in the figure BF the opposite sides AB, FC, & AF, BC, are pple (*Hyp. 2 & Prep. 2.*).

1. The figure BF is a Pgr. D. 35. B. 1.

It will be proved after the same manner,

2. That the figure AE is a Pgr.

But the Pgrs BF, AE, are upon the same base AB and between the same plles AB, FE, (*Hyp. 2 & Prep. 1.*).

3. Consequently, the Pgr BF is = to the Pgr AE. P. 35. B. 1.

But the straight lines AC, BD, are the diagonals of the Pgrs BF, AE, (*Prep. 1 & 2.*).

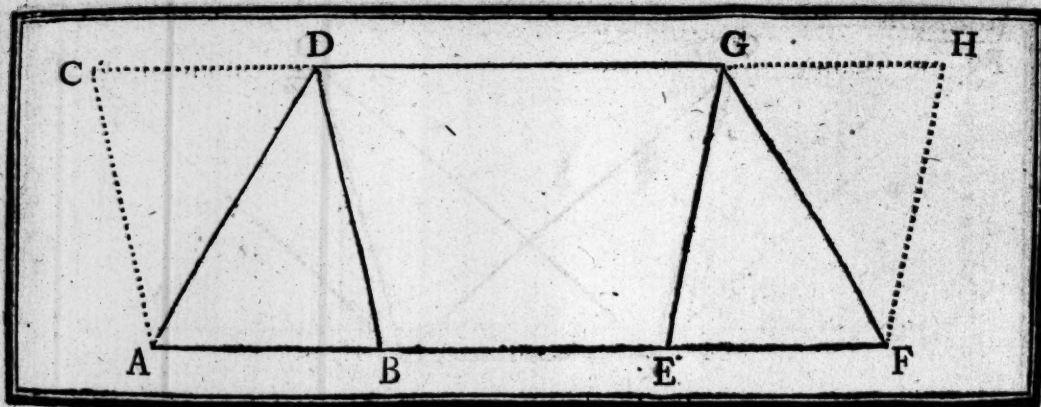
4. Wherefore those diagonals AC, BD, divide the Pgrs BF, AE, into two equal parts. P. 34. B. 1.

5. Consequently, the Δ ACB is the half of the Pgr BF, & the Δ ADB the half of the pgr AE.

Since then the whole Pgrs BF, AE, are equal to one another (*Arg. 3.*), & the Δ ACB, ADB, are the halves of those Pgrs (*Arg. 5.*).

6. It is evident that the Δ ACB, ADB, are also = to one another. Ax. 7.

Which was to be demonstrated.



PROPOSITION XXXVIII. THEOREM XXVIII.

TRIANGLES (ADB, EGF,) upon equal bases (AB, EF,) & between the same parallels (AF, DG,) are equal to one another.

Hypothesis.

- I. ADB, EGF, are two Δ .
- II. And those two Δ are upon = bases AB, EF,
& between the same plles AF, DG.

Thesis.

The Δ ADB is = to the Δ EGF.

Preparation.

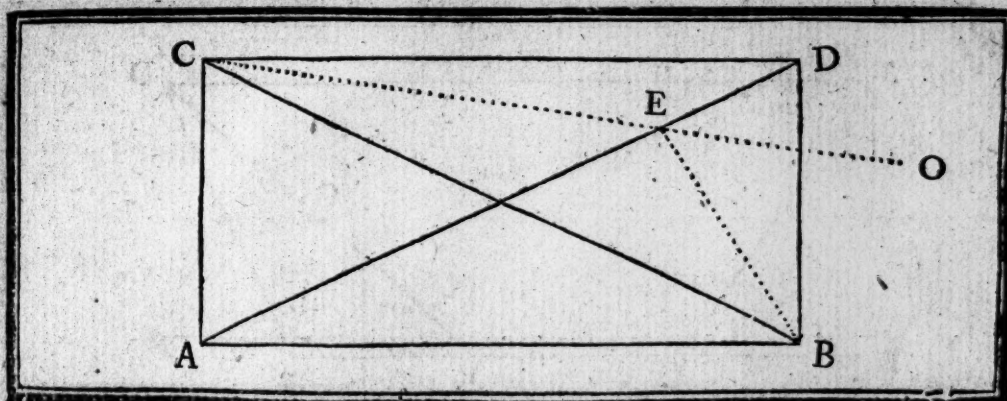
1. Produce the straight line DG both ways to the points H, C. *Pof. 2.*
2. Thro' the points A & F, draw the straight lines AC, FH,
p'le to the sides BD, EG; which will meet the produced line *P. 31. B. 1.*
DG, somewhere in C & in H.

DEMONSTRATION.

BECAUSE in the figure BC, the opposite sides AB, CD, & AC, BD,
are p'le (*Hyp. 2 & Prep. 2.*).

1. The figure BC is a Pgr. *D. 35. B. 1.*
It may be proved after the same manner.
2. That the figure EH is a Pgr.
But the pgrs BC, EH, (*Arg. 1 & 2.*) are upon = bases AB, EF, &
between the same plles AF, CH, (*Hyp. 2.*).
3. Consequently, the Pgr BC, is = to the Pgr EH. *P. 36. B. 1.*
But the straight lines AD, FG, being the diagonals of the Pgrs BC,
EH, (*Prep. 1 & 2.*).
4. Those straight lines AD, FG, divide the Pgrs BC, EH, into two
equal parts. *P. 34. B. 1.*
5. Wherefore, the Δ ADB; is half of the Pgr BC, & the Δ EGF is the
half of the Pgr EH.
Since then the whole Pgrs BC, EH, are = to one another (*Arg. 3.*),
and the Δ ADB, EGF, are the halves of those Pgrs (*Arg. 5.*).
6. It follows, that those Δ ADB, EGF, are also = to one another. *Ax. 7.*

Which was to be demonstrated.



PROPOSITION XXXIX. THEOREM XXIX.

EQUAL triangles ($\triangle ACB$, $\triangle ADB$,) upon the same base (AB) & upon the same side of it, are between the same parallels (AB , CD ,).

Hypothesis.

- I. The $\triangle ACB$, $\triangle ADB$, are equal.
- II. And those $\triangle$ are upon the same base AB .

Thesis.

The $\triangle ACB$, $\triangle ADB$, are between the same plles AB , CD .

DEMONSTRATION.

If not,

The straight lines AB , CD , are not pple, & there may be drawn thro' the point C , some other straight line CO , pple to AB .

Preparation.

1. Draw then thro' the point C , the straight line CO pple to AB ; *P. 31. B. 1.* which will cut the straight line AD , somewhere in E .
2. From the point B , to the point of intersection E , draw the straight line BE . *Pos. 1.*

BECAUSE the two $\triangle ACB$, $\triangle AEB$, are upon the same base AB , (*Hyp. 2.*), & between the same plles AB , CO , (*Prep. 1.*).

1. The $\triangle ACB$ is $=$ to the $\triangle AEB$. *P. 37. B. 1.*

But the $\triangle ADB$ being $=$ to the $\triangle ACB$ (*Hyp. 1.*), & the $\triangle AEB$ being $=$ to the same $\triangle ACB$ (*Arg. 1.*).

2. The $\triangle ADB$ is $=$ to the $\triangle AEB$. *Ax. 1.*

But the $\triangle ADB$ being the whole, & the $\triangle AEB$ its part.

3. It follows, that the whole is equal to its part.

4. Which is impossible. *Ax. 8.*

5. Consequently, the straight line CO is not pple to AB .

It may be proved after the same manner, that no other straight line but CD , can be pple to AB .

6. Consequently, the straight line CD , drawn thro' the vertices of the $\triangle ACB$, $\triangle ADB$, is pple to the base AB .

Which was to be demonstrated.



Hypothesis.

Thesis.

DEMONSTRATION.

The straight lines BF, AD, are not p^lle, & there may be drawn thro' the point A some other straight line AO p^lle to BF.

Preparation.

- BECAUSE the $\triangle BAC, EGF$, are upon the equal bases BC, EF , (*Hyp.* 2.), & between the same p^lles BF, AO , (*Prop.* 1.).

- i. The $\triangle BAC$ is $\equiv$ to the $\triangle EGF$.
But the $\triangle EDF$ is $\equiv$ to the $\triangle BAC$ (*Hyp. 1.*), & the $\triangle EGF$ is $\equiv$ to the same $\triangle BAC$ (*Arg. 1.*).

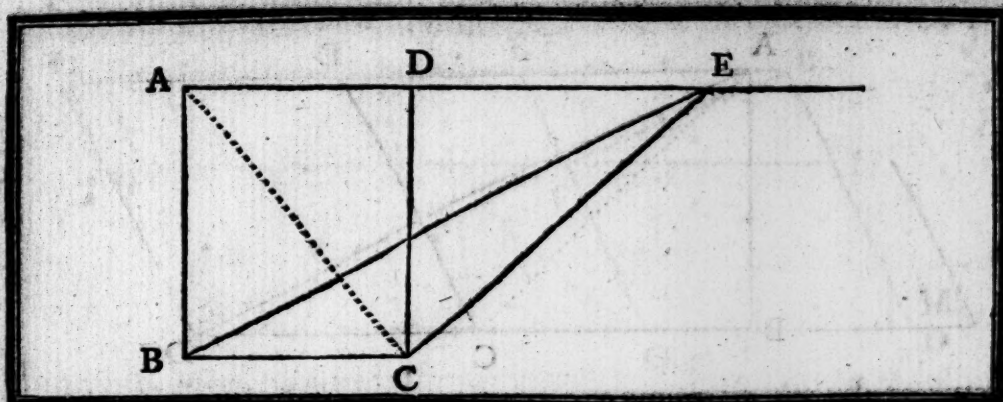
2. Wherefore the $\triangle EDF$ is $=$ to the $\triangle EGF$.
But the $\triangle EDF$ being the whole & the $\triangle EGF$ its part.
3. It follows, that the whole is $=$ to its part.
4. Which is impossible.

5. Consequently, AO is not p_lle to BF.

It will be proved after the same manner, that no other straight line but AD can be p^{er}ple to BF.

6. Consequently, the straight line AD, drawn thro' the summits of the $\triangle$ BAC, EDF, is p^{ar}lle to the straight line BF.

Which was to be demonstrated.



PROPOSITION XLI. THEOREM XXXI.

IF a parallelogram (BD) and a triangle (BEC) be upon the same base (BC), and between the same parallels (BC, AE,); the parallelogram shall be double of the triangle.

Hypothesis.

- I. BD is a Pgr & BEC a Δ .
- II. Those figures are upon the same base BC, & between the same plles BC, AE.

Thesis.

The Pgr BD is double of the Δ BEC.

Preparation.

From the point A to the point C, draw the straight line AC. *Pos. 1.*

DEMONSTRATION.

BECAUSE the Δ BAC, BEC, are upon the same base BC, & between the same plles BC, AE (*Hyp. 2.*).

1. The Δ BAC is = to the Δ BEC.

P. 37. B. 1.

But the straight line AC being the diagonal of the Pgr BD (*Prep.*).

2. This diagonal divides the Pgr into two equal parts.
3. Consequently, the Pgr BD is double of the Δ BAC.

P. 34. B. 1.

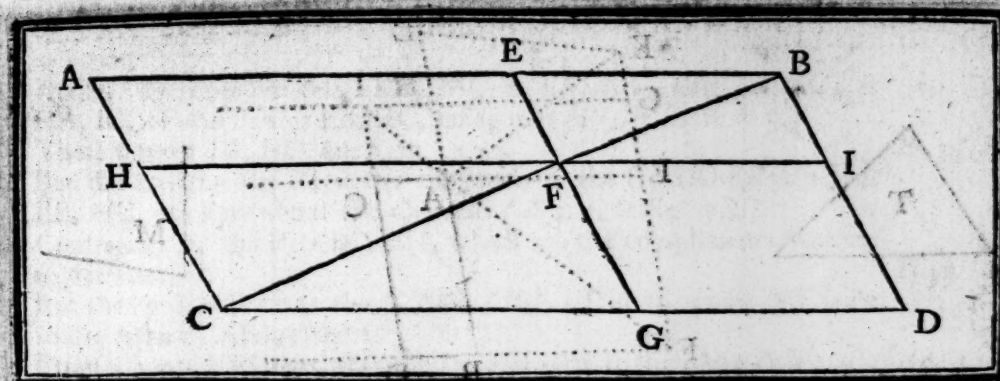
But this Δ BAC being = to the Δ BEC (*Arg. 1.*).

4. The Pgr BD is also double of the Δ BEC.

Ax. 1.

Which was to be demonstrated.





PROPOSITION XLIII. THEOREM XXXII.

THE complements (AF, FD,) of the parallelograms (HG, EI,) about the diagonal (BC) of any parallelogram (AD), are equal to one another.

Hypothesis.

- I. AD is a Pgr, whose diagonal is BC.
- II. HG, EI, are the Pgrs about the diagonal.

Thesis.

The Pgrs AF, FD, which are the complements of the Pgrs HG, EI, are = to one another.

DEMONSTRATION.

BECAUSE AD is a Pgr, whose diagonal is BC (*Hyp.* 1.).

1. This diagonal divides the Pgr into two equal parts.

P. 34. *B.* 1.

2. Consequently, the $\triangle CAB$ is = to the $\triangle BDC$.

Likewise, EI being a Pgr, whose diagonal is BF (*Hyp.* 2.).

3. It divides also the Pgr into two equal parts.

P. 34. *B.* 1.

4. Wherefore the $\triangle FEB$ is = to the $\triangle BIF$.

In fine, HG is a Pgr, whose diagonal is FC (*Hyp.* 2.).

5. Which consequently divides it into two equal parts.

P. 34. *B.* 1.

6. Consequently, the $\triangle CHF$ is = to the $\triangle FGC$.

Since then the $\triangle FEB$ is = to the $\triangle BIF$ (*Arg.* 4.), & the $\triangle CHF$ = to the $\triangle FGC$ (*Arg.* 6.).

7. The $\triangle FEB$, together with the $\triangle CHF$ is = to the $\triangle BIF$, together with the $\triangle FGC$.

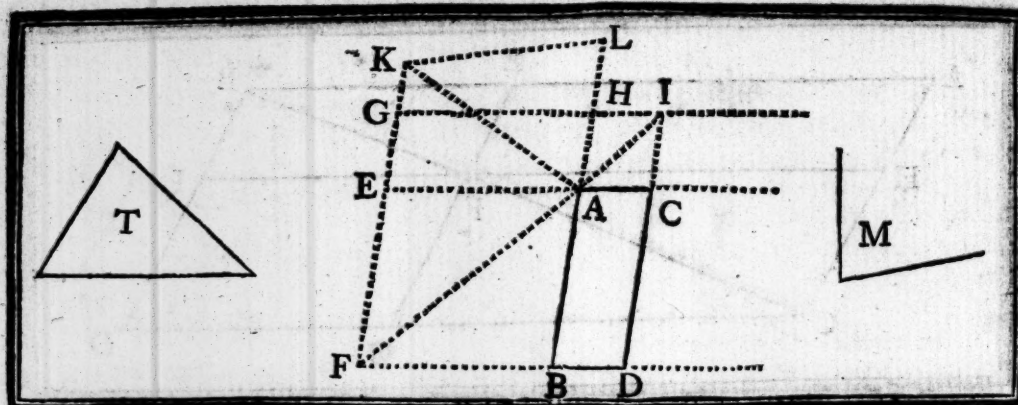
Ax. 2.

But the whole $\triangle CAB$, BDC , being = to one another (*Arg.* 2.); if there be taken away from both, the $\triangle FED + CHF$, & the $\triangle BIF + FGC$, which are equal (*Arg.* 7.).

8. The remaining Pgrs AF, FD, which are the complements of the Pgrs HG, EI, will be also = to one another.

Ax. 3.

Which was to be demonstrated.



PROPOSITION XLIV. PROBLEM XII.

UPON a given straight line (AB), to make a parallelogram (BC) which shall be equal to a given triangle (T), and have one of its angles as (BAC) equal to a given rectilineal angle (M).

Given

- I. The straight line AB.
- II. The $\triangle T$.
- III. The rectilineal $\angle M$.

Sought

A Pgr made upon a straight line AB
 = to the $\triangle T$, having one of its $\angle$
 BAC = to the given $\angle M$.

Resolution.

1. Produce the straight line AB indefinitely.
2. Take AL = to one of the sides of the given $\triangle T$.
3. Make the $\triangle AKL$ = to the given $\triangle T$.
4. Describe the Pgr EH = to the $\triangle AKL$, having an $\angle HAE$ = to the given $\angle M$.
5. Thro' the point B, draw a straight line BF pple to EA or GH.
6. Produce GH indefinitely, as also GE, until it meets BF in F.
7. Thro' the points F & A, draw the straight line FA, which when produced will meet GH produced, somewhere in I.
8. Thro' the point I, draw the straight line ID pple to HB or GF.
9. Produce FB, EA, until they meet ID in the points D & C.

Pof. 2.
 P. 3. B. 1.
 P. 22. B. 1.
 P. 42. B. 1.
 P. 31. B. 1.
 Pof. 2.
 Pof. 1.
 P. 31. B. 1.
 Pof. 2.

DEMONSTRATION.

BECAUSE in the figure DG the opposite sides GI, FD, & GF, ID, are pple (Ref. 5. 6. 8. & 9.).

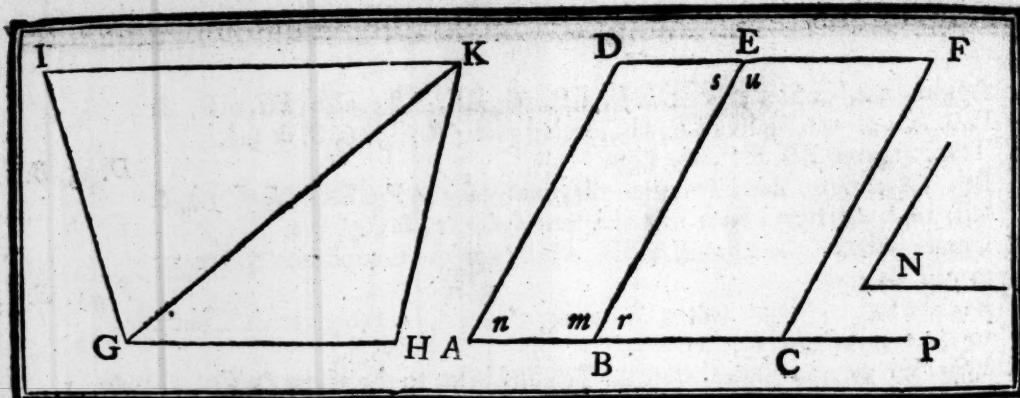
- I. The figure DG is a Pgr.

D. 35. B. 1.

- Again, the opposite sides EA, FB, & EF, AB; also HI, AC, & HA, IC, of the figures EB, HC, being p^{lle} (*Ref.* 5. 6. 8. & 9.).
2. Those figures EB, HC, are Pgrs. *D.* 35. *B.* 1.
 But the straight line FI is the diagonal of the Pgr DG (*Ref.* 7.), & EB, HC, are Pgrs about this diagonal (*Arg.* 2, & *Ref.* 7.).
3. Consequently, the Pgrs BC, EH, which are the compliments, are = to one another. *P.* 43. *B.* 1.
 But the Pgr EH is = to the $\triangle AKL$ (*Ref.* 4.), & the given $\triangle T$ is = to the same $\triangle AKL$ (*Ref.* 3.).
4. From whence it follows, that the Pgr EH is = to the given $\triangle T$. *Ax.* 1.
 The Pgr EH being therefore = to the given $\triangle T$ (*Arg.* 4.), & this same Pgr EH being = to the Pgr BC (*Arg.* 3.).
5. The Pgr BC is = to the given $\triangle T$. *Ax.* 1.
 Moreover, because the $\sphericalangle$ HAE, BAC, are vertically opposite.
6. Those $\sphericalangle$ are = to one another. *P.* 15. *B.* 1.
 Wherefore, $\sphericalangle$ HAE being = to the given $\sphericalangle$ M (*Ref.* 4.).
7. The $\sphericalangle$ BAC is also = to this given $\sphericalangle$ M. *Ax.* 1.
8. Therefore, upon the given straight line AB, there has been made a Pgr BC = to the given $\triangle T$ (*Arg.* 5.), & which has an $\sphericalangle$ BAC = to the given $\sphericalangle$ M (*Arg.* 7.).

Which was to be done.





PROPOSITION XLV. PROBLEM XIII.

TO describe a parallelogram (AF), equal to a rectilineal figure (IH); and having an angle (n) equal to a given rectilineal angle (N).

- Given
 I. A rectilineal figure IH.
 II. A rectilineal $\angle N$.

Sought
 The construction of a Pgr = to the rectilineal figure IH, & having an $\angle n$ = to a given $\angle N$.

Resolution.

1. Draw the diagonal GK. Pos. 1.
2. Upon an indefinite straight line AP, make the Pgr AE = to the $\triangle GHK$, having an $\angle n$ = to the given $\angle N$. P. 42. B. 1.
3. Upon the side BE of the Pgr AE, make the Pgr DF = to the $\triangle GIK$; having an $\angle r$ = to the given $\angle N$. P. 44. B. 1.

DEMONSTRATION.

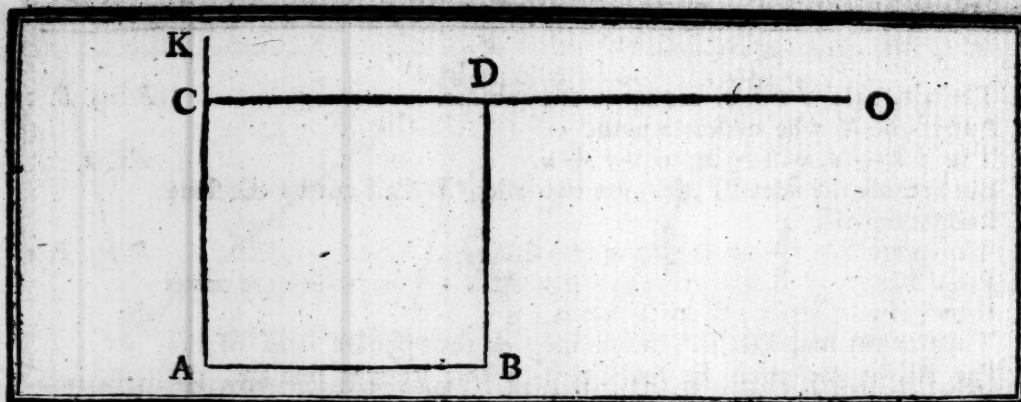
BECAUSE $\angle N$ is = to each of the $\angle n$ & r (Ref. 2 & 3.).

1. The $\angle n$ is = to the $\angle r$. Ax. 1.
 If the $\angle m$ be added to both.
2. The $\angle n + m$ will be = to the $\angle r + m$. Ax. 2.
 But because the sides AD, BE, are p^les (Ref. 2.) cut by the same straight line AB.
3. The two interior $\angle n + m$, are = to two $\angle$. P. 29. B. 1.
4. Consequently, the adjacent $\angle r + m$, which are = to them (Arg. 2.), are also = to two $\angle$. Ax. 1.
 The straight lines AB, BC, which meet on the opposite sides of the line BE at the point B, making with this straight line BE the sum of the adjacent $\angle r + m$ = to two $\angle$ (Arg. 4.).
5. Those straight lines AB, BC, form but one & the same straight line AC. P. 14. B. 1.
 Moreover, the straight lines DE, AC, being two p^les (Ref. 2.) cut by the same straight line BE.

6. The alternate $\angle r$ & s , are $\equiv$ to one another.
And if the $\angle u$ be added to both.
7. The $\angle r + u$, will be $\equiv$ to $\angle s + u$.
But because the sides EF, BC, are two plles (*Ref. 3.*) cut by the same straight line BE.
8. The interior $\angle r + u$, are $\equiv$ to two $\angle$.
P. 29. B. 1.
9. From whence it follows, that the adjacent $\angle s + u$, which are $\equiv$ to them (*Arg. 7.*), are also $\equiv$ to two $\angle$.
Ax. 1.
The straight lines DE, EF, which meet on the opposite sides of the line BE at the point E, making with this straight line BE, the sum of the adjacent $\angle s + u \equiv$ to two $\angle$ (*Arg. 9.*).
10. Those straight lines DE, EF, form but one and the same straight line DF.
P. 14. B. 1.
But since the straight lines AD, BE, & BE, CF, are the opposite sides of the Pgrs AE, BF, (*Ref. 2 & 3.*)
11. The straight line AD is $\equiv$ & pple to BE, & BE is $\equiv$ & pple to CF.
P. 34. B. 1.
12. Consequently, AD is $\equiv$ & pple to CF. - - - - -
P. 30. B. 1.
Moreover, those $\equiv$ and pple straight lines AD, CF, are joined by the straight lines AC, DF, (*Arg. 5 & 10.*)
Ax. 1.
13. Consequently, the figure AF is a Pgr. - - - - -
P. 33. B. 1.
And because the Pgr BF is $\equiv$ to the $\triangle GIK$ (*Ref. 3.*), the Pgr AE is $\equiv$ to the $\triangle GHK$, & $\angle n \equiv$ to the given $\angle N$ (*Ref. 2.*).
D. 35. B. 1.
14. The whole Pgr AF is $\equiv$ to the rectilinear figure IH; & has an $\angle n \equiv$ to the given $\angle N$.
Ax. 2.

Which was to be demonstrated.





PROPOSITION. XLVI. PROBLEM XIV.

UPON a given straight line (AB) to describe a square (AD).

Given
The straight line AB.

Sought
A square made upon the straight line AB.

Resolution.

1. At the point A, erect upon the straight line AB the perpendicular AK. P. 11. B. 1.
2. From the straight line AK cut off a part AC = to AB. P. 3. B. 1.
3. Thro' the point C, draw the straight line CO p^lle to AB. P. 31. B. 1.
4. And thro' the point B, draw the straight line BD p^lle to AC, which will cut CO somewhere in D. P. 31. B. 1.

DEMONSTRATION.

BECAUSE in the figure AD the opposite sides AB, CD, & AC, BD, are p^lle (Ref. 3 & 4.).

1. The figure AD is a Pgr. D. 35. B. 1.
2. Consequently, the opposite sides AB, CD, & AC, BD, are = to one another. P. 34. B. 1.
But AC is = to AB (Ref. 2.).
3. Consequently, the four sides AB, CD, AC, BD, are = to one another. Ax. 1.
Again, because the straight lines AB, CD, are p^lle (Ref. 3.).
4. The interior opposite $\angle$ A & ACD, are = to two $\angle$. P. 29. B. 1.
But the $\angle$ A being a $\angle$ (Ref. 1.).
5. It is evident, that $\angle$ ACD is also a $\angle$. C. N.
Moreover, because AD is a Pgr (Arg. 1.).
6. The opposite $\angle$ are = to one another. P. 34. B. 1.
7. Wherefore, the $\angle$ BDC & B opposite to the right $\angle$ A & ACD, are also $\angle$.
The figure AD being therefore an equilateral Pgr (Arg. 3.), & rectangular (Arg. 7.).
8. It follows, that this figure AD described upon the straight line AB, is a square. D. 30. B. 1.

Which was to be done.

COROLLARY I.

EVERY parallelogram, that has two equal sides AB, AC , including a right angle, is a square; for drawing thro' the points C & B the straight lines CD, BD , parallel to the two sides AB, AC , the square AD will be described (*D. 30. B. 1.*).

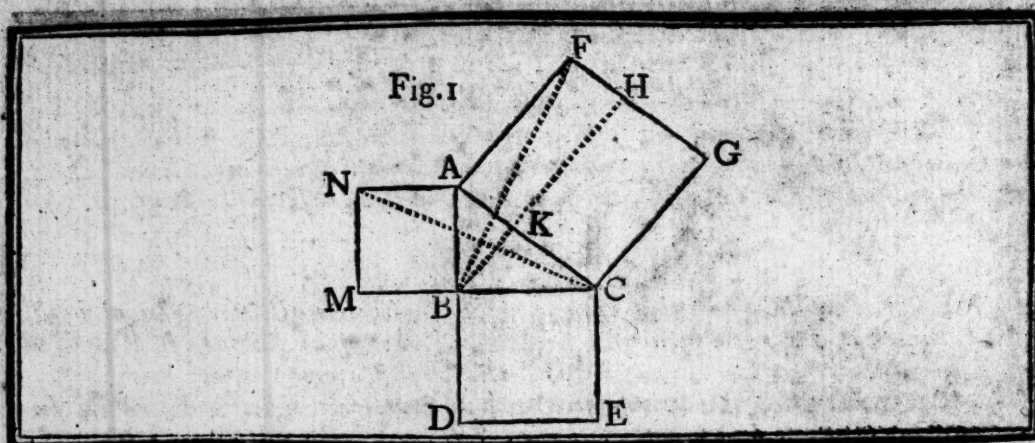
COROLLARY II.

EVERY parallelogram that has one right angle, has all its angles right angles. For since the opposite angles A & BDC , are equal (*P. 34. B. 1.*), & the angle A is a right angle, the angle BDC will be also a right angle: moreover, the lines AB, CD , & AC, BD , being parallels; the interior angles A & ACD , likewise A & B , are equal to two right angles (*P. 29. B. 1.*); but the angle A being a right angle, it is manifest that the angles ACD & B , are also right angles.

COROLLARY III.

THE squares described on equal straight lines, are equal to one another, & reciprocally, equal squares are described on equal straight lines.





PROPOSITION XLVII. THEOREM XXXIII.

IN any right angled triangle (ABC); the square which is described upon the side (AC) subtending the right angle, is equal to the squares made upon the sides (AB, BC,) including the right angle.

Hypothesis.

The $\triangle ABC$ is Rgle, or $\angle ABC$ is a $\angle$.

Thesis.

The $\square$ of the side AC is = to the $\square$ of AB, together with the $\square$ of BC.

Preparation.

1. On the three sides AC, AB, BC, describe (Fig. 1.) the $\square$ AG, AM, CD. P. 46. B. 1.
2. Thro' the point B, draw the straight line BH pple to CG. P. 31. B. 1.
3. From the point B to the point F, draw the straight line BF. } Pos. 1.
4. From the point C to the point N, draw the straight line CN. }

DEMONSTRATION.

BECAUSE the figure AM is a $\square$ (Prep. 1.).

1. The $\angle ABM$ is a $\angle$. D. 30. B. 1.

But $\angle ABC$ being also a $\angle$ (Hyp.).

2. The two adjacent $\angle ABM, ABC$, are = to two $\angle$. Ax. 2.

The straight lines MB, BC, which meet on the opposite sides of the line AB at the point B, making with this straight line AB the sum of the adjacent $\angle ABM, ABC$, = to two $\angle$ (Arg. 2.).

3. These straight lines MB, BC, are in one and the same straight line MC, which is pple to NA. P. 14. B. 1.
P. 28. B. 1.

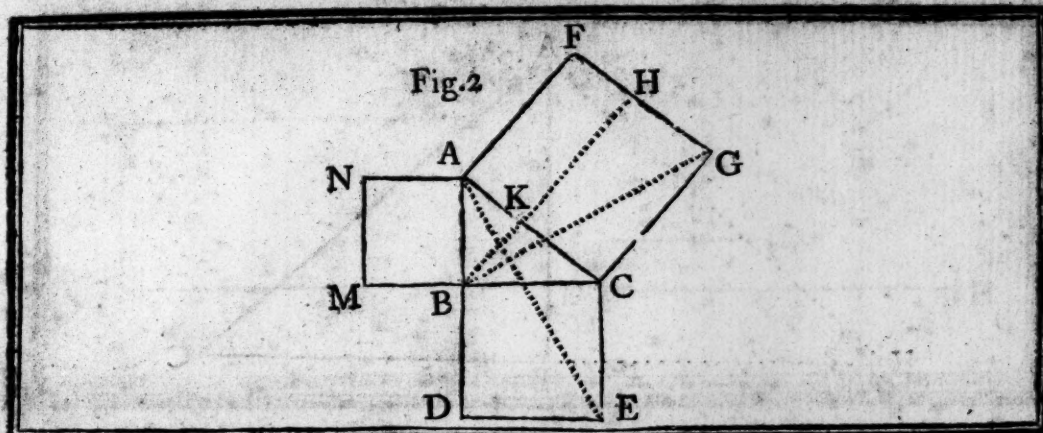
In like manner it may be demonstrated.

4. That AB, BD, are in one & the same straight line AD, which is pple to CE.

Moreover, because AG, AM, are $\square$ (Prep. 1.).

5. The $\angle FAC, NAB$, are = to one another, (being right angles) & the sides AF, AC, & AB, AN, are also = to one another. D. 30. B. 1.

Therefore, if to those equal $\angle FAC, NAB, \angle CAB$ be added.



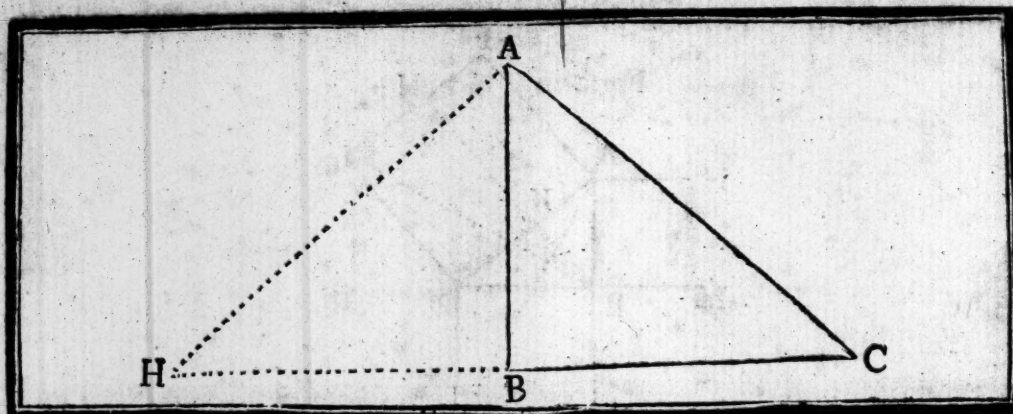
6. The whole $\forall$ FAB will be $=$ to the whole $\forall$ NAC. Ax. 2.
 Since then in the $\triangle$ AFB, ACN, the sides AF, AB, & AC, AN, are
 $=$ each to each (*Arg. 5.*), & the $\forall$ FAB is $=$ to the $\forall$ NAC,
 (*Arg. 6.*).
7. The $\triangle$ AFB will be $=$ to the $\triangle$ ACN. P. 4. B. 1.
 But the $\triangle$ AFB & the Pgr AH, are upon the same base AF & be-
 tween the same plles AF, BH, (*Prep. 2.*).
8. From whence it follows, that the Pgr AH is double of the $\triangle$ AFB. P. 41. B. 1.
 Likewise, the $\triangle$ ACN & the $\square$ AM being upon the same base AN,
 and between the same plles AN, MC, (*Arg. 3.*).
9. The $\square$ AM is double of the $\triangle$ ACN. P. 41. B. 1.
 The $\triangle$ AFB, ACN, being therefore $=$ to one another (*Arg. 7.*).
 and the Pgr AH & the $\square$ AM their doubles (*Arg. 8 & 9.*).
10. It follows, that the Pgr AH is $=$ to the $\square$ AM. Ax. 6.

In the same manner, by drawing (*Fig. 2.*) the lines BG, AE, it is demonstrated, that the Pgr CH is = to the $\square$ CD.

11. But the Pgr AH, together with the Pgr CH, form the $\square$ AG.
12. Wherefore, this $\square$ AG is $=$ to the sum of the $\square$ AM & CD. Ax. 1.
But since the $\square$ AG is the $\square$ made upon the side AC, & the $\square$ AM and CD the $\square$ upon the sides which include the $\angle$ ABC.
13. The $\square$ made upon the side AC is $=$ to the $\square$ made upon AB & BC taken together.

Which was to be demonstrated.





PROPOSITION XLVIII. THEOREM XXXIV.

IF the square described upon one of the sides (CA) of a triangle (CBA) be equal to the squares described upon the other two sides of it (AB, BC,); the angle (ABC) included by these two sides (AB, BC,), is a right angle.

Hypothesis.

The $\square$ of CA is = to the $\square$ of AB,
together with the $\square$ of BC.

Thesis.

The $\angle$ ABC included by the
sides AB, BC, is $\angle$.

Preparation.

1. At the point B, in the straight line BA, erect the perpendicular BH.
2. Make BH = BC.
3. From the point H to the point A, draw the straight line HA.

P. 11. B. 1.

P. 3. B. 1.

Pos. 1.

DEMONSTRATION.

BECAUSE BH is = to BC (*Prep. 2.*).

1. The $\square$ of BH will be = to the $\square$ of BC.

If the $\square$ of AB be added to both.

2. The $\square$ of AB & BH, will be = to the $\square$ of AB & BC.

But the $\triangle$ HBA being Rgle in B (*Prep. 1.*).

3. It follows, that the $\square$ of HA is = to the $\square$ of AB & BH.

Since then the $\square$ of CA is = to the $\square$ of AB & BC (*Hyp. 1.*), the

$\square$ of HA = to the $\square$ of AB & BH (*Arg. 3.*), & the $\square$ of AB & BH,

are = to the $\square$ of AB & BC, (*Arg. 2.*).

4. The $\square$ of CA must necessarily be = to the $\square$ of HA.

5. Consequently, CA is = to HA.

But in the $\triangle$ CBA, HBA, the side CA is = to the side HA, (*Arg. 5.*), AB is common to the two $\triangle$, & the base BC is = to the base BH (*Prep. 2.*).

6. Wherefore, the $\angle$ ABC, ABH, included by the equal sides AB, BC, and AB, BH, are = to one another.

But the $\angle$ ABH is a $\angle$ (*Prep. 1.*).

7. Consequently, the $\angle$ ABC will be also a $\angle$.

Which was to be demonstrated.

{ P. 46. B. 1.
Cor. 3.

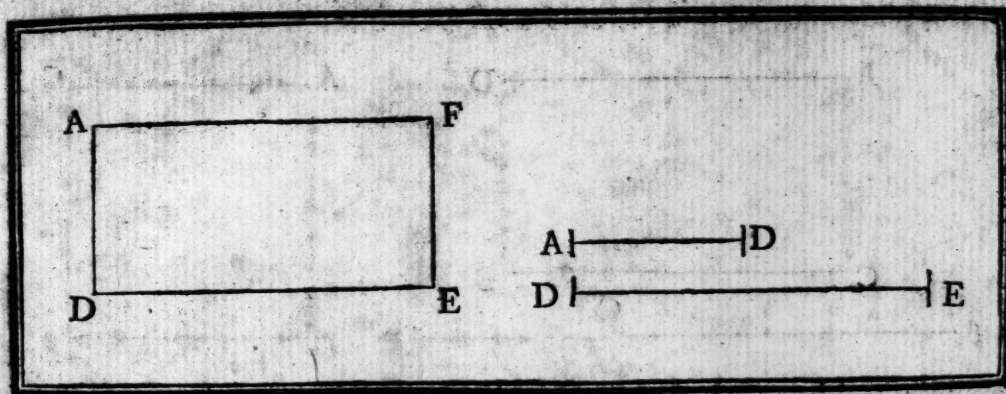
Ax. 2.

P. 47. B. 1.

Ax. 1.

{ P. 46. B. 1.
Cor. 3.

P. 8. B. 1.



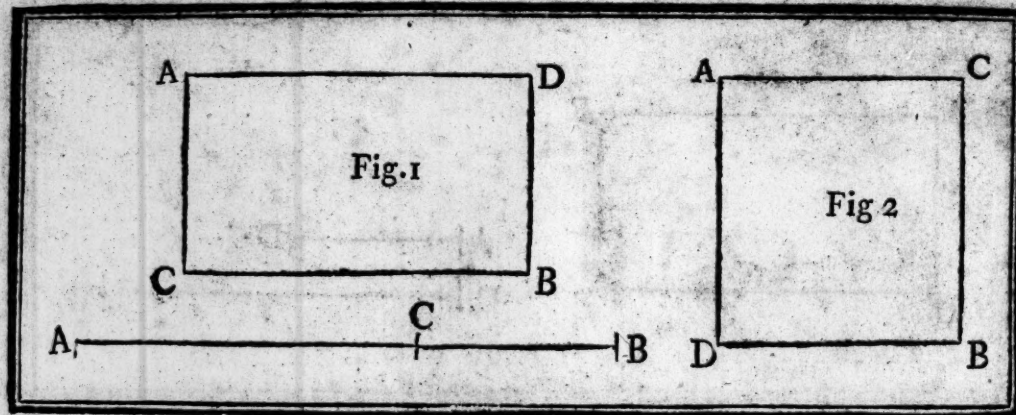
DEFINITIONS.

I.

EVERY right angled parallelogram (DF), is said to be contained by any two of the straight lines (AD, DE,) which include one of the right angles (ADE).

1. *A right angled parallelogram may be thus denoted, because a right angle & the two sides which include it, are what determine this figure. When the length of the sides AD, DE, including the right angle is fixed, the magnitude of the rectangle is determined, its construction being completed by drawing thro' the extremities A & E of those sides, the lines (AD, DE,) parallel to them, according to D. 35 & P. 31. B. I.*
2. *A right angled parallelogram DF is for brevity sake often denoted by the three letters about the right angle, in this manner; the Rgle Pgr ADE. It is also represented thus: The Rgle Pgr AD, DE, that is, the Rgle Pgr resulting from the two sides AD & DE, which form a right angle; & is expressed thus: The Rgle Pgr under AD & DE, or the Rgle Pgr of AD & DE.*

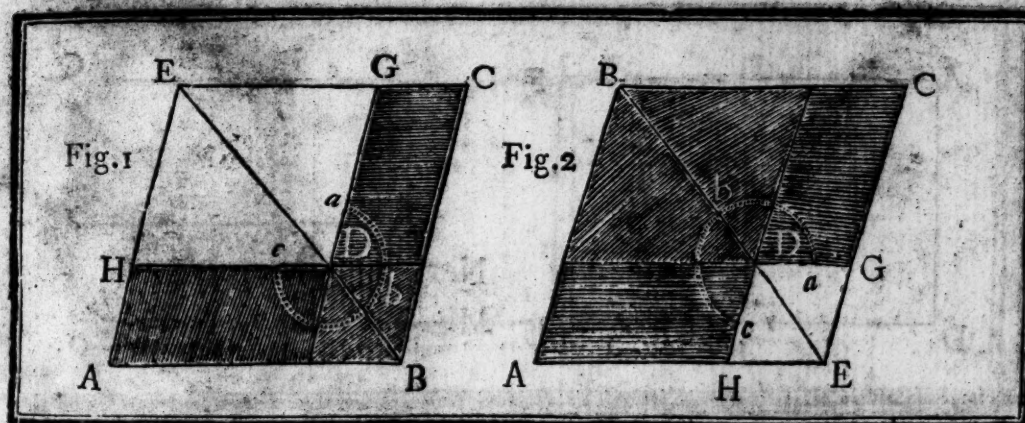




DEFINITIONS.

3. **SOMETIMES** the parts of a straight line serve to denote a right angled parallelogram, for example (Fig. 1.), the straight line AB being divided in C, there may be described (P. 31. B. 1.), with those two lines AC, CB, a right angled parallelogram, by joining them at right angles, & this parallelogram is expressed thus: The RglePgr AC, CB, or simply the Rgle Pgr ACB, the letter that marks the point which is common to the two lines, being put between the other two letters; in like manner, by the Rgle Pgr ABC, is to be understood the parallelogram described according to the same rules, one of whose sides is AB & the other BC.
4. When the lines AD & DB, including the right angle, are equal (Fig. 2.), the parallelogram DC is a square (D. 30. B. 1.). As in this case one of the sides DB with the right angle, determine the square, which may be described from those data by P. 31. B. 1. This square may be expressed thus: The $\square$ of DB, or the $\square$ of AD..





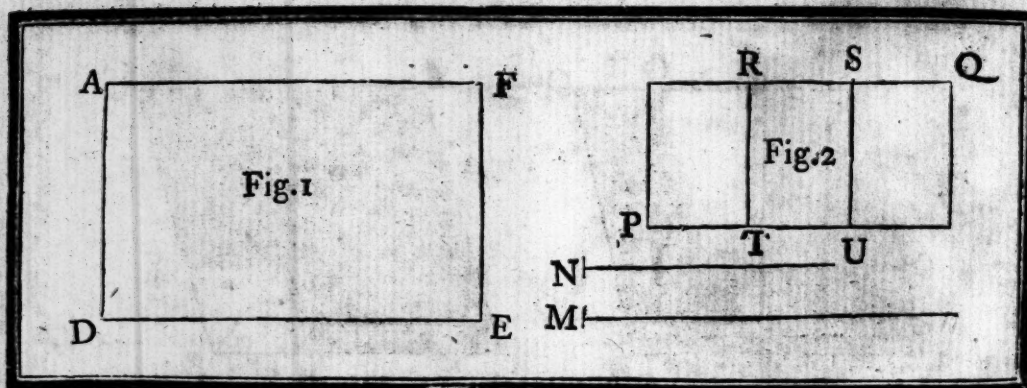
DEFINITIONS.

II.

THE figure (ABCGDH) composed of a parallelogram (DB) about the diagonal (BE), together with the two complements (AD, DC,) is called a Gnomon.

The Gnomon is marked by an arc of a circle (*abc*), which passes thro' the two complements (AD, DC,) & the Pgr about the diagonal. There may be formed in every parallelogram two different gnomons; one, by taking away (Fig. 1.) from the whole Pgr, the greater Pgr ED about the diagonal; the other, by taking away (Fig. 2.) the lesser Pgr ED about the diagonal.





A X I O M S.

I.

TH E whole is equal to all its parts taken together.

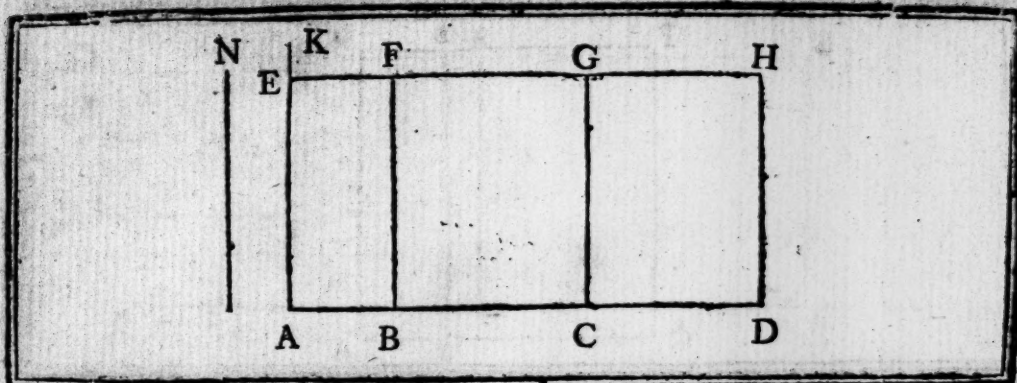
The whole Pgr PQ (Fig. 2.) is equal to all its parts, the Pgrs PR, TS, VQ, taken together.

II.

RI G H T angled parallelograms contained by equal sides, are equal.

The Rgle Pgr DF (Fig. 1.) is contained by the straight lines AD, DE; consequently, if the straight line N is equal to AD, & the straight line M is equal to DE, the Rgle formed by the straight lines N & M, will be necessarily equal to the Rgle DF.





PROPOSITION I. THEOREM I.

IF there be two straight lines (AD & N), one of which (AD) is divided into any number of parts (AB, BC, CD,) ; the rectangle contained by these straight lines (AD & N) is equal to the rectangles contained by the undivided line (N), and the several parts (AB, BC, CD,) of the divided line (AD).

Hypothesis.

Thesis.

AD & N are two straight lines, one of which AD is divided into several parts AB, BC, CD.

The Rgle AD . N is = to the Rgles AB . N + BC . N + CD . N.

Preparation.

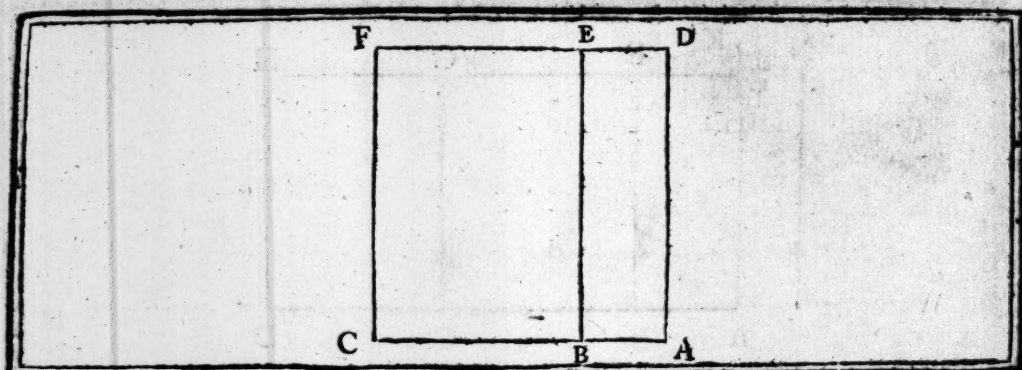
1. At the point A in the straight line AD, erect the $\perp$ AK. P. 11. B. 1.
2. From AK, cut off a part EA = N. P. 3. B. 1.
3. Thro' the points D & E, draw the straight lines DH, EH, p^lle to AE, AD. P. 31. B. 1.
4. And Thro' the points of division B & C, draw the straight lines BF, CG, p^lle to AE or DH.

DEMONSTRATION.

1. **T**HE Rgle AH is = to the Rgles AF, BG, CH, taken together. Ax. 1. B. 2.
But because the Rgle AH is contained by the straight lines EA, AD, (Prep. 3.), & AE is = to N (Prep. 2.).
2. This Rgle AH is contained by the straight lines AD & N. Ax. 2. B. 2.
Likewise, because the Rgle AF is contained by the straight lines EA, AB, (Prep. 4.), & EA is = to N (Prep. 2.).
3. This Rgle AF is contained by the straight lines AB & N. Ax. 2. B. 2.
4. In like manner, the Rgle BG is contained by the straight lines BC & N, because it is contained by the straight lines FB & BC, & that FB = N. P. 34. B. 1.
And so of all the others.
5. Consequently, the Rgle contained by the straight lines AD & N is = to the Rgles contained by the straight lines AB & N, BC & N, CD & N, taken together.
That is the Rgle AD . N is = to the Rgles AB . N + BC . N + CD . N. Ax. 1. B. 1.

Which was to be demonstrated.

K



PROPOSITION II. THEOREM II.

IF a straight line (AC) be divided into any two parts (AB, BC,) ; the rectangle contained by the whole line (CA), and each of the parts (AB, BC,) , are together equal to the square of the whole line (AC).

Hypothesis.
AC is a straight line divided into
two parts AB, BC.

Thesis.
The Rgle CAB + Rgle ACB,
are = to the $\square$ of AC.

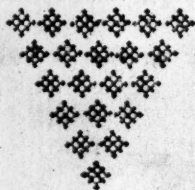
Preparation.

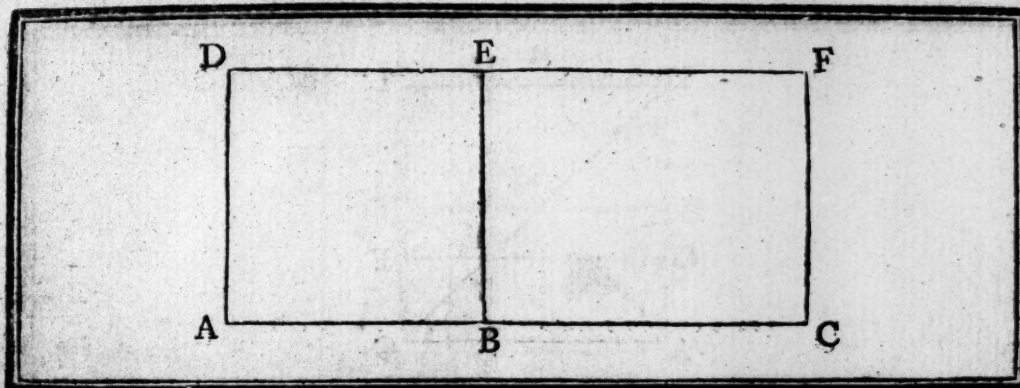
1. Upon the straight line AC, describe the $\square$ AF. P. 46. B. 1.
2. Thro' the point of section B, draw the straight line BE p^{lle} to AD or CF. P. 31. B. 1.

DEMONSTRATION.

1. **T**HE whole Rgle AF is = to the Rgles AE, BF, taken together. Ax. 1. B. 2.
But this Rgle AF is the $\square$ of the line AC (*Prep.* 1.).
2. Consequently, the Rgles AE, BF, taken together, are = to the $\square$ of the line AC. Ax. 1. B. 1.
3. But the Rgle AE is contained by the straight lines CA, AB, because it is contained by the straight lines DA, AB, of which DA = CA, (*Prep.* 1.). Ax. 2. B. 2.
4. Likewise, BF is a Rgle contained by the straight lines AC, CB, because it is contained by the straight lines EB, BC, of which EB = AC, (*Prep.* 1 & 2.). P. 34. B. 1.
5. Wherefore, the Rgle contained by the straight lines CA, AB, together with the Rgle contained by the straight lines AC, CB, is = to the $\square$ of the straight line AC ; or the Rgle CAB + the Rgle ACB, are = to the $\square$ of AC. Ax. 1. B. 1.

Which was to be demonstrated.





PROPOSITION III. THEOREM III.

IF a straight line (AC) be divided into two parts in (B); the rectangle contained by the whole line (AC) & of one of the parts (AB), is equal to the rectangle contained by the two parts (AB, BC,) together with the square of the aforefaid part (AB).

Hypothesis.

AC is a straight line divided
into any two parts AB, BC.

Thesis.

The Rgle CAB is = to the
Rgle ABC + the $\square$ of AB.

Preparation.

1. Upon the straight line AB, describe the $\square$ AE.
2. Produce the line DE indefinitely to F.
3. Thro' the point C, draw the straight line CF pple to AD or BE and produce it, until it meets DF in F.

P. 46. B. 1.
Pos. 2.

Pos. 2.

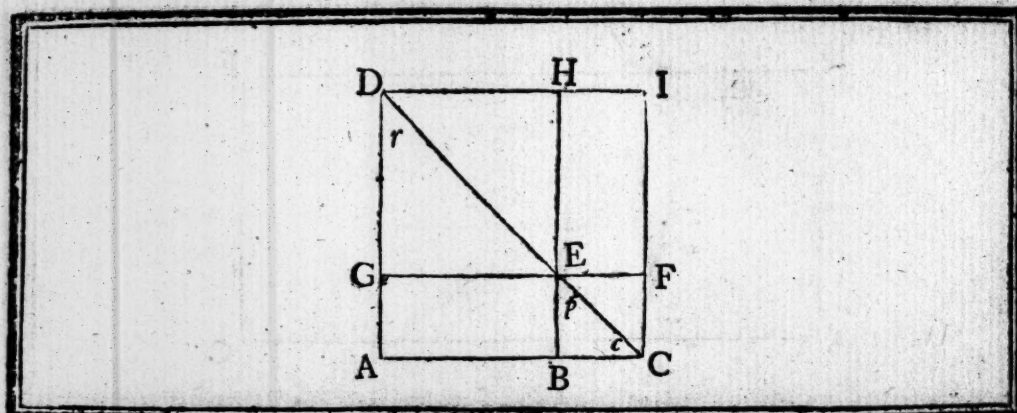
DEMONSTRATION.

1. **T**H E Rgle AF is = to the Rgles AE & BF taken together. Ax. 1. B. 2.
2. But the Rgle AF is contained by the straight lines CA, AB; because it is contained by CA & AD, of which AD = AB (Prep. 1.). Ax. 2. B. 2.
3. And the Rgle BF is contained by AB, BC; because it is contained by EB, BC, of which EB = AB (Prep. 1.).
Moreover, the Rgle AE being the $\square$ of the straight line AB, (Prep. 1.).
4. The Rgle of CA . AB, is = to the Rgle of AB . BC together with the $\square$ of AB; or the Rgle CAB is = to the Rgle ABC + the $\square$ of AB. Ax. 1. B. 1.

Which was to be demonstrated.

K 2





PROPOSITION IV. THEOREM IV.

IF a straight line (AC) be divided into any two parts (AB, BC,); the square of the whole line (AC) is equal to the squares of the two parts (AB, BC,) together with twice the rectangle contained by the parts (AB, BC,).

Hypothesis.

AC is a straight line divided
into any two parts AB, BC.

Thesis.

The $\square$ of AC is = to the $\square$ of AB +
the $\square$ of BC + 2 Rgles ABC.

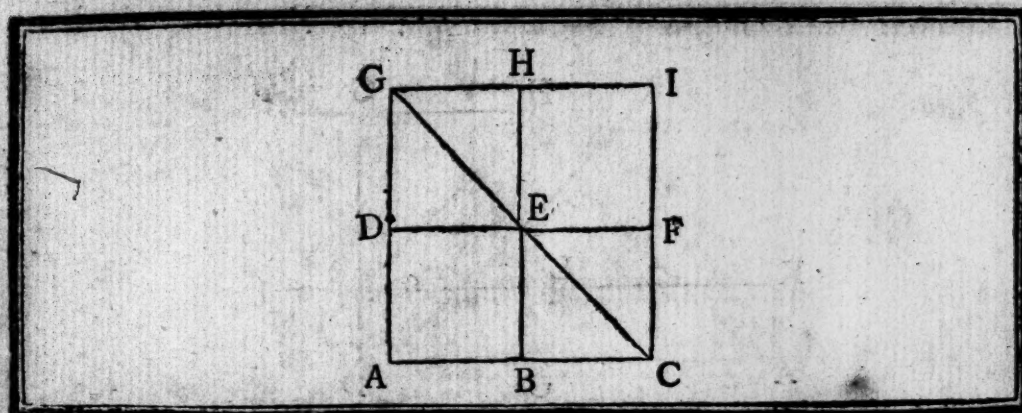
Preparation.

1. Upon AC, describe the $\square$ AI. P. 46. B. 1.
2. Thro' the point of division B, draw BH pple to CI or AD. P. 31. B. 1.
3. Draw the diagonal CD, which will cut BH somewhere in E. Pos. 1.
4. Thro' the point E, draw GF pple to the opposite sides DI or AC. P. 31. B. 1.

DEMONSTRATION.

BECAUSE the lines AD, BH, CI; likewise AC, GF, DI, are
pples (Prep. 1. 2. & 4.).

1. The four figures AE, EI, BF, GH, are Pgrs. D. 35. B. 1.
And since each of those figures include one of the right angles of the $\square$ AI.
2. Those Pgrs are also Rgles. { P. 46. B. 1.
Moreover, because the sides DA, AC, of the $\square$ AI, are equal, { Cor. 2.
(D. 30. B. 1.).
3. The $\angle r$ is = to the $\angle c$. P. 5. B. 1.
And because the straight lines AD, BH, are pples (Prep. 2.) cut by the straight line DC (Prep. 3.).
4. The interior $\angle r$ is = to its exterior opposite $\angle p$. P. 29. B. 1.
5. Consequently, $\angle c = \angle p$. Ax. 1. B. 1.
6. Wherefore, the side BE is = to the side BC. P. 6. B. 1.
7. And the Rgle BF is a $\square$, viz. the $\square$ of BC. D. 30. B. 1.
8. It may be proved in the same manner, that the Pgr GH is a $\square$, viz. the $\square$ of AB, because GE = AB. P. 34. B. 1.
Moreover, BE being = to BC (Arg. 6.);
9. The Rgle AE, or the Rgle of AB. BE, will be = to the Rgle of AB. BC. Ax. 2. B. 2.
But the Rgle AE is = to the Rgle EI (P. 43. B. 1.).
10. From whence it follows, that the Rgle EI is also = to the Rgle of AB. BC. Ax. 1. B. 1.



11. Consequently, the two Rgles AE, EI, taken together, are $\equiv$ to twice the Rgle of the parts AB, BC.
 Since then the two $\square$ GH & BF are the squares of the two parts AB & BC (*Arg.* 7. & 8.), & the Rgles AE, EI, taken together, are $\equiv$ to twice the Rgle of the parts AB, BC.
12. It follows, that the $\square$ of the whole line AC is $\equiv$ to the $\square$ of AB + the $\square$ of BC + 2 Rgles ABC.

Which was to be demonstrated.

COROLLARY I.

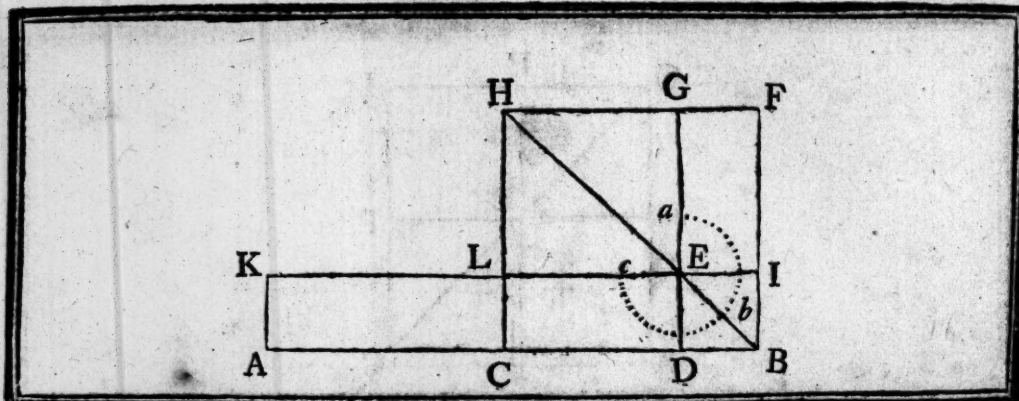
WHEN two straight lines HB, DF, p^{ar}lle to the sides of a square intersect each other in a point E of the diagonal, the Rgles BF, DH, formed about the diagonal, are squares.

COROLLARY II.

IF the line AC be divided into two equal parts in B, the complements AE, EI, are squares, & those complements equal to one another, are also equal to the squares about the diagonal, & the the square of the whole line AC is four times the square of one of the parts AB or BC.

For BF, DH, are squares (by the precedent Corollary), & are equal to one another, because $BC = AB = DE$. Moreover, AE being $\equiv$ to BF, & EI being $\equiv$ to BF (*P.* 36. *B.* 1.), the complements AE, EI, are also squares; & since they are equal to one another, the $\square$ of AC $= 4 \square$ of AB $= 4 \square$ of BC.





PROPOSITION V. THEOREM V.

IF a straightline (AB) be divided equally in (C) & unequally in (D); the rectangle contained by the unequal parts (AD, DB,) together with the square of the part (CD), between the points of section (C & D), is equal to the square of the half (AC or CB) of the whole straight line (AB).

Hypothesis.

AB is a straight line divided equally in C, & unequally in D.

Thesis.

The Rgle ADB + the $\square$ of CD, are = to the $\square$ of CB.

Preparation.

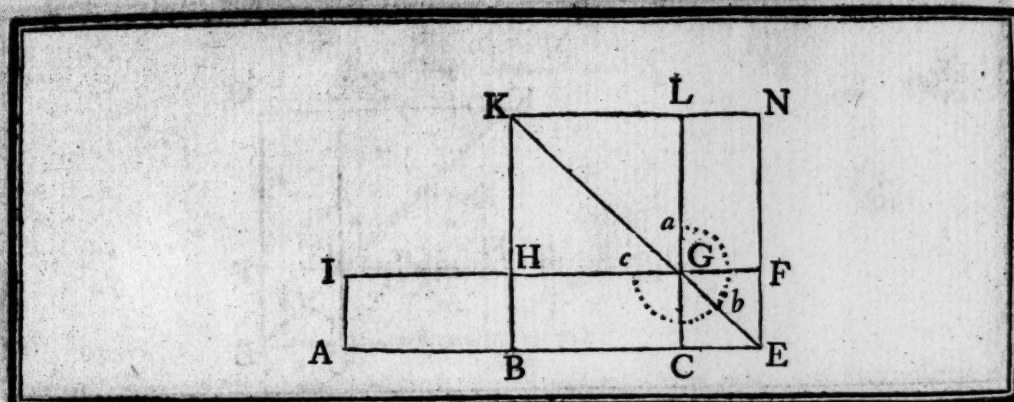
1. Upon the straight line CB, describe the $\square$ CF. P. 46. B. 1.
2. Thro' the point of section D, draw DG pple to BF or CH. P. 31. B. 1.
3. Draw the diagonal BH. Pos. 1.
4. Thro' the point of section E, draw IL pple to BC or FH, & thro' the point A, the straight line AK pple to CL. P. 31. B. 1.

DEMONSTRATION.

BECAUSE the figure CF is a square (*Prep. 1.*).

1. The Rgles LG, DI, about the diagonal are $\square$. { P. 4. B. 2.
Cor. 1.
2. Namely DI the $\square$ of DB, & LG the $\square$ of CD; because LE = CD. P. 34. B. 1.
3. Moreover, the complement CE is = to the complement EF. P. 43. B. 1.
Let the square DI be added to both.
4. The Rgle CI will be = to the Rgle DF. Ax. 2. B. 1.
But because AC is = to CB (*Hyp.*).
5. The Rgle AL is = to the Rgle CI. Ax. 2. B. 2.
6. Consequently, the Rgle AL is = to the Rgle DF. Ax. 1. B. 1.
Therefore, if the Rgle CE be added to both.
7. The Rgle AE will be = to the Rgles DF, CE, i. e. to the Gnomon *abc*. Ax. 2. B. 1.
8. But the Rgle AE is contained by AD, DB; because it is contained by AD, DE, of which DE = DB (*Arg. 1.*). Ax. 2. B. 2.
9. Consequently, the Rgle of AD. DB, is also = to the Gnomon *abc*. Ax. 1. B. 1.
Adding to both the $\square$ LG, which is the $\square$ of CD (*Arg. 2.*).
10. The Rgle AD. DB, together with the $\square$ of CD, will be = to the Gnomon *abc*, together with the $\square$ LG. Ax. 2. B. 1.
But this Gnomon *abc* together with the $\square$ LG, is = to the $\square$ CF, which is the $\square$ of the half CB, of the whole line AB (*Prep. 1.*).
11. Wherefore, the Rgle ADB + the $\square$ of CD, are = to the $\square$ of CB. Ax. 1. B. 1.

Which was to be demonstrated.



PROPOSITION VI. THEOREM VI.

IF a straight line (AC) be bisected in (B), & produced to any point E; the rectangle contained by the whole line thus produced (AE), & the part of it produced (EC), together with the square of the half (BC), is equal to the square of the straight line (BE) made up of the half (BC) & the part produced (CE).

Hypothesis.

- I. AC is a straight line bisected in B.
 II. And which is produced to the point E.

Thefis.

The Rgle AEC + the $\square$ of BC,
 is = to the $\square$ of BE.

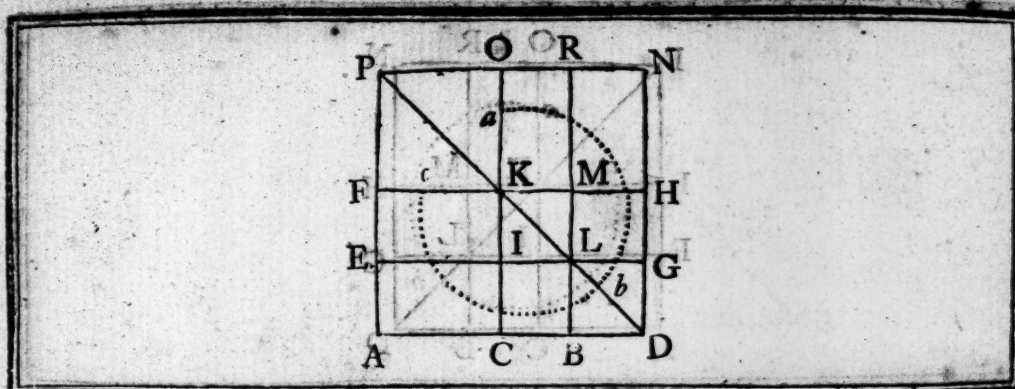
Preparation.

1. Upon the straight line BE, describe the $\square$ BN. P. 46. B. 1.
2. Thro' the point C, draw CL pple to EN or BK. P. 31. B. 1.
3. Draw the diagonal EK. Pof. 1.
4. Thro' the point G, draw FH pple to EB or NK. } P. 31. B. 1.
5. And thro' the point A, draw the straight line AI pple to BK.

DEMONSTRATION.

BECAUSE the figure BN is a square (*Prep. 1.*).

1. The Rgles CF, HL, about the diagonal are squares. - - - - - { P. 4. B. 2.
 And because HG is = to BC (*P. 34. B. 1.*). { Cor. 1.
2. The $\square$ HL is = to the $\square$ of BC. - - - - - { P. 46. B. 1.
 Moreover, AB being = to BC (*Hyp. 1.*). { Cor. 3.
3. The Rgle AH is = to the Rgle BG. Ax. 2. B. 2.
 But the Rgle BG is = to the Rgle GN (*P. 43. B. 1.*).
4. Therefore, the Rgle AH is also = to the Rgle GN. Ax. 1. B. 1.
 And if the Rgle BF be added to both.
5. The Rgle AF will be = to the Rgles GN, BF, *i. e.* to the Gnomon *abc*. Ax. 2. B. 1.
6. But this Rgle AF is contained by AE, EC; because EC = EF (*Arg. 1.*).
7. Consequently, the Rgle AE. EC, is also = to the Gnomon *abc*. Ax. 1. B. 1.
 Therefore, if the $\square$ HL, which is the $\square$ of BC (*Arg. 2.*), be added to both.
8. The Rgle AE. EC, together with the $\square$ of BC, will be = to the Gnomon *abc*, together with the $\square$ HL. Ax. 2. B. 1.
 But the Gnomon *abc* & the $\square$ HL form the $\square$ of BE, (*Prep. 1.*).
9. Consequently, the Rgle AEC + the $\square$ of BC is = to the $\square$ of BE. Ax. 1. B. 1.
 Which was to be demonstrated.



PROPOSITION VIII. THEOREM-VIII.

IF a straight line (AB) be divided into any two parts (AC, CB); four times the rectangle contained by the whole line (AB) & one of the parts (BC), together with the square of the other part (AC), is equal to the square of the straight line (AD), which is made up of the whole (AB), & the part produced (BD) equal to the part (BC).

Hypothesis.

AB is a straight line divided in C, & produced to D, so that BD = BC.

Thesis.

Four times the Rgle ABC + the $\square$ of AC are = to the $\square$ of AD.

Preparation.

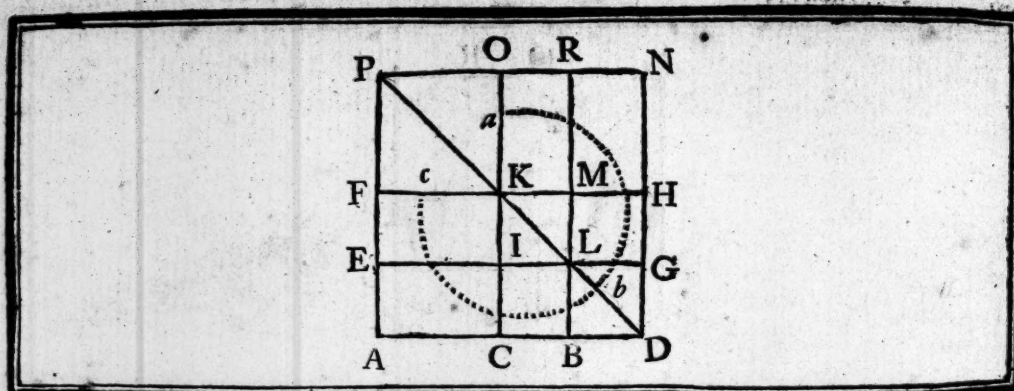
1. Upon AD, describe the $\square$ AN. P. 46. B. 1.
2. Thro' the points B & C, draw BR & CO p^lle to DN or AP. P. 31. B. 1.
3. Draw the diagonal DP. Pos. 1.
4. Thro' the points L & K, draw GE & HF p^lle to DA or NP. P. 31. B. 1.

DEMONSTRATION.

BECAUSE the figure AN is a square (Prep. 1.).

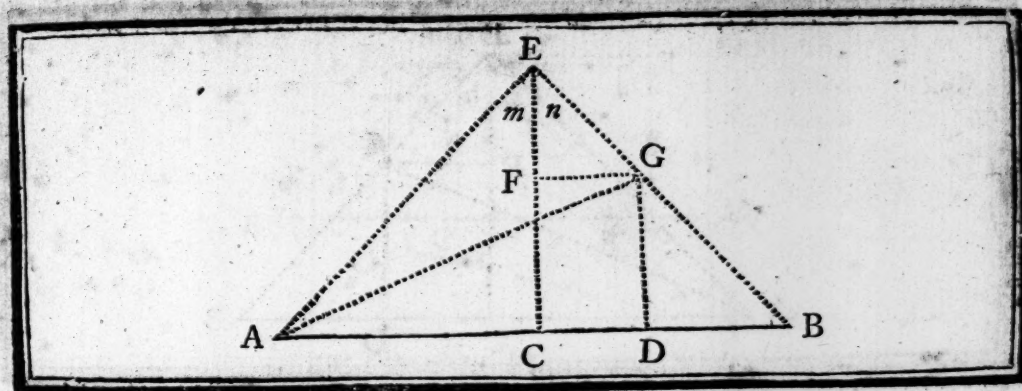
1. The Rgles about the diagonal CH, ER, FO, are squares. { P. 4. B. 2.
Cor. 1.
- And because in the $\square$ CH, the side CD is bisected in B (Hyp.).
2. The Rgles BG, CL, LH, IM, are four equal squares.
3. And the $\square$ CH is = to four times the $\square$ CL. { P. 4. B. 2.
Cor. 2.
- Moreover, because ER is a square (Arg. 1.).
4. The Rgle EK is = to the Rgle KR. P. 43. B. 1.
- But since IK = IC (Arg. 2.), & CO p^lle to AP (Prep. 2.).
5. The Rgle AI is = to the Rgle EK. P. 36. B. 1.
6. Consequently, the Rgle AI is also = to the Rgle KR. Ax. 1. B. 1.
- Likewise, because KM = MH (Arg. 2.), & HF p^lle to NP (Prep. 4.).
7. The Rgle KR is = to the Rgle MN. P. 36. B. 1.
8. Wherefore, the Rgles AI, EK, KR, MN, are = to one another. Ax. 1. B. 1.

L



9. Consequently, their sum is = to four times the Rgle AI.
 If the $\square$ CH which is = to four times the $\square$ CL (*Arg. 3*) be added to both.
10. The Gnomon *abc* which results on one side, is = to four times the Rgle AI & to four times the $\square$ CL, *i. e.* to four times the Rgle AL, the Rgle AI + the $\square$ CL being = to the Rgle AL. *Ax. 2. B. 1.*
 Adding to both the $\square$ of AC, which is = to the $\square$ FO, because $AC = FK$ (*P. 34. B. 1.*).
11. Four times the Rgle AL & the $\square$ of AC will be = to the $\square$ AN. *Ax. 2. B. 1.*
 But the Rgle AL is = to the Rgle contained by AB, BC, because $BC = BL$ (*Arg. 2.*), & the $\square$ AN is = to the $\square$ of AD (*Prep. 1.*).
12. Wherefore, four times the Rgle ABC + the $\square$ of AC, are = to the $\square$ of AD. *Ax. 1. B. 1.*





PROPOSITION IX. THEOREM IX.

IF a straight line (AB) be divided into two equal parts (AC, CB,) & into two unequal parts (AD, DB,) ; the squares of the two unequal parts (AD, DB,) are together double of the the square of the half (AC) of the whole line (AB) & of the square of the part (CD) between the points of section (C & D).

Hypothesis.

AB is a straight line divided equally in C & unequally in D.

Thesis.

The $\square$ of AD + the $\square$ of DB, are double of the $\square$ of AC + the $\square$ of CD.

Preparation.

1. At the point C in the line AB, erect the $\perp$ CE.
2. Make CE = to AC or BC.
3. From the points A & B to the point E, draw AE, BE.
4. Thro' the points D & G, draw the straight lines DG & GF p^{er}ple to CE & AB.

P. 11. B. 1.

P. 3. B. 1.

Pos. 1.

P. 31. B. 1.

DEMONSTRATION.

BECAUSE CE is = to AC (Prep. 2.).

1. The $\angle$ CAE is = to the $\angle$ m.

P. 5. B. 1.

But the $\angle$ ECA is a $\perp$ (Prep. 1.).

2. Wherefore, the two other $\angle$ CAE & m together, make also a $\perp$.

P. 32. B. 1.

3. Consequently, each of them is half a $\perp$; because they are = to one another (Arg. 1.).

It may be proved after the same manner that :

4. Each of the $\angle$ CBE & n is half a $\perp$.

5. Consequently, the whole $\angle$ m + n is = to a $\perp$.

Ax. 2. B. 1.

Again, $\angle$ n being half a $\perp$ (Arg. 4.), & $\angle$ EFG a $\perp$; being = to its interior opposite one ECB (P. 29. B. 1.), which is a $\perp$, (Prep. 1.).

6. The $\angle$ EGF is also half a $\perp$.

P. 32. B. 1.

7. Consequently, EF is = to FG.

P. 6. B. 1.

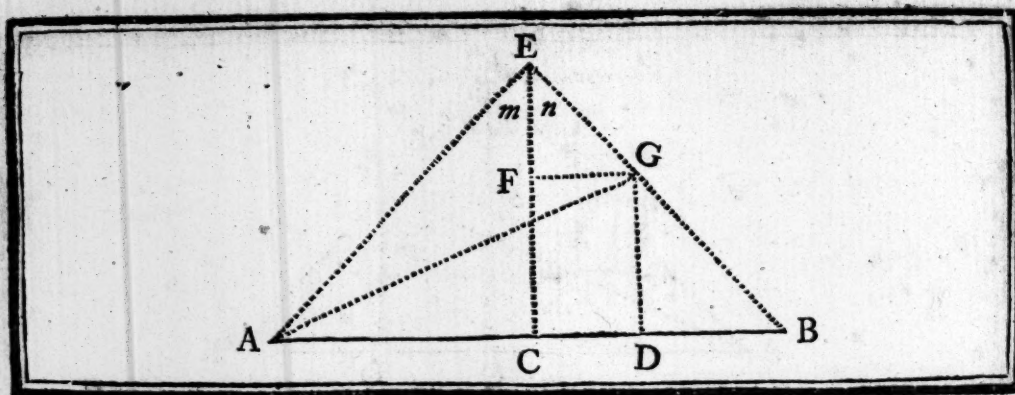
It is proved in the same manner that :

8. The $\angle$ BGD is = to half a $\perp$, & DG = DB.

Since then the $\square$ of AE is = to the $\square$ of AC together, with the $\square$ of CE (P. 47. B. 1.), & AC = CE (Prep. 2.).

9. The $\square$ of AE is double of the $\square$ of AC.

L 2

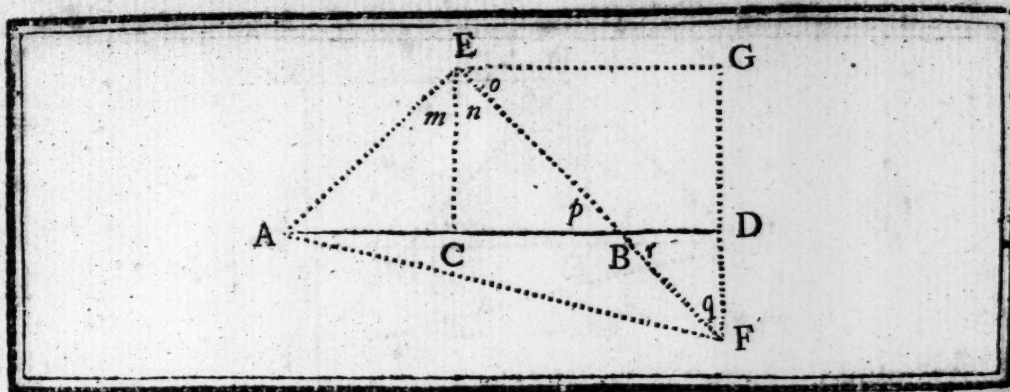


For the same reason :

10. The $\square$ of EG is double of the $\square$ of FG, *i. e.* of the $\square$ of CD, because $FG = CD$. P. 34. B. I.
11. Consequently, the $\square$ of AE & the $\square$ of EG taken together, are double of the $\square$ of AC & of the $\square$ of CD. Ax. 2. B. I.
And because the $\square$ of AE & the $\square$ of EG taken together, are = to the $\square$ of AG (P. 47. B. I. & Arg. 5.).
12. The $\square$ of AG is also double of the $\square$ of AC & of the $\square$ of CD. Ax. 1. B. I.
But $\angle ECA$ being = to a $\angle$ (Prep. 1.), & $\angle GDC =$ to $\angle ECA$, (P. 29. B. I.).
13. The $\square$ of AG is = to the $\square$ of AD & to the $\square$ of DG. P. 47. B. I.
14. Or the $\square$ of AG is = to the $\square$ of AD & to the $\square$ of DB taken together, because DB is = to DG (Arg. 8.).
- 15 Wherefore, the $\square$ of AD & the $\square$ of DB taken together, are double of the $\square$ of AC & of the $\square$ of CD; or the $\square$ of AD + the $\square$ of DB, are double of the $\square$ of AC + the $\square$ of CD. Ax. 1. B. I.

Which was to be demonstrated.





PROPOSITION X. THEOREM X.

IF a straight line (AB) be bisected in (C) & produced to any point (D), the square of the whole line thus produced (AD) & the square of the part of it produced (BD), are together double of the square of the half (AC) of the whole line (AB), & of the square of the line (CD) made up of the half (CB) & the part produced (BD),

Hypothesis.

AB is a straight line bisected in C
and produced to the point D.

Thesis.

The $\square$ of AD + the $\square$ of BD, are double of the $\square$ of AC + the $\square$ of CD.

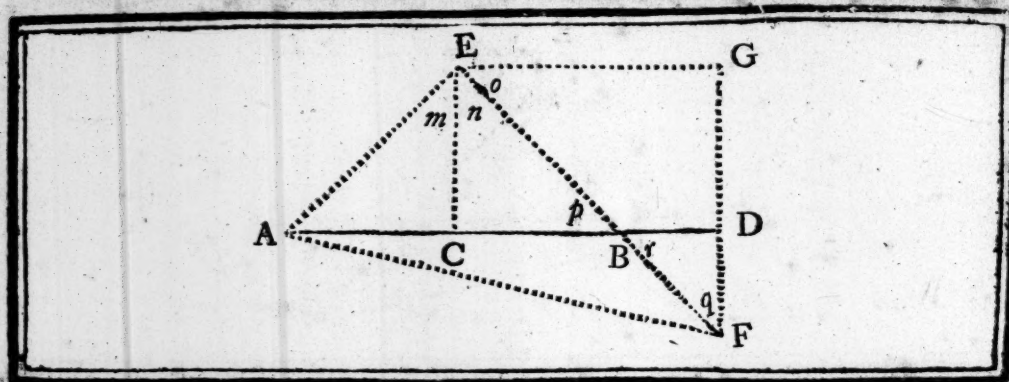
Preparation.

1. At the point C in the line AB, erect the $\perp$ CE. P. 11. B. 1.
2. Make CE = AC or BC. P. 3. B. 1.
3. From the points A & B to the point E, draw AE & BE. Pos. 1.
4. Thro' the points E & D, draw EG, DG, p^lle to AD & CE, P. 31. B. 1.
and produce DG until it meets EB produced, in F. Pos. 2.

DEMONSTRATION.

BECAUSE in the $\triangle$ ACE the side AC is = to CE (Prep. 2.).

1. The $\sphericalangle$ CAE is = to $\sphericalangle$ m. P. 5. B. 1.
But $\sphericalangle$ ACE is a $\perp$ (Prep. 1.).
2. Hence each of the $\sphericalangle$ CAE & m is half a $\perp$. P. 32. B. 1.
It is proved in the same manner that :
3. Each of the $\sphericalangle$ p & n is half a $\perp$.
4. Consequently, $\sphericalangle$ m + n will be = to $\perp$. Ax. 2. B. 1.
Moreover, $\sphericalangle$ p being half a $\perp$ (Arg. 3.).
5. The $\sphericalangle$ r will be also half a $\perp$. P. 15. B. 1.
But the $\sphericalangle$ BDF being a $\perp$ (P. 29. B. 1.), because it is the alternate of $\sphericalangle$ ECD which is a $\perp$ (Prep. 1.).
6. The $\sphericalangle$ q is also half a $\perp$. P. 32. B. 1.
7. Consequently, the side BD is = to the side DF. P. 6. B. 1.
Likewise, $\sphericalangle$ q being half a $\perp$ (Arg. 6.), & $\sphericalangle$ G a $\perp$, as being diagonally opposite to $\sphericalangle$ ECD (P. 34. B. 1.).
8. The $\sphericalangle$ o is half a $\perp$. P. 32. B. 1.
9. Therefore EG = GF. P. 6. B. 1.



Also AC being $\equiv$ to CE (*Prep. 2.*).

10. The $\square$ of AC is $\equiv$ to the $\square$ of CE.

{ *P. 46. B. 1.*
Cor. 3.

11. Consequently, the $\square$ of AC & of CE are double of the $\square$ of AC.
 And those $\square$ of AC & CE being $\equiv$ to the $\square$ of AE (*P. 47. B. 1.*).

12. The $\square$ of AE will be also double of the $\square$ of AC.

Ax. 6. B. 1.

It is proved after the same manner that:

13. The $\square$ of EF is double of the $\square$ of EG; *i. e.* of the $\square$ of CD,
 because $EG = CD$.

P. 34. B. 1.

14. Consequently, the $\square$ of AE together with the $\square$ of EF, are double of the $\square$ of AC & of the $\square$ of CD.

But the $\square$ of AE & the $\square$ of EF being $\equiv$ to the $\square$ of AF,
 (*P. 47. B. 1.*).

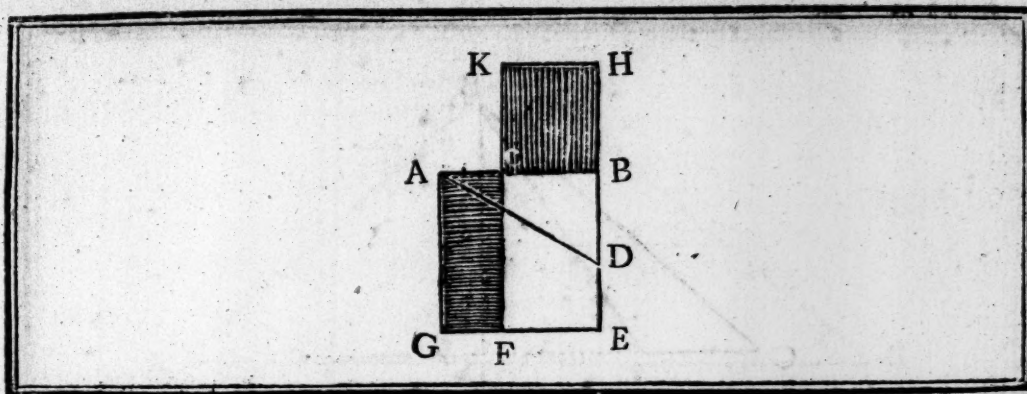
15. The $\square$ of AF is double of the $\square$ of AC & of the $\square$ of CD.

And this same $\square$ of AF being also $\equiv$ to the $\square$ of AD & to the $\square$ of DF (*P. 47. B. 1.*), or or BD, since $DF = BD$ (*Arg. 7.*).

16. It follows, that the $\square$ of AD + the $\square$ of BD, are double of the $\square$ of AC + the $\square$ of CD.

Which was to be demonstrated.





PROPOSITION. XI. PROBLEM I.

TO divide a given straight line (AB) into two parts, so that the rectangle contained by the the whole (BA) & one of the parts (AC) shall be equal to the square of the other part (CB).

Given
The straight line AB.

Sought
The point of intersection C, such that the Rgle BAC shall be = to the $\square$ of CB.

Resolution.

1. Upon the straight line AB, describe the $\square$ AE. P. 46. B. 1.
2. Bisect the side BE in D, & draw thro' the point D to the point A the straight line DA. P. 10. B. 1.
3. Upon EB produced, take DH = DA. Pos. 1.
4. Upon the straight line BH, describe the $\square$ CH. P. 3. B. 1.
5. And produce the side KC to F. P. 46. B. 1.

Pos. 2.

DEMONSTRATION.

BECAUSE the straight line BE is bisected in D & produced to the point H.

1. The Rgle EH. HB + the $\square$ of BD is = to the $\square$ of DH. P. 6. B. 2.
2. And this $\square$ of DH is = to the $\square$ of DA, because DH = DA (Ref. 3.). P. 46. B. 1.
3. Consequently, the Rgle EH. HB + the $\square$ of BD is = to the $\square$ of DA. Cor. 3.

But this same $\square$ of DA is = to the $\square$ of AB + the $\square$ of BD (P. 47. B. 1.).

4. Wherefore, the Rgle EH. HB + the $\square$ of BD is = to the $\square$ of AB + the $\square$ of BD. Ax. 1. B. 1.

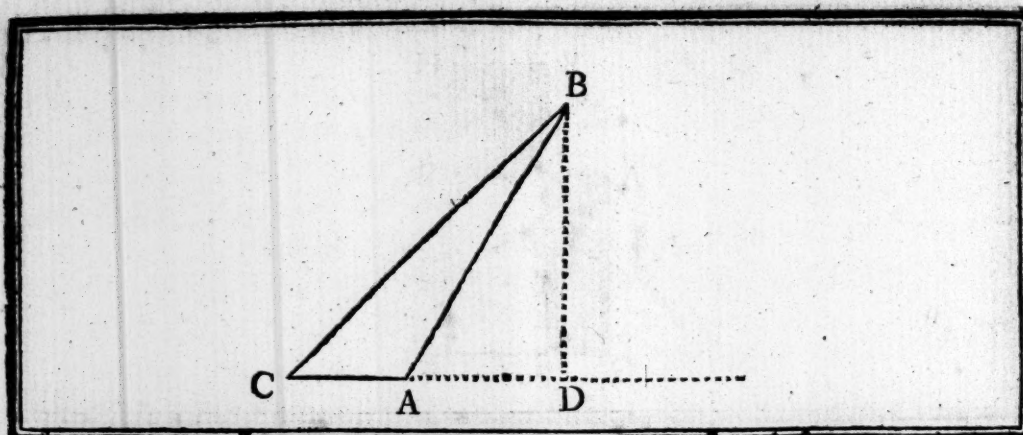
Therefore if the $\square$ of BD be taken away from both sides.

5. The Rgle EH. HB will be = to the $\square$ of AB. Ax. 3. B. 1.
- And if from the Rgle EH. HB which is = to the Rgle FH (Ref. 4. 5.) and from the $\square$ of AB which is = to the $\square$ AE (Ref. 1.) the Rgle FB be taken away.

6. There will remain the $\square$ CH = to the Rgle GC. Ax. 3. B. 1.
- This $\square$ CH being therefore = to the $\square$ of BC (Ref. 4.), & the Rgle GC = to the Rgle BA. AC; because AG = AB (Ref. 1.).

7. It follows, that the straight line AB is divided in C, so that the Rgle BAC is = to the $\square$ of CB. Ax. 1. B. 1.

Which was to be done.



PROPOSITION XII. THEOREM XI.

IN any obtuse angled triangle (CBA) ; if a perpendicular be drawn from one of the acute angles (B) to the opposite side (CA) produced ; the square of the side (BC) subtending the obtuse angle (A), is greater than the squares of the sides (AB, CA,) containing the obtuse angle, by twice the rectangle contained by the side (CA), upon which when produced the perpendicular falls, & the straight line (AD) intercepted between the perpendicular & the obtuse angle (A).

Hypothesis.

- I. CBA is an obtuse angled Δ .
 II. BD the $\perp$ drawn from the vertex of the $\angle$ B to the opposite side CA produced.

Thesis.

The $\square$ of BC is = to the $\square$ of AB
 + the $\square$ of AC + 2 Rgles CAD.

DEMONSTRATION.

BECAUSE the straight line CD is divided into two parts CA, AD,
 (Hyp. 2.).

1. The $\square$ of CD is = to double the Rgle CA . AD together with the $\square$ of CA & of AD.

P. 4. B. 2.

Therefore if the $\square$ of BD be added to both sides.

2. The $\square$ of CD + the $\square$ of BD, will be = to double the Rgle CA . AD + the $\square$ of CA + the $\square$ of AD + the $\square$ of BD.

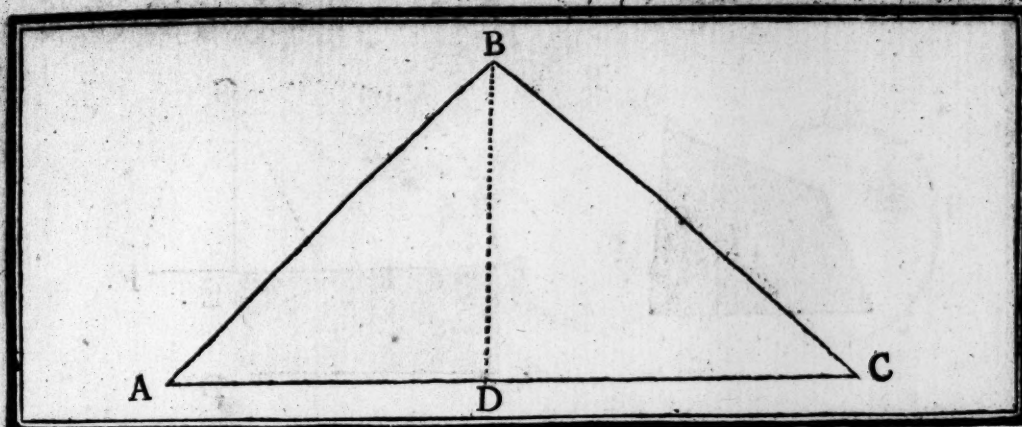
Ax. 2. B. 1.

But the $\square$ of CD together with the $\square$ of BD is = to the $\square$ of BC, and the $\square$ of AD together with the $\square$ of BD is = to the $\square$ of AB, (P. 47. B. 1.).

3. Consequently, the $\square$ of BC is = to double the Rgle CAD + the $\square$ of CA + the $\square$ of AB.

Ax. 1. B. 1.

Which was to be demonstrated.



PROPOSITION XIII. THEOREM XII.

IN every acute angled triangle (CBA); the square of the side (BA) subtending one of the acute angles (C), is less than the squares of the sides (CB, CA,) containing that angle, by twice the rectangle contained by one of those sides (AC) & the straight line (CD) intercepted between the perpendicular (BD) let fall upon it from the opposite angle (B), & the acute angle (C).

Hypothesis.

- I. CBA is an acute angled $\triangle$.
- II. BD the $\perp$ let fall upon AC from the opposite angle B.

Thesis.

The $\square$ of BA + twice the Rgle ACD is = to the $\square$ of CA + the $\square$ of CB.

DEMONSTRATION.

BECAUSE the straight line CA is divided into two parts CD, DA, (Hyp. 2.).

1. The $\square$ of CA together with the $\square$ of CD is = to twice the Rgle AC. CD together with the $\square$ of AD.

P. 7. B. 2.

Therefore if the $\square$ of DB be added to both sides:

2. The $\square$ of CA + the $\square$ of CD + the $\square$ of DB will be = to twice the Rgle AC. CD + the $\square$ of AD + the $\square$ of DB.

Ax. 2. B. 1.

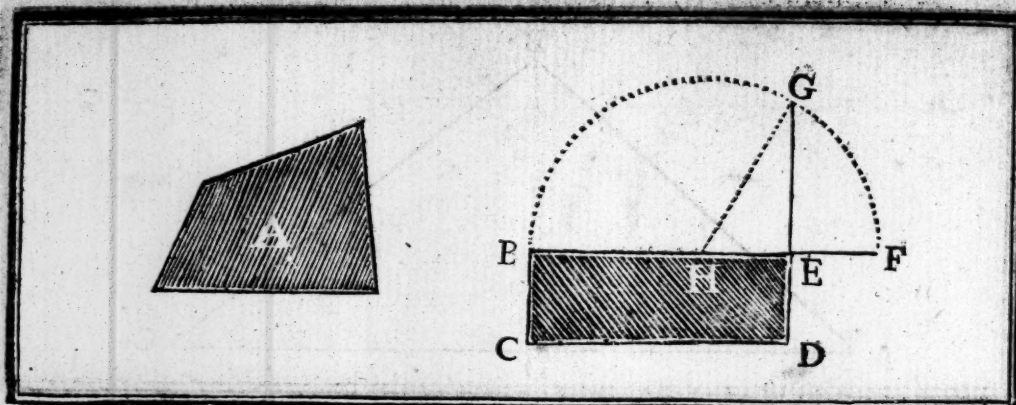
But the $\square$ of CD + the $\square$ of DB is = to the $\square$ of CB, & the $\square$ of AD + the $\square$ of DB is = to the $\square$ of BA (P. 47. B. 1.).

3. Wherefore the $\square$ of BA + twice the Rgle ACD is = to the $\square$ of CA + the $\square$ of CB.

Ax. 1. B. 1.

Which was to be demonstrated.

M



PROPOSITION XIV. PROBLEM II.

TO describe a square that shall be equal to a given rectilinear figure (A).
 Given Sought
The rectilinear figure A. *The construction of a square =*
to a given rectilinear figure A.

Resolution.

1. Describe the Rgle Pgr CE = to the figure A. P. 45. B. 1.
2. Produce the side BE, & make EF = to ED. P. 3. B. 1.
3. Bisect the straight line BF in H. P. 10. B. 1.
4. From the center H at the distance HB, describe the $\odot$ BGF. Pos. 3.
5. Produce the side DE, until it cuts the $\odot$ BGF in G. Pos. 1.

Preparation.

From the point H to the point G, draw the straight line HG. Pos. 1.

DEMONSTRATION.

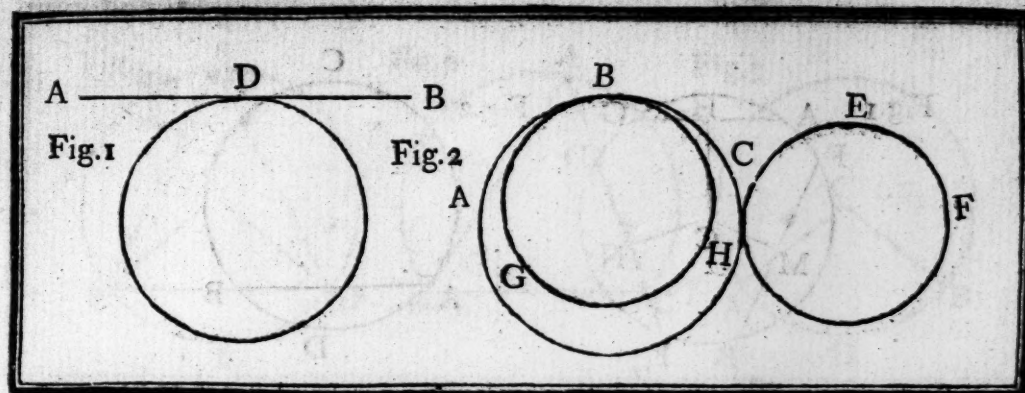
BECAUSE BF is divided equally in H & unequally in E (Ref. 3 & 2.).

1. The Rgle BE . EF together with the $\square$ of HE is = to the $\square$ of HF. P. 5. B. 2.
2. And because HF = HG (D. 15. B. 1.), the $\square$ of HF = the $\square$ of HG, P. 46. B. 1.
 the Rgle BE . EF + the $\square$ HE is = to the $\square$ of HG. Cor. 3.
 But the $\square$ of HG being = to the $\square$ HE + the $\square$ of EG (P. 47. B. 1.).
3. The Rgle BE . EF + the $\square$ of HE is also = to the $\square$ of HE + the $\square$ of EG. Ax. 1. B. 1.
- Therefore, if the $\square$ of HE be taken away from both sides :
4. The Rgle BE . EF will be = to the $\square$ of EG. Ax. 3. B. 1.
- And this Rgle BE EF being moreover = to the Rgle BE . ED ; because EF = ED (Ref. 2.).
5. The Rgle BE . ED will be also = to the $\square$ of EG. Ax. 1. B. 1.
- But the Rgle BE . ED is = to the given figure A (Ref. 1.).
6. Consequently, the $\square$ of EG will be also = to this given figure A. Ax. 1. B. 1.

Which was to be done.

REMARK.

IF the point H falls upon the point E, the straight lines BE, EF, ED, will be each equal to EG, & the Rgle Pgr CE itself, will be the square sought (Cor. 1. & 3. of P. 46. B. 1.).



DEFINITIONS.

I.

A Straight line (ADB) is said to *touch a circle* when it meets the circle & being produced does not cut it. *Fig. 1.*

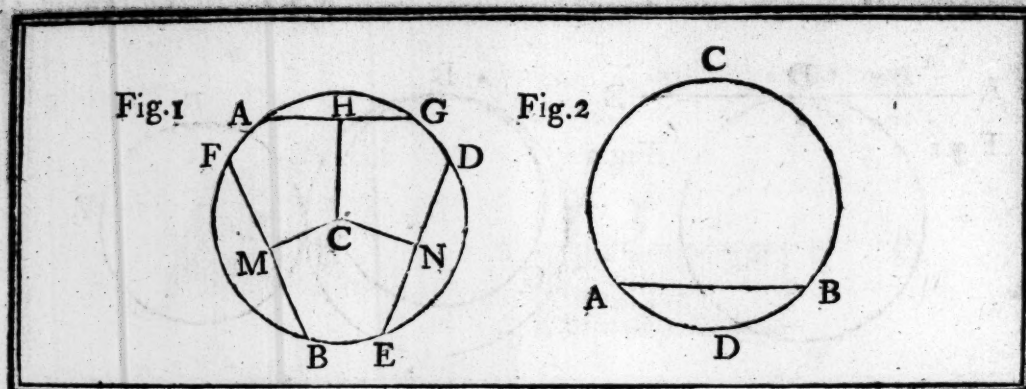
II.

Circles are said to *touch one another* when their circumferences (ABC, CEF, or ABC, GBH) meet but do not cut one another. *Fig. 2.*

III.

Two circles touch each other externally, *when one (CEF) falls without the other (ABC): but two circles touch each other internally, when one (GBH) falls within the other (ABC).* *Fig. 2.*





DEFINITIONS.

IV.

THE distance of a straight line (FB) from the center of a circle, is the perpendicular (CM) let fall from the center of the circle (C) upon this straight line (FB); for which reason two straight lines (FB, DE,) are said to be equally distant from the center of a circle, when the perpendiculars (CM, CN,) let fall upon those lines (FB, DE,) from the center (C), are equal. Fig. 1.

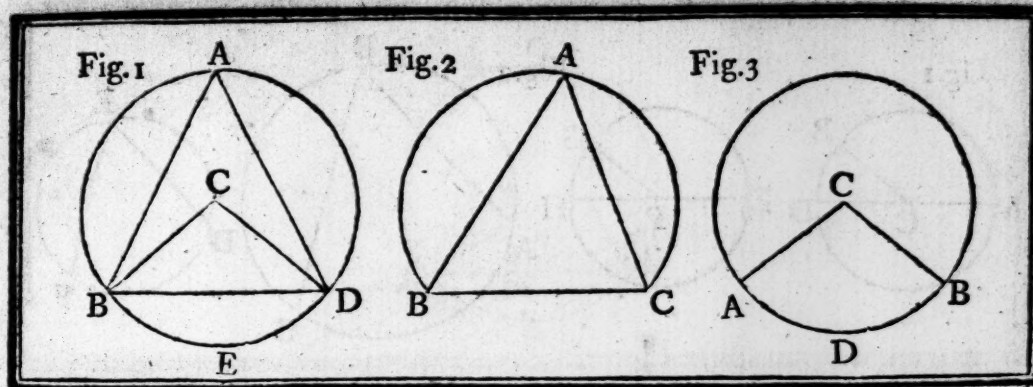
V.

And a straight line (AG) is said to be farther from the center of the circle than (BF or ED), when the perpendicular (CH) drawn to this line from the center (C), is greater than (CM or CN). Fig. 1.

VI.

The angle of a segment, is the angle (CAB or DAB) formed by the arch (CA or DA) of the segment (ACB or ADB) & by its chord (AB). Fig. 2.





DEFINITIONS.

VII.

A *Angle in a segment*, is the angle (BAC) contained by two straight lines (AB, AC,) drawn from any point (A) of the arch of the segment, to the extremities (B & C) of the chord (BC) which is the base of the segment. *Fig. 2.* *When the straight lines (AB, AD,) are drawn from a point (A) in the circumference of the circle, the angle (BAD) is an angle at the circumference: but when the straight lines (CB, CD,) are drawn from the center, the angle (BCD) is an angle at the center.* *Fig. 1.*

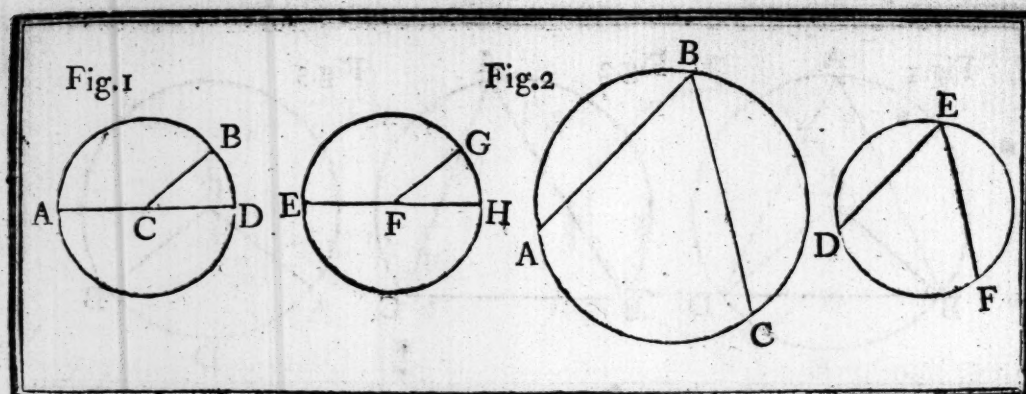
VIII.

An angle is said to *insist or stand upon the arch of a circle*, when the straight lines (AB, AD, or CB, CD,) which form this angle (BAD, or BCD,) are drawn; either from the same point (A) in the circumference; or from its center (C), to the extremities (B & D) of the arch (BED). *Fig. 1.*

IX.

A sector of a circle, is the figure contained by two rays (CA, CB,) & the arch (ADB) between those two rays. *Fig. 3.*





A X I O M S.

I.

EQUAL circles (ABD, EGH,) are those of which the diameters (AD, EH,) or the rays (CB, FG,) are equal. *Fig. 1.*

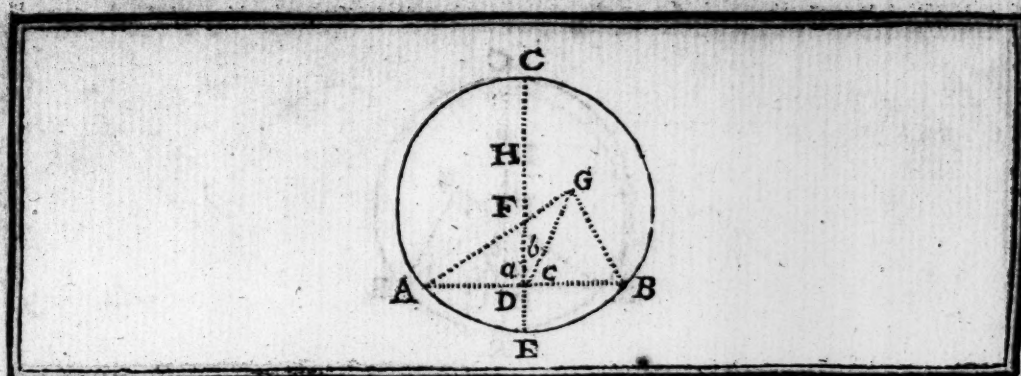
If the circles be applied to one another, so that their centers coincide, when their rays are equal, the circles must likewise coincide.

II.

Similar segments of circles (ABC, DEF,) are those in which the angles (ABC, DEF,) are equal. *Fig. 2.*

Circles are similar figures. If then the two segments (ABC, DEF,) be taken away by substituting the equal angles (ABC, DEF,) those segments are similar.





PROPOSITION I. PROBLEM I.

TO find the center (F) of a given circle (ACBE).

Given
The $\odot$ ACBE.

Sought
The center F of this $\odot$.

Resolution.

1. Draw the chord AB.
 2. Bisect it in the point D.
 3. At the point D in AB, erect the $\perp$ DE & produce it to E.
 4. Bisect CE in F.
- The point F will be the center sought of the given $\odot$ ACBE.

Ref. 1.
P. 10. B. 1.
P. 11. B. 1.
P. 10. B. 1.

DEMONSTRATION.

If not,

Some other point as H or G taken in the line, or without the line EC, will be the center sought of the $\odot$ ACBE.

Case I.

BECAUSE the center to be in EC at a point H different from F. BECAUSE the center of the $\odot$ is in the line EC, at a point H different from F (*Sup. 1.*).

1. The rays HE & HC are $=$ to one another.
But FE being $=$ to FC (*Ref. 4.*) & $HC < FC$ (*Ax. 8. B. 1.*).
2. HC will be also $< FE$, & a fortiori $< HE$.
3. Therefore HE is not $=$ to HC.
4. Consequently, the point H taken in the line EC different from the point F, cannot be the center of the $\odot$ ACBE.

D. 15. B. 1.

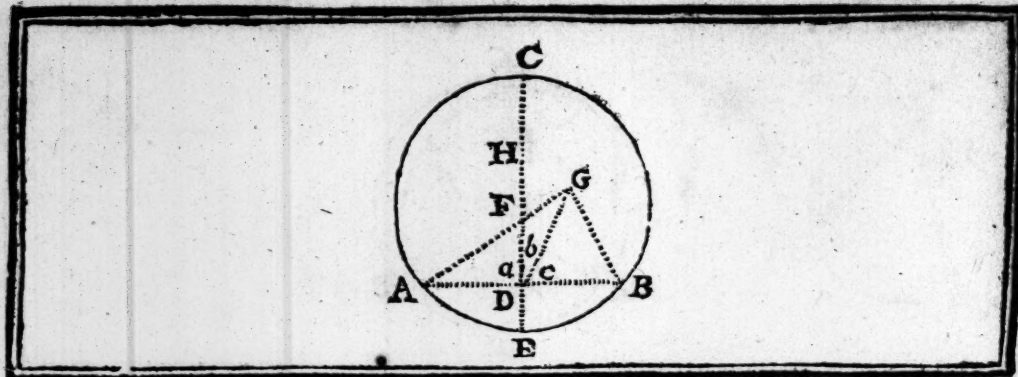
Case II.

Suppose the center to be without the line EC in the point G.

Preparation.

BECAUSE Draw from the center G, the straight lines GA, GD, GB. BECAUSE in the $\triangle AGD$, DGB , the side GA is $=$ to side GB (*Prep. & D. 15. B. 1.*), the side GD common to the two $\triangle$, & the base AD $=$ to the base DB (*Ref. 2.*).

Ref. 1.

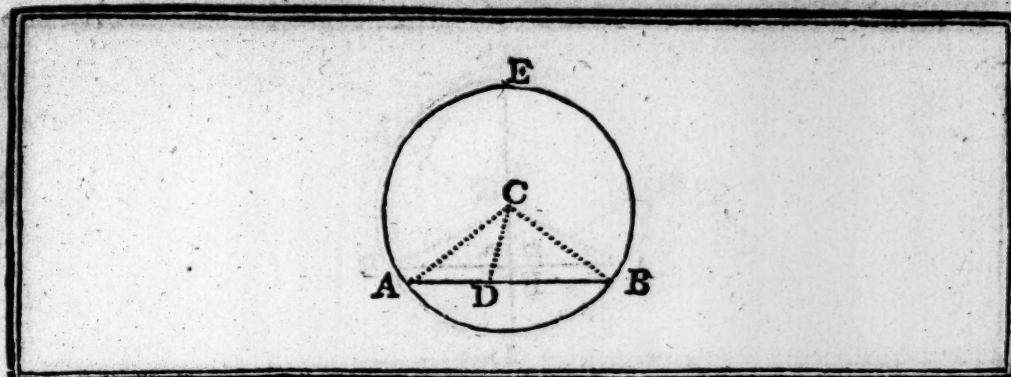


1. The adjacent $\angle a + b \& c$ to which the equal sides GA, GB, are opposite, are $\equiv$ to one another. P. 8. B. 1.
2. Therefore $\angle a + b$ is a $\angle$. D. 10. B. 1.
But $\angle a$ being also a $\angle$ (Ref. 3.).
3. It follows, that $\angle a + b$ is $\equiv$ to $\angle a$, which is impossible. Ax. 8. B. 1.
4. Therefore the point G taken without the line EC, cannot be the center of the $\odot$ ACBE.
Consequently, since the center is not in the line EC, at a point H different from F (Case 1.) nor without the line EC in a point G (Case. 11.)
5. The center sought of the $\odot$ ACBE, will be necessarily in F.
Which was to be done.

COROLLARY.

If in a circle ACBE, a chord EC bisects another chord AB at right angles; this chord CE is a diameter, & consequently passes thro' the center of the circle, (D. 17. B. 1.).





PROPOSITION II. THEOREM I.

IF any two points (A & B) be taken in the circumference of a circle (AEB); the straight line (AB) which joins them, shall fall within the circle.

Hypothesis.

The two points A & B are taken in the $\odot$ AEB.

Thesis.

The straight line AB falls within the $\odot$ AEB.

Preparation.

1. Find the center C of $\odot$ AEB.
2. Draw the straight lines CA, CD, CB.

P. 1. B. 3.
Pos. 1.

DEMONSTRATION.

BECAUSE in the $\triangle$ ACB, the side CA is = to the side CB, (Prep. 2. & D. 15. B. 1.).

1. The $\angle$ CAD, CBD, are = to one another.

P. 5. B. 1.

But $\angle$ CDA being an exterior $\angle$ of $\triangle$ CBD.

2. It is $>$ than its interior CBD.

P. 16. B. 1.

And because the $\angle$ CBD is = to the $\angle$ CAD (Arg. 1.).

3. This $\angle$ CDA will be also $>$ than $\angle$ CAD.

4. Consequently, the side CA opposite to the greater $\angle$ CDA, is $>$ the side CD opposite to the lesser $\angle$ CAD.

P. 19. B. 1.

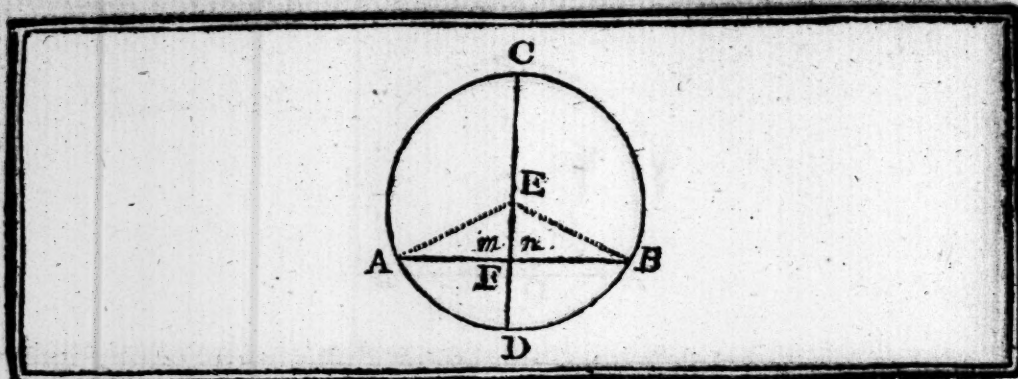
5. From whence it follows, that the extremity D of this side CD falls within the $\odot$ AEB.

And as the same may be demonstrated with respect to any other point in the line AB.

6. It is evident that the whole line AB falls within the $\odot$ AEB.

Which was to be demonstrated.

N



PROPOSITION III. THEOREM II.

IF a diameter (CD) bisects a chord (AB) in (F); it shall cut it at right angles, & *reciprocally* if a diameter (CD) cuts a chord (AB) at right angles, it shall bisect it.

I.

Hypothesis.

CD is a diameter of the $\odot$ ACBD, which bisects AB in F.

Thesis.

The diameter CD is $\perp$ upon the chord AB.

Preparation.

Draw the rays EA, EB.

Pos. 1.

DEMONSTRATION.

IN the $\triangle$ AEF, BEF, the side EA is $=$ to the side EB (*Prep.* & D. 15. B. 1.), the side EF is common to the two $\triangle$, & the base AF is $=$ to the base BF (*Hyp.*).

1. Consequently, the adjacent $\angle$ m & n , to which the equal sides EA, EB, are opposite, are $=$ to one another.
2. Wherefore, the straight line CD, which stands upon AB making the adjacent $\angle$ m & n $=$ to one another, is $\perp$ upon AB.

P. 8. B. 1.

D. 10. B. 1.

Which was to be demonstrated.

II.

Hypothesis.

CD is a diameter of the $\odot$ ACBD, $\perp$ upon the chord AB; or which makes $\angle$ $m = \angle$ n .

Thesis.

AF is $=$ to FB.

DEMONSTRATION.

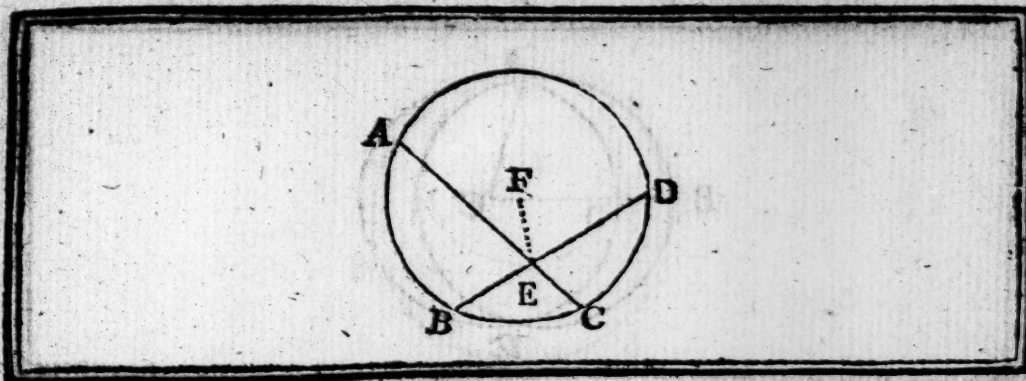
THE sides EA, EB, of the $\triangle$ AEB being $=$ to one another (*Prep.* & D. 15. B. 1.).

1. The $\angle$ EAF, EBF, will be also $=$ to one another. Since then in the $\triangle$ AEF, BEF, the $\angle$ EAF, EBF, are $=$ (*Arg.* 1.), as also the $\angle$ m & n (*Hyp.*), & the side EF common to the two $\triangle$.
2. The base AF will be $=$ to the base FB.

P. 5. B. 1.

P. 26. B. 1.

Which was to be demonstrated.



PROPOSITION IV. THEOREM III.

In a circle (ADCB) two chords (AC, DB,) cut one another, they are divided into two unequal parts.

Hypothesis.

The two chords AC, DB, of the $\odot$ ADCB cut one another in the point E.

Thesis.

These chords are divided into two unequal parts.

DEMONSTRATION.

If not,

The chords AC, DB, bisect one another.

Preparation.

From the center F to the point E, draw the portion of the diameter FE.

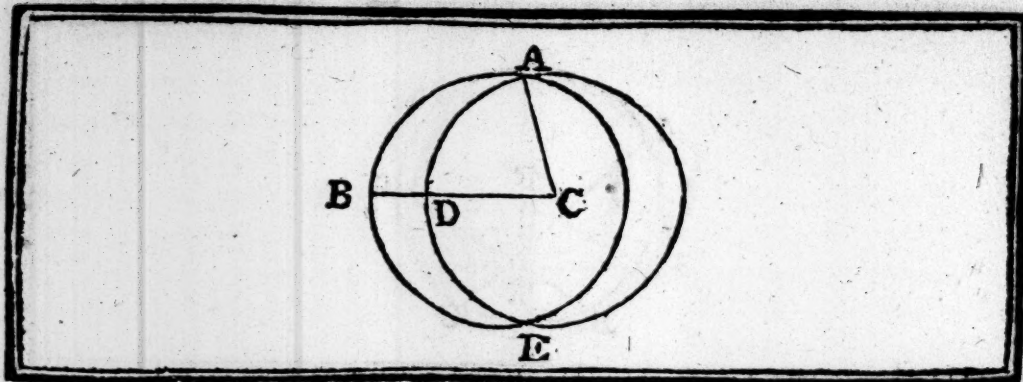
Pos. 1.

BECAUSE the diameter, or its part FE, bisects each of the chords AC, DB, of the $\odot$ ADCB (*Sup.*).

1. This straight line FE is $\perp$ upon each of the chords AC, DB. *P. 3. B. 1.*
2. Consequently, the $\angle$ FEB, FEA, are $=$ to one another; which *Ax. 10. B. 1.*
is impossible. *Ax. 8. B. 1.*
3. Wherefore, the two chords AC, DB, are divided into two unequal parts.

Which was to be demonstrated.





PROPOSITION V. THEOREM IV.

IF two circles (ABE, ADE,) cut one another, they shall not have the same center (C).

Hypothesis.

ABE, ADE, are two $\odot$ which cut one another in the points A & E.

Thesis.

Those two $\odot$ have different centers.

DEMONSTRATION.

If not,

The circles ABE, ADE, have the same center C.

Preparation.

1. From the point C to the point of section A, draw the ray CA. } *Pos. 1.*
2. And from the same point C, draw the straight line CB; which cuts the two $\odot$ in D & B.

BECAUSE the straight lines CA, CD, are drawn from the center C to the $\odot$ ADE (*Prep. 1. & 2.*).

1. These straight lines CA, CD, are = to one another.

D. 15. B. 1.

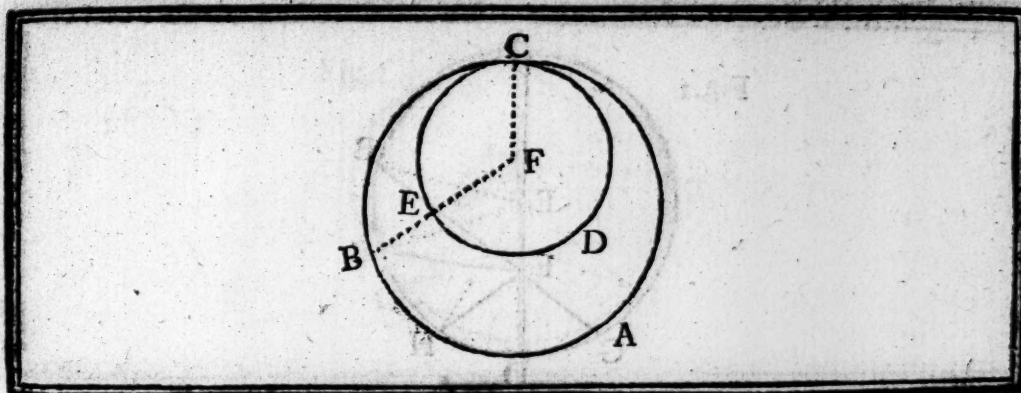
It is proved in the same manner, that :

2. The straight lines CA, CB, are = to one another.
3. Consequently, CB will be = to CD ; which is impossible.
4. Therefore, the two circles ABE, ADE, have not the same center.

Ax. 8. B. 1.

Which was to be demonstrated.





PROPOSITION VI. THEOREM V.

IF two circles (BCA, ECD,) touch one another internally in (C); they shall not have the same center (F).

Hypothesis.

The $\odot$ ECD touches the $\odot$ BCA internally in C.

Thesis.

Those two $\odot$ have different centers.

DEMONSTRATION.

If not,

The $\odot$ BCA, ECD, have the same center F.

Preparation.

Draw the rays FB, FC.

Pos. 1.

BECAUSE the point F is the center of the $\odot$ BCA (Sup.).

1. The rays FB, FC, are = to one another.

Again, the point F being also the center of $\odot$ ECD (Sup.)

2. The rays FE, FC, are = to one another.

3. Consequently, FB = FE (Ax. 1. B. 1.); which is impossible.

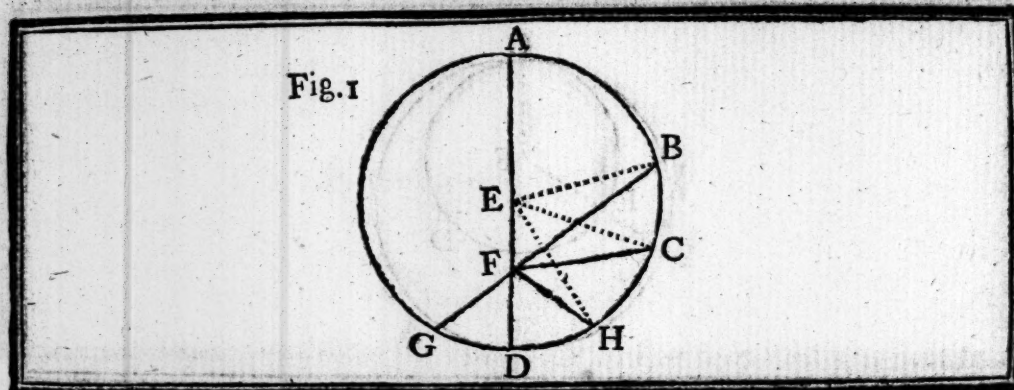
4. Wherefore, the two $\odot$ BCA, ECD, have not the same center.

} D. 15. B. 1.

Ax. 8. B. 1.

Which was to be demonstrated.





PROPOSITION VII. THEOREM VI.

IF any point (F) be taken in a circle (AHG) which is not the center (E), of all the straight lines (FA, FB, FC, FH,) which can be drawn from it to the circumference, the greatest is (FA) in which the center is, & the part (FD) of that diameter is the least, & of any others, that (FB or FC) which is nearer to the line (FA) which passes thro' the center is always greater than one (FC or FH) more remote, & from the same point (F) there can be drawn only two straight lines (FH, FG,) that are equal to one another, one upon each side of the shortest line (FD).

Hypothesis.

- I. The point F taken in the $\odot$ AHG is not the center E.
- II. The straight line FA, drawn from the point F, passes thro' the center E of the $\odot$ AHG.
- III. And the straight lines FB, FC, FH, are drawn from the point F to the $\odot$ AHG.

Thesis.

- I. FA is the greatest of all the straight lines which can be drawn from the point F to the $\odot$ AHG.
- II. FD is the least.
- III. And of any others FB or FC which is nearer to FA is $>$ FC or FH more remote.
- IV. From the point F there can be drawn only two $=$ straight lines FH, FG, one upon each side of the shortest FD.

I. Preparation.

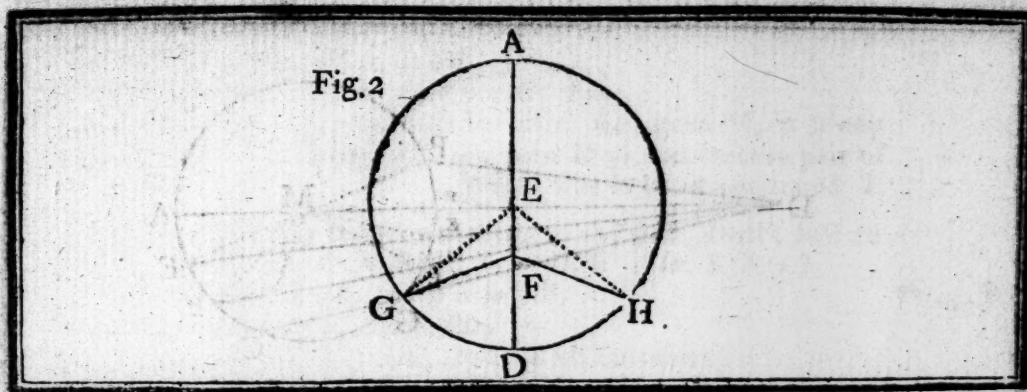
Draw the rays EB, EC, EH, &c. Fig. 1.

DEMONSTRATION.

1. **T**HE two sides FE + EB of the $\triangle$ FEB are $>$ the third FB. P. 20. B. 1. But EB is $=$ to EA (D. 15. B. 1.).
2. Therefore, FE + EA, or FA is $>$ FB. It is proved in the same manner that :
3. The straight line FA, is the greatest of all the straight lines drawn from the point F to the $\odot$ AHG.

Which was to be demonstrated. I.

4. Again, the two sides FE + FH of the $\triangle$ FEH are $>$ the third EH. P. 20. B. 1. And ED being $=$ to EH (D. 15. B. 1.).



5. The straight lines $FE + FH$ are also $> ED$.
Therefore, taking away from both sides the part FE :
6. The straight line FH will be $> FD$; or $FD < FH$. *Ax. 5. B. 1.*
It is proved in the same manner that :
7. The straight line FD , which is the produced part of FA , is the least of all the straight lines drawn from the point F to the $\odot AHG$.
Which was to be demonstrated. II.

Moreover, the side FE being common to the two $\triangle FEB$, FEC , the side $EB =$ the side EC (*D. 15. B. 1.*), & the $\angle FEB > \angle FEC$ (*Ax. 8. B. 1.*).

8. The base FB will be $>$ the base FC . *P. 24. B. 1.*
For the same reason :
9. The straight line FC is $> FH$.
10. Consequently, the straight line FB or FC which is nearer the line FA , which passes thro' the center, is $>$ FC or FH more remote.
Which was to be demonstrated. III.

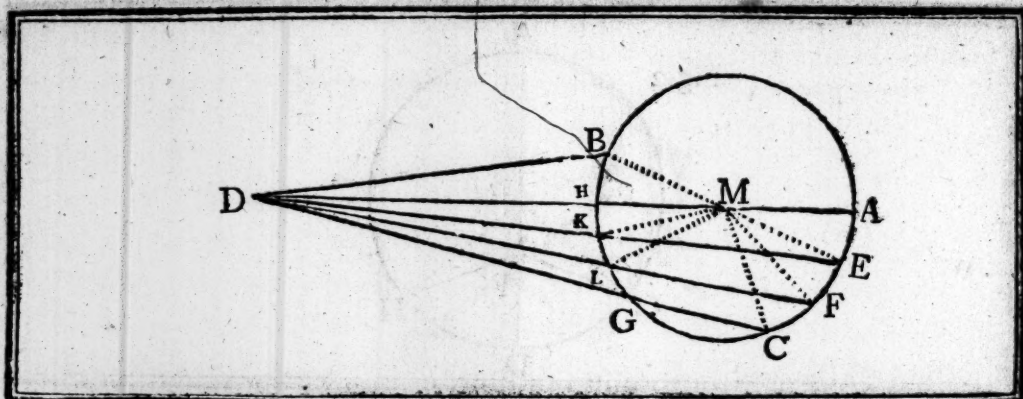
II. Preparation. Fig. 2.

1. Make $\angle FEG =$ to $\angle FEH$, & produce EG until it meets the $\odot AHG$. *P. 23. B. 1.*
2. From the point F to the point G , draw the straight line FG . *Pos. 1.*

Then, EF being common to the two $\triangle FEH$, FEG , the side $EH =$ the side EG (*D. 15. B. 1.*), & the $\angle FEH =$ to the $\angle FEG$ (*II. Prep. 1.*).

11. The base FH will be $=$ to the base FG . *P. 4. B. 1.*
But because any other straight line, different from FG , is either nearer the line FD , or more remote from it, than FG .
12. Such a straight line will be also $<$ or $>$ FG (*Arg. 10.*).
13. Wherefore, from the same point F , there can be drawn only two straight lines FH , FG , that are $=$ to one another, one upon each side of the shortest line FD .

Which was to be demonstrated. IV.



PROPOSITION VIII. THEOREM VII.

IF a point (D) be taken without a circle (BGCA), & straight lines (DA, DE, DF, DC,) be drawn from it to the circumference, whereof one (DA) passes thro' the center (M); of those which fall upon the concave circumference, the greatest is that (DA) which passes thro' the center; & of the rest, that (DE or DF) which is nearer to that (DA) thro' the center, is always greater than (DF or DC) the more remote: but of those (DH, DK, DL, DG,) which fall upon the convex circumference, the least is that (DH) which produced passes thro' the center: & of the rest, that (DK or DL) which is nearer to the least (DH) is always less than (DL or DG) the more remote: & only two equal straight lines (DK, DB,) can be drawn from the point (D) unto the circumference, one upon each side of (DH) the least.

Hypothesis.

Thesis.

- | | |
|---|--|
| <p><i>I. The point D is taken without a
⊙ BGCA in the same plane.</i></p> <p><i>II. The straight lines DA, DE,
DF, DC, are drawn from this
point to the concave part of
the ⊙ BGCA.</i></p> <p><i>III. And those straight lines cut the
convex part in the points H,
K, L, G.</i></p> | <p><i>I. DA which passes thro' the center M is
the greatest of all the straight lines
DA, DE, DF, DC.</i></p> <p><i>II. DE or DF, which is nearer to DA is >
DF or DC, the more remote.</i></p> <p><i>III. DH which when produced passes thro'
center M is the least of all the straight
lines DH, DK, DL, DG.</i></p> <p><i>IV. DK or DL which is nearer to the line
DH, is < DL or DG the more remote.</i></p> <p><i>V. From the point D only two equal straight
lines DK, DB, can be drawn, one upon
each side of DH the least.</i></p> |
|---|--|

I. Preparation.

Draw the rays ME, MF, MC, MK, ML.

DEMONSTRATION.

- T**H E two sides DM + ME of the $\triangle$ DME are > the third DE. *P. 20. B. 1.*
And because ME = MA (*D. 15. B. 1.*).

2. $DM + MA$ or DA will be $> DE$.

It is demonstrated after the same manner that :

3. The straight line DA , which passes thro' the center M , is $>$ any other straight line drawn from the point D to the concave part of the $\odot BGCA$. Which was to be demonstrated I.

Moreover, DM being common to the two $\triangle DME, DMF$, $ME = MF$ (*D. 15. B. 1.*), & $\angle DME > \angle DMF$ (*Ax. 8. B. 1.*).

4. The base DE will be also $>$ the base DF .

P. 24. B. 1.

In like manner it may be shewn that :

5. The straight line DF is $> DC$, & so of all the others.
6. Consequently, the straight lines DE or DF , which is nearer the line DA , which passes thro' the center, is $>$ DF or DC more remote. Which was to be demonstrated. II.

7. Again, the sides $DK + KM$ of the $\triangle DKM$ are $>$ the third DM . *P. 20. B. 1.*
If the equal parts MK, MH , (*D. 15. B. 1.*) be taken away.

8. The remainder DK will be $> DH$, or $DH < DK$.

It may be proved in the same manner, that :

9. The straight line DH is $< DL$, & so of all the others.
10. Consequently, the straight line DH , which produced passes thro' the center M , is the least of all the straight lines drawn from the point D to the convex part of the $\odot BGCA$.
Which was to be demonstrated. III.

Also, DK, MK , being drawn from the extremities D & M of the side DM of the $\triangle DLM$ to a point K , taken within this $\triangle$ (*Hyp. 3.*).

11. It follows, that $DK + MK < DL + ML$.

P. 21. B. 1.

And taking away the equal parts MK, ML , (*D. 15. B. 1.*).

12. The straight line DK will be $< DL$.

In like manner it may be shewn, that :

13. The straight line DL is $< DG$, & so of all the others.
14. Consequently; the straight lines DK or DL , which are nearer the line DH , which produced passes thro' the center, are $< DL$ or DG the more remote. Which was to be demonstrated. IV.

II. Preparation.

1. Make $\angle DMB = \angle DMK$, & produce MB till it meets the $\odot$. *P. 23. B. 1.*

2. From the point D to the point B , draw the straight line DB . *Pos. 1.*

Then, the side DM being common to the two $\triangle DKM, DMB$, the side $MK =$ the side MB (*D. 15. B. 1.*), & $\angle DMK = \angle DMB$ (*II. Prep. 1.*).

15. The base DK will be $=$ to the base DB .

P. 4. B. 1.

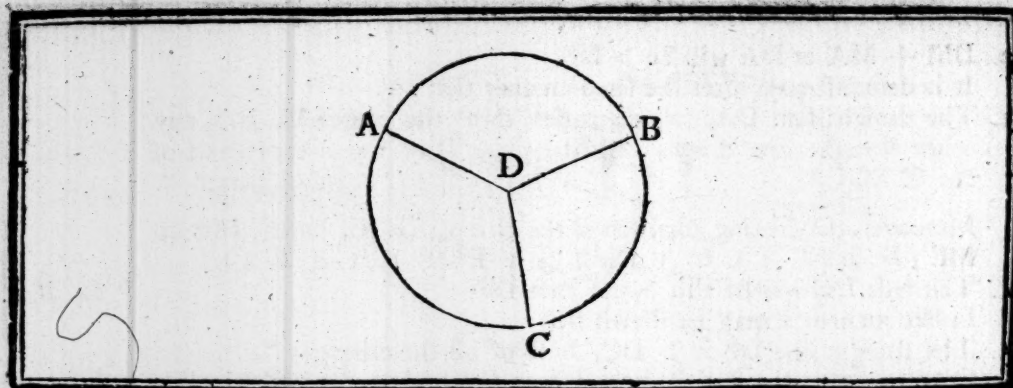
But because any other straight line different from DB , is either nearer the line DH or more remote from it, than DB .

16. Such a straight line will be also $<$ or $>$ BD (*Arg. 14.*).

17. Wherefore, from the point D , only two $=$ straight lines DK, DB , can be drawn, one upon each side of DH .

Which was to be demonstrated. V.

Q



PROPOSITION IX. THEOREM VIII.

IF a point (D) be taken within a circle (ABC), from which there fall more than two equal straight lines (DA, DB, DC,) to the circumference; that point is the center of the circle.

Hypothesis.

From the point D, taken within a $\odot$ ABC, there fall more than two equal straight lines DA, DB, DC, to the $\odot$ ABC.

Thesis.

The point D is the center of the $\odot$ ABC.

DEMONSTRATION.

If not,

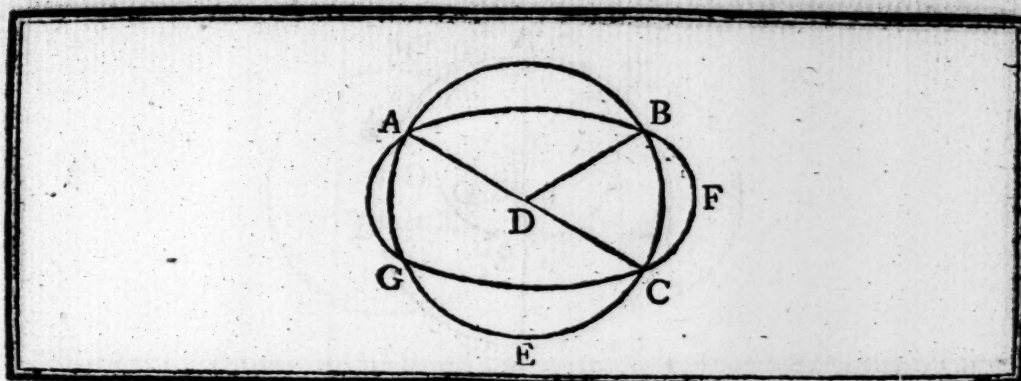
Some other point will be the center.

BECAUSE the point D is not the center (*Sup.*), & from this point D there fall more than two equal straight lines DA, DB, DC, to the $\odot$ ABC (*Hyp.*).

1. It follows, that from a point D, which is not the center, there can be drawn more than two equal straight lines; which is impossible. *P. 7. B. 3.*
2. Consequently, the point D is the center of the $\odot$ ABC.

Which was to be demonstrated





PROPOSITION X. THEOREM IX.

ONE circumference of a circle (ABCEG) cannot cut another (ABFCG) in more than two points (A & B).

Hypothesis.

The two $\odot$ ABCEG, ABFCG, cut one another.

Thesis.

They cut one another only in two points A & B.

DEMONSTRATION.

If not,

They cut each other in more than two points, as A, B, C, &c.

Preparation.

1. Find the center D of the $\odot$ ABCEG.
2. From the center D to the points of section A, B, C, &c. draw the rays DA, DB, DC.

P. 1. B. 3.

Pos. 1.

BECAUSE the point D is taken within the $\odot$ ABFCG, & that more than two straight lines DA, DB, DC, drawn from this point to the circumference of the $\odot$ ABFCG, are equal to one another, (Prep. 1. & D. 15. B. 1.).

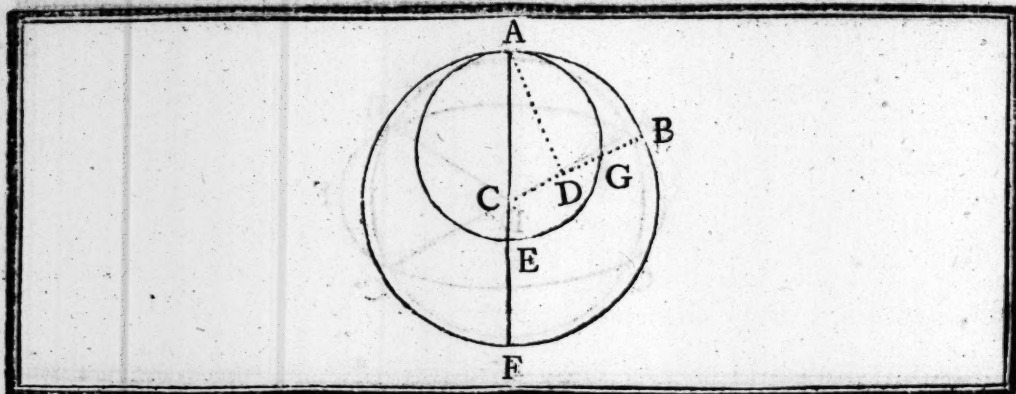
1. The point D is the center of this $\odot$.
But this point D being also the center of the $\odot$ ABCEG (Prep. 1.).
2. It would follow, that two $\odot$ ABFCG, ABCEG, which cut one another, have a common center D; which is impossible.
3. Consequently, two $\odot$ ABCEG, ABFCG, cannot cut one another in more than two points.

P. 9. B. 3.

P. 5. B. 3.

Which was to be demonstrated.





PROPOSITION XI. THEOREM X.

IF two circles touch each other internally in (A); the straight line which joins their centers being produced, shall pass thro' the point of contact (A).

Hypothesis.

The straight line CA joins the centers of the two $\odot$ AGE, ABF, which touch each other internally in A.

Thesis.

This straight line CA being produced, passes thro' the point of contact A of these two $\odot$.

DEMONSTRATION.

If not,

The straight line which joins the centers, will fall otherwise, as the straight line CGB.

Preparation.

From the centers C & D to the point of contact A, draw the lines CA, DA.

Pos. 1.

BECAUSE in the $\triangle CDA$, the two sides CD & DA taken together, are $<$ the third CA (*P. 20. B. 1.*), & that $CA = CB$ (*D. 15. B. 1.*).

1. The straight lines CD + DA will be also $>$ CB.

Therefore, if the common part CD be taken away from both sides.

2. The straight line DA will be $>$ DB.

Ax. 5. B. 1.

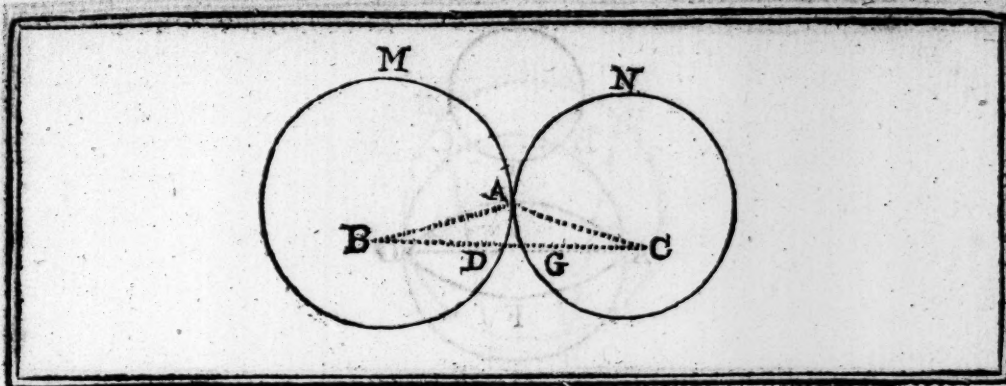
But the straight line DA being $=$ to DG (*Prep. & D. 15. B. 1.*).

3. DG will be also $>$ DB, which is impossible.

Ax. 8. B. 1.

4. Wherefore, the straight line CA, which joins the centers of the $\odot$ AGE, ABF, which touch each other internally, being produced, will pass thro' the point of contact A.

Which was to be demonstrated.



PROPOSITION XII. THEOREM XI.

IF two circles (DAM; GAN,) touch each other externally; the straight line (BC), which joins their centers, shall pass thro' the point of contact (A).

Hypothesis.

The straight line BC joins the centers of the two $\odot$ DAM, GAN, which touch each other externally in A.

Thesis.

This straight line BC passes thro' the point of contact of the two $\odot$.

DEMONSTRATION.

If not,

This straight line, which joins the centers, will pass otherwise, as BDGC.

Preparation.

Draw from the centers B & C to the point of contact A, the rays BA, CA.

Pos. 1.

BECAUSE BA is = to BD, & CA = to CG (*D. 15. B. 1.*).

1. The straight lines BA + CA are = to the straight lines BD + CG. *Ax. 2. B. 1.*

And if the part DC be added to the straight lines BD + CG.

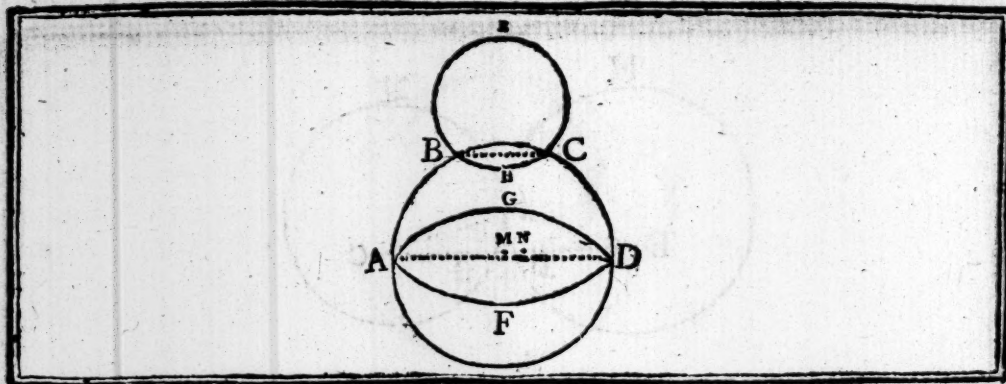
2. BD + DG + CG, or the base BC of the $\triangle$ BAC is > the two sides BA + CA, which is impossible.

P. 20. B. 1.

3. Therefore, the straight line BC, which joins the centers, will pass thro' the point of contact A.

Which was to be demonstrated.





PROPOSITION XIII. THEOREM XII.

TWO circles (ABCD, AGDF or ABCD, BECH,) which touch each other; whether internally; or externally: cannot touch in more points than one.

Hypothesis.

Thesis.

- I. $\odot$ ABCD touches $\odot$ AGDF internally. The $\odot$ ABCD, AGDF, or ABCD,
 II. $\odot$ ABCD touches $\odot$ BECH externally. BECH, touch only in one point.

If not,

DEMONSTRATION.

1. Either the $\odot$ ABCD, AGDF, touch each other internally in more points than one, as in A & in D.
2. Or the $\odot$ ABCD, BECH, touch each other externally in more points than one, as in B & in C.

I. Preparation.

1. Find the centers M & N of the $\odot$ ABCD, AGDF. P. 1. B. 3.
2. Thro' the centers, draw the line MN, & produce it to the $\odot$. Pos. 1. & 2.

BECAUSE MN joins the centers M & N of the two $\odot$ ABCD, AGDF, (Prep. 2.) which touch on the inside (Sup. 1.).

1. This straight line will pass thro' the points of contact A & D. P. 11. B. 3.
 But AM is = to MD (I. Prep. 2. & D. 15. B. 1.).
2. Therefore, the straight line AM is > ND, & AN is much > ND. Ax. 8. B. 1.
 But since AN is = to ND (I. Prep. 2. & D. 15. B. 1.).
3. The line AN will be > ND & = to ND; which is impossible.
4. Consequently, two $\odot$ ABCD, AGDF, which touch each other internally, cannot touch each other in more points than one.

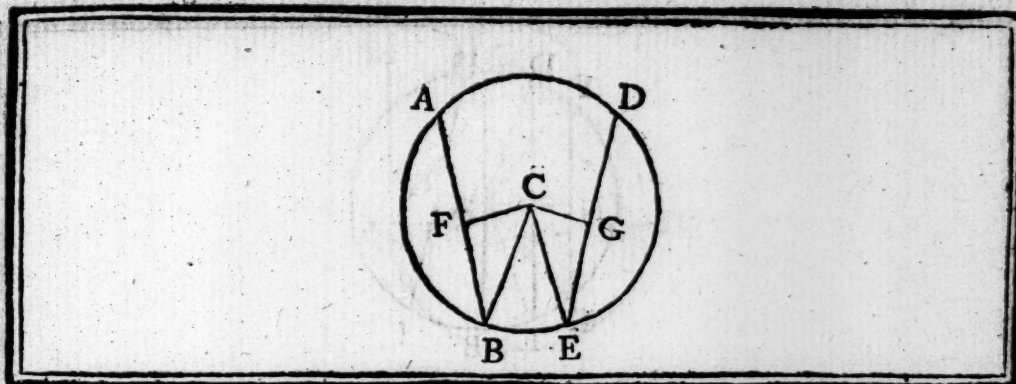
II. Preparation.

Thro' the points of contact B & C of the $\odot$ ABCD, BECH, draw the straight line BC. Pos. 1.

BECAUSE the line BC joins the two points B & C in the $\odot$ of the $\odot$ ABCD, BECH, (II. Prep.).

1. This straight line will fall within the two $\odot$ ABCD, BECH. P. 2. B. 3.
 But the $\odot$ BECH touching externally the $\odot$ ABCD (Sup. 2.).
2. BC, drawn in the $\odot$ BECH, will fall without the $\odot$ ABCD. D. 3. B. 3.
3. Consequently, BC will, at the same time, fall within the $\odot$ ABCD (Arg. 1.), & without the same $\odot$ (Arg. 2.); which is impossible.
4. Wherefore, two $\odot$ ABCD, BECH, which touch each other externally, cannot touch each other in more points than one.

Which was to be demonstrated.



PROPOSITION XIV. THEOREM XIII.

IN a circle (ABED) the equal chords (AB, DE,) are equally distant from the center (C); & the chords (AB, DE,) equally distant from the center (C), are equal to one another.

Hypothesis.

The chords AB, DE, are equal.

CASE I.

Thesis.

They are equally distant from the center C.

Preparation.

1. Find the center C of the $\odot$ ABED. P. 1. B. 3.
2. Let fall upon the chords AB, DE, the $\perp$ CF, CG. P. 12. B. 1.
3. From the center C to the points E & B, draw the rays CE, CB. Pos. 1.

DEMONSTRATION.

THE chords AB, DE, being = to one another (*Hyp.*) & bisected in F & G (*Prep. 2. & P. 3. B. 3.*).

1. Their halves FB, GE, are also equal. Ax 7. B. 1.
2. Consequently, the $\square$ of FB is = to the $\square$ of GE. { P. 46. B. 1.
But because $\square$ of CB = $\square$ of CE (*Prep. 3. & P. 46. Cor. 3.*) { Cor. 3.
3. It follows, that $\square$ of FB + $\square$ of FC is = to $\square$ of GE + $\square$ of CG. { P. 47. B. 1.
Therefore the equal $\square$ of FB & of GE (*Arg. 2.*) being taken away. { Ax. 1. B. 1.
4. The $\square$ of FC will be = the $\square$ of GC (*Ax. 3. B. 1.*), or FC = GC. { P. 46. B. 1.
5. Consequently, the chords AB, DE, are equally distant from the center C of the $\odot$ ABED. { Cor. 3.

Which was to be demonstrated. D. 4. B. 3.

Hypothesis.

CASE II.

Thesis.

The chords AB, DE, are equally distant from the center C of the $\odot$ ABED.

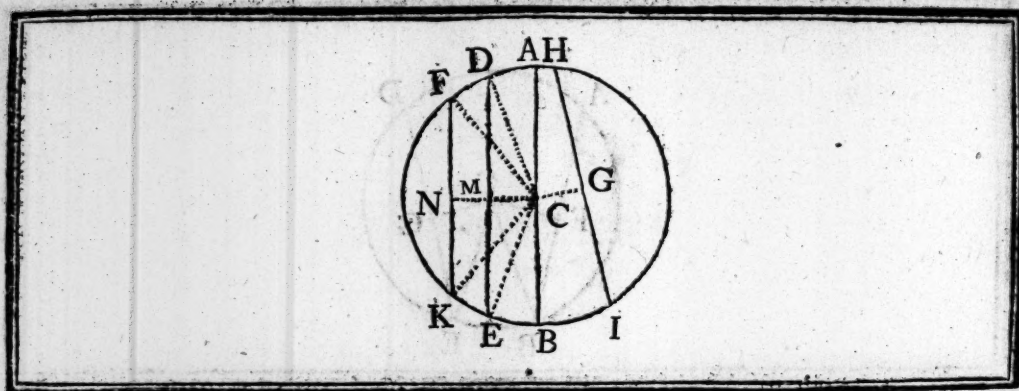
These chords are equal.

DEMONSTRATION.

BECAUSE FC = GC (*Hyp. & D. 4. B. 3.*), & CB = CE (*Prep. 3. & D. 15, B. 1.*).

1. The $\square$ of FC = the $\square$ of GC, & the $\square$ of CB = the $\square$ of CE. { P. 46. B. 1.
2. Consequently, $\square$ of FC + $\square$ of FB, = $\square$ of GC + $\square$ of GE. { Cor. 3.
Therefore, the equal $\square$ of FC & of GC (*Arg. 1.*) being taken away. { P. 47. B. 1.
3. The $\square$ of FB will be = the $\square$ of GE (*Ax. 3. B. 1.*) or FB = GE. { Ax. 1. B. 1.
4. Consequently, FB, GE, being the semichords (*Prep. 2. P. 3. B. 3.*), { P. 46. B. 1.
the whole chords AB, DE, are also = to one another. { Cor. 3.

Which was to be demonstrated. Ax. 6. B. 1.



PROPOSITION XV. THEOREM XIV.

THE diameter (AB) is the greatest straight line in a circle (AIK); & of all others that (HI), which is nearer the diameter, is always greater than one (FK) more remote.

Hypothesis.

- I. AB is the diameter of the $\odot$ AIK.
- II. The chord HI is nearer the diameter than the chord FK.

Thesis.

- I. The diameter AB is $>$ each of the chords HI, FK.
- II. The chord HI is $>$ the chord FK.

Preparation.

1. From the center C let fall upon HI & FK the $\perp$ CG, CN. P. 12. B. 1.
2. From CN, the greatest of those $\perp$, take away a part CM = to CG. P. 3. B. 1.
3. At the point M in CN, erect the $\perp$ DM & produce it to E. P. 11. B. 1.
4. Draw the rays CD, CF, CE, CK. Pos. 1.

DEMONSTRATION.

BECAUSE the straight lines CD, CE, CA, CB, are = to one another (*Prep. 4. & D. 15. B. 1.*).

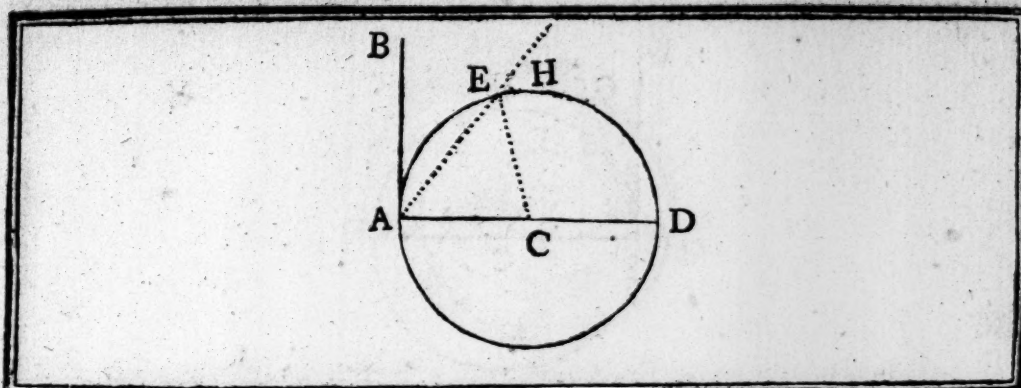
1. It follows, that $CD + CE$ is = to $CA + CB$ or AB. Ax. 2. B. 1.
But $CD + CE$ is $>$ DE (*P. 20. B. 1.*).
2. Wherefore, AB is also $>$ DE or $>$ HI, because $HI = DE$ (*P. 2.*). { D. 4. B. 3.
P. 14. B. 3.
3. It may be proved after the same manner, that AB is also $>$ FK.

Which was to be demonstrated. I.

Moreover, the $\triangle CDE$, CFK, having two sides CD, CE, = to the two sides CF, CK, each to each (*Prep. 4. & D. 15. B. 1.*), & the $\angle DCE > \angle FCK$ (*Ax. 8. B. 1.*).

4. The base DE will be $>$ the base FK. P. 24. B. 1.
5. And because HI is = to DE (*Prep. 2.*), HI is also $>$ FK. { D. 4. B. 3.
P. 14. B. 3.

Which was to be demonstrated. II.



PROPOSITION XVI. THEOREM XV.

THE straight line (AB) perpendicular to the diameter of a circle (AHD) at the extremity of it (A), falls without the circle; & no straight line can be drawn between this perpendicular (AB) & the circumference from the extremity, so as not to cut the circle; also the angle (HAD) formed by a part of the circumference (HEA) & the diameter (AD), is greater than any acute rectilinear angle; & the angle (HAB) formed by the perpendicular (AB) & the same part of the circumference (HEA), is less than any acute rectilinear angle.

Hypothesis.

- I. AB is drawn perpendicular to the extremity A of the diameter.
- II. And makes with the arch HEA the mixtilineal $\angle$ HAB.
- III. The diameter AD makes with the same arch HEA the mixtilineal $\angle$ HAD.

Thesis.

- I. The $\perp$ AB falls without the $\odot$ AHD.
- II. No straight line can be drawn between the $\perp$ AB & the arch HEA.
- III. The mixtilineal $\angle$ HAD is $>$ any acute rectilinear $\angle$.
- IV. The mixtilineal $\angle$ HAB is $<$ any acute rectilinear $\angle$.

DEMONSTRATION.

I. If not,

The $\perp$ AB will fall within the $\odot$ AHD, & will cut it somewhere in E, as AE.

Preparation.

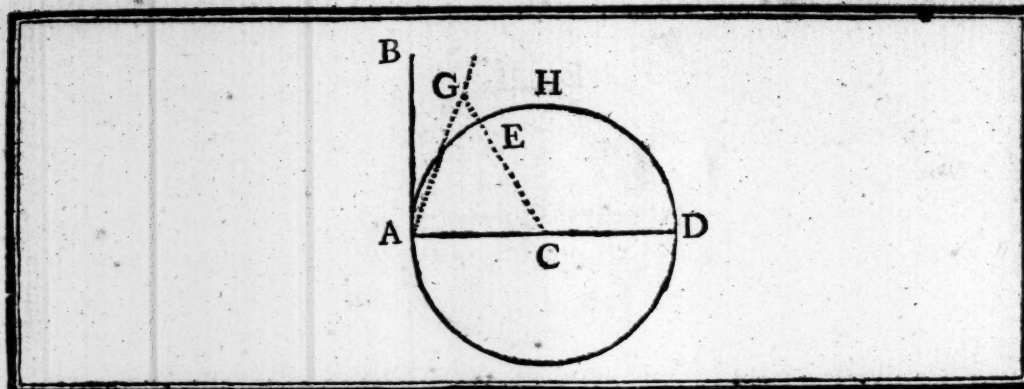
From the center C to the point of section E, draw the ray CE. *Pos.* I.

BECAUSE CA is $=$ to CE (*D. 15. B. 1.*).

1. The $\angle$ CAE will be $=$ to the $\angle$ CEA. *P. 5. B. 1.*
2. And because the $\angle$ CAE is a $\angle$ (*Sup.*); $\angle$ CEA is also a $\angle$. *Ax. 1. B. 1.*
3. Wherefore, the two $\angle$ CAE + CEA, of the $\triangle$ AEC will not be $<$ $2\angle$; which is impossible. *P. 17. B. 1.*
4. Therefore, the $\perp$ AB falls without the circle.

Which was to be demonstrated. I.

P



II. If not,

There may be drawn a straight line, as AG, between the $\perp$ AB & the circumference of the $\odot$ AHD.

Preparation.

From the center C, let fall upon AG, the $\perp$ CG.

P. 12. B. 1.

BECAUSE $\angle CGA$ is a $\perp$; & $\angle CAG < \perp$ (Ax. 8. B. 1.) as being but a part of the $\angle CAB$ (Hyp. 1.).

1. It follows, that the side CA is $>$ the side CG.

P. 19. B. 1.

But CA being $=$ to CE (D. 15. B. 1.).

2. The straight line CE will be also $>$ CG; which is impossible.

Ax. 8. B. 1.

3. Therefore, no straight line can be drawn between the $\perp$ AB & the $\odot$ AHD.

Which was to be demonstrated. II.

III. & IV. If not,

There may be drawn a straight line, as AG, which makes with the diameter AD & with the $\perp$ AB, an acute rectilinear $\angle GAD >$ the mixtilinear $\angle HAD$, & an acute rectilinear $\angle GAB <$ the mixtilinear $\angle EAB$.

BECAUSE then the straight line AG, drawn from the extremity A of the diameter AD, makes with the diameter & with the $\perp$ AB, an acute rectilinear $\angle GAD >$ the mixtilinear $\angle HAD$, & a rectilinear $\angle GAB <$ the mixtilinear $\angle EAB$ (Sup.).

1. This straight line AG will necessarily fall on the extremity A of the diameter AD, between the $\perp$ AB & the circumference of the $\odot$ AHD; which is impossible.

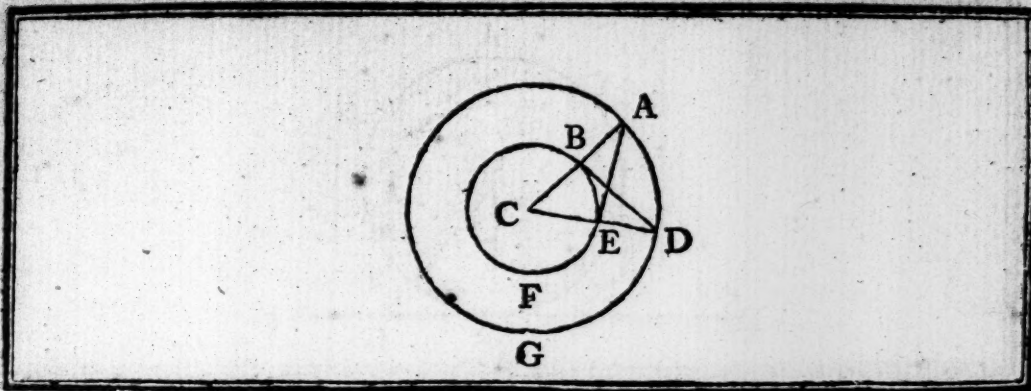
II. Case.

2. Therefore, the mixtilinear $\angle HAD$ is $>$, & the mixtilinear $\angle HAB <$ any acute rectilinear $\angle$.

Which was to be demonstrated. III. & IV.

COROLLARY.

A Straight line, drawn at right angles to the diameter of a circle from the extremity of it, touches the circle only in one point.



PROPOSITION XVII. PROBLEM II.

FROM a given point (A) without a circle (BEF), to draw a tangent (AE) to this circle.

Given

The point A without the $\odot$ BEF.

Sought

The tangent AE, drawn from the point A to the $\odot$ BEF.

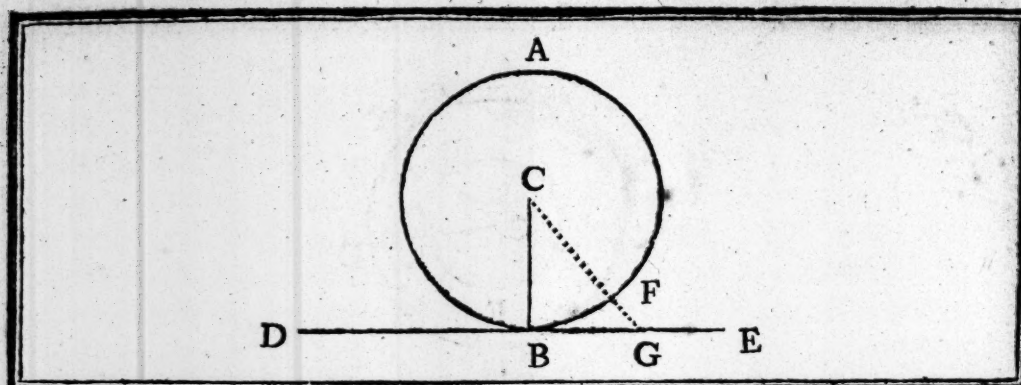
Resolution.

1. Find the center C of the $\odot$ BEF, & draw CA. P. 1. B. 3.
2. From the center C at the distance CA, describe the $\odot$ ADG. Pos. 3.
3. At the point B in the line CA, where it cuts the $\odot$ BEF, erect the $\perp$ BD. P. 11. B. 1.
4. From the center C to the point D, where the $\perp$ BD cuts the $\odot$ ADG, draw the ray CD. Pos. 1.
5. From the point A to the point E, where CD cuts the $\odot$ BEF, draw the straight line AE, which will be the tangent sought.

DEMONSTRATION.

BECAUSE in the $\triangle$ CBD, CEA, the side CB is $=$ to the side CE, the side CA $=$ to the side CD (D. 15. B. 1.), & the $\angle$ BCD common to the two $\triangle$.

1. The $\angle$ CBD, CEA, opposite to the equal sides CD, CA, are $=$ to one another. P. 4. B. 1.
2. Wherefore, $\angle$ CBD being a $\perp$ (Ref. 3.), $\angle$ CEA will be also a $\perp$. Ax. 1. B. 1.
3. Consequently, the straight line AE, drawn from the given point A, is a tangent of the $\odot$ BEF. P. 16. B. 1.
Cor. D. 1. B. 3.



PROPOSITION XVIII. THEOREM XVI.

IF a straight line (DE) touches a circle (AFB) in a point (B); the ray (CB), drawn from the center to the point of contact (B), shall be perpendicular to the tangent (DE).

Hypothesis.

- I. *The straight line DE touches the*
 $\odot$ AFB *in the point B.*
- II. *And the ray CB passes thro' the*
point of contact B.

Thesis.

The ray CB is $\perp$ upon the tangent DE.

DEMONSTRATION.

If not,

There may be let fall from the center C, another straight line CG $\perp$ upon the tangent DE.

Preparation.

Let fall then from the center C upon the tangent DE, the $\perp$ CG. *P. 12. B. 1.*

BECAUSE the $\sphericalangle$ BGC of the $\triangle$ BCG is a $\perp$ (*Prep.*).

1. The $\sphericalangle$ CBG will be $<$ a $\perp$.
2. Consequently, CB is $>$ CG.
 And CF being $=$ CB (*D. 15. B. 1.*).
3. The straight line CF is also $>$ CG; which is impossible.
4. Wherefore, the ray CB is $\perp$ upon the tangent DE.

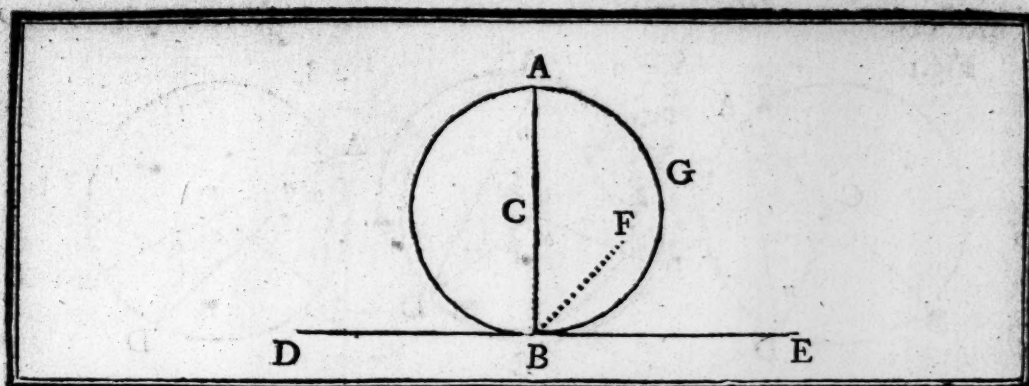
P. 17. B. 1.

P. 19. B. 1.

Ax. 8. B. 1.

Which was to be demonstrated.





PROPOSITION XIX, THEOREM XVII.

IF a straight line (DE) touches a circle (AGB in B), & from the point of contact (B) a perpendicular (BA) be drawn to the touching line; the center (C) of the circle, shall be in that line.

Hypothesis.

- I. The straight line DE touches the $\odot$ AGB.
- II. And BA is the $\perp$ erected from the point of contact B in this line.

Thesis.

The straight line BA passes thro' the center C of the $\odot$ AGB.

DEMONSTRATION.

If not,

The center will be in a point F without the straight line BA.

Preparation.

Draw then from the point of contact B to the center F, the straight line BF.

Pos. 1.

BECAUSE the straight line BF is drawn from the point of contact B to the center F of the $\odot$ AGB (*Prep.*).

1. The $\angle$ FBE is a $\perp$.

But $\angle$ ABE being also a $\perp$ (*Hyp.* 2.).

2. The $\angle$ ABE is $\equiv$ to the $\angle$ FBE; which is impossible
3. Wherefore, the center C will be necessarily in the straight line BA.

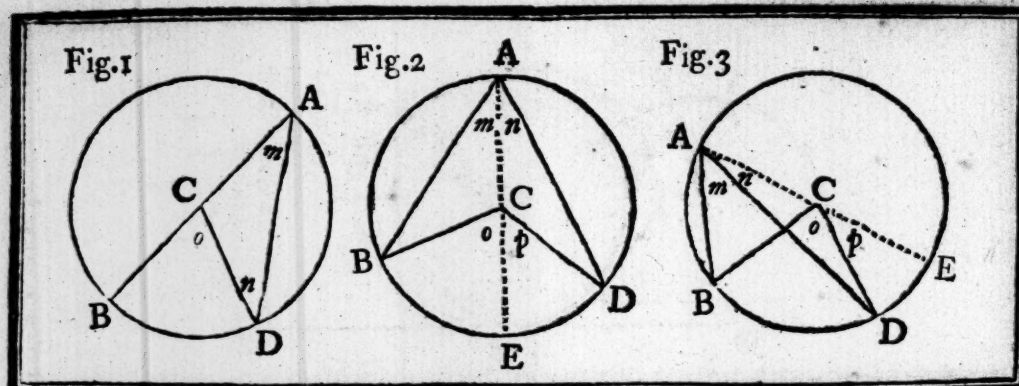
P. 18. B. 3.

{ Ax. 10. B. 1.

{ Ax. 8. B. 1.

Which was to be demonstrated.





PROPOSITION XX. THEOREM XVIII.
THE angle (BCD) at the center of a circle, is double of the angle (BAD) at the circumference, when both angles stand upon the same arch (BD).

Hypothesis.

- I. The $\angle$ BCD is at the center & $\angle$ BAD at the $\circ$.
 II. The sides BC, CD, & BA, AD, of those $\angle$ s, stand upon the same arch BD.

Thesis.

The $\angle$ BCD at the center is double of the $\angle$ BAD at the $\circ$.

DEMONSTRATION.

CASE I.

If the center C, is in one of the sides AB of the $\angle$ at the $\circ$ (Fig. 1.).

BECAUSE in the $\triangle$ CAD the side CA is = to the side CD (D. 15. B. 1.).

1. The $\angle$ m is = to the $\angle$ n, & $\angle$ m + n is double of $\angle$ m. { P. 5. B. 1.
Ax. 2. B. 1.
 But $\angle$ o is = to $\angle$ m + n (P. 32. B. 1.).
2. Therefore, $\angle$ o is double of $\angle$ m, or $\angle$ BCD is double of $\angle$ BAD. Ax. 6. B. 1.

CASE II.

If the center C falls within the $\angle$ at the $\circ$ (Fig. 2.).

Preparation.

Draw the diameter ACE.

Pos. 1.

IT may be proved as in the first case.

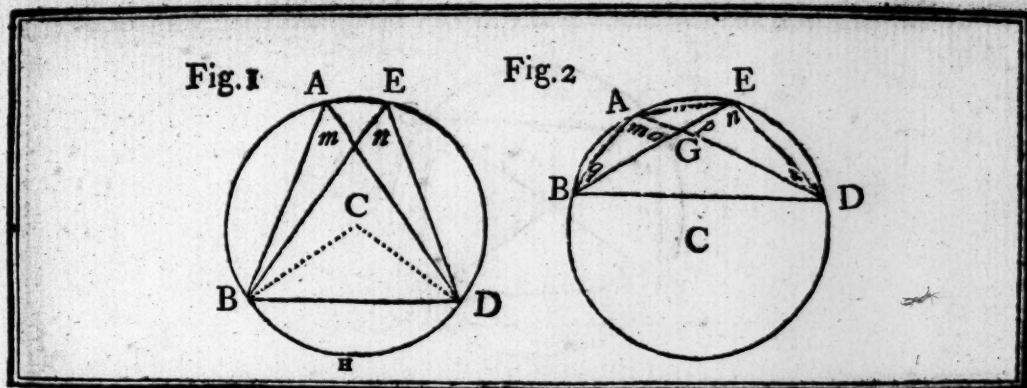
1. That the $\angle$ o is double of the $\angle$ m, & $\angle$ p double of the $\angle$ n.
2. From whence it follows, that $\angle$ o + p is double of the $\angle$ m + n, or $\angle$ BCD is double of the $\angle$ BAD. Ax. 8. B. 1.

CASE III.

If the center C falls without the $\angle$ at the $\circ$ (Fig. 3.).

THE diameter ACE being drawn, it is demonstrated as in the first case, that:

1. The $\angle$ p is double of the $\angle$ n, & $\angle$ o + p is double of the $\angle$ m + n. Therefore, the $\angle$ p being taken away from one side, & the $\angle$ n from the other.
2. The $\angle$ o will be double of the $\angle$ m, or $\angle$ BCD is double of $\angle$ BAD. Ax. 3. B. 1.
 Which was to be demonstrated.



PROPOSITION XXI. THEOREM XIX.

THE angles (m & n) in the same segment of a circle (BAED), are equal to one another.

Hypothesis.

The $\angle m$ & n are in the same segment of the $\odot$ BAED.

Thesis.

$\angle m$ is = to $\angle n$.

DEMONSTRATION.

CASE I.

If the segment BAED is $>$ the semi $\odot$ (Fig. 1.).

Preparation.

1. Find the center C of the $\odot$ BAED.
2. And draw the rays CB, CD.

P. 1. B. 3.
Pos. 1.

BECAUSE $\angle BCD$ is double of each of the $\angle m$ & n (P. 20. B. 3.).
1. It follows, that $\angle m$ is = to $\angle n$.

Ax. 7. B. 1.

CASE II.

If the segment BAED is $<$ the semi $\odot$ (Fig. 2.).

Preparation.

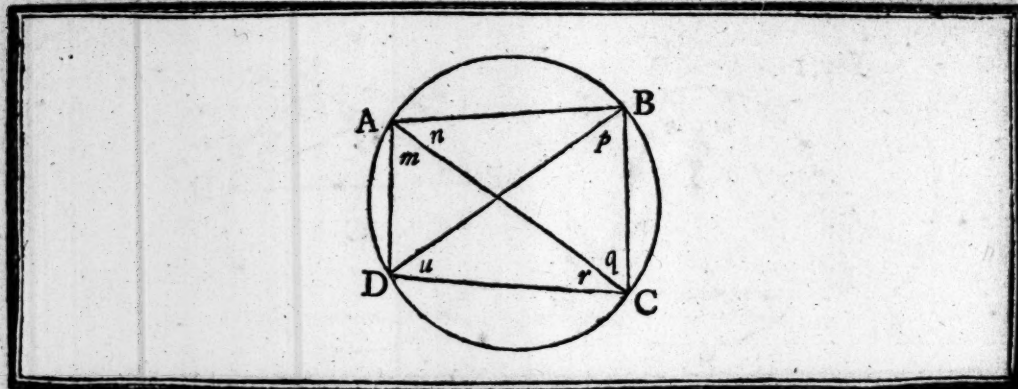
Draw the straight line AE.

Pos. 1.

1. **T**HE three $\angle m + o + q$ of the $\triangle BAG$, are = to the three $\angle p + n + r$ of the $\triangle GED$.
But $\angle q$ is = to $\angle r$ (Case 1.), & $\angle o$ = to $\angle p$ (P. 15. B. 1.).
Therefore, the $\angle q + o$ being taken away from one side, & their equals $\angle p + r$ from the other.
2. The remaining $\angle m$ & n will be = to one another,
Which was to be demonstrated.

P. 32. B. 1.
Ax. 1. B. 1.

Ax. 3. B. 1.



PROPOSITION XXII. THEOREM XX.

THE opposite angles (BAD, BCD, or ABC, ADC,) of any quadrilateral figure (DABC) inscribed in a circle, are together, equal to two right angles.

Hypothesis.

The figure DABC is a quadrilateral figure inscribed in a $\odot$.

Thesis.

The opposite $\angle$ BAD + BCD, or ABC + ADC, are = to 2 $\angle$.

Preparation.

Draw the diagonals AC, BD.

Pos. 1.

DEMONSTRATION.

BECAUSE the $\angle u + n$ are the $\angle$ at the $\odot$, in the same segment DABC.

1. These $\angle u$ & n are = to one another.

P. 21. B. 3.

It is proved in the same manner, that :

2. The $\angle p$ & m are = to one another.

3. Wherefore, the $\angle u + p$ are = to the $\angle n + m$ or to the $\angle$ BAD. Ax. 2. B. 1.

Therefore, if the $\angle r + q$ or BCD be added to both sides.

4. The $\angle u + p + (r + q)$ are = to the $\angle$ BAD + BCD.

Ax. 2. B. 1.

But the three $\angle u + p + (r + q)$ of the $\triangle$ DBC being = to 2 $\angle$ (P. 32. B. 1.).

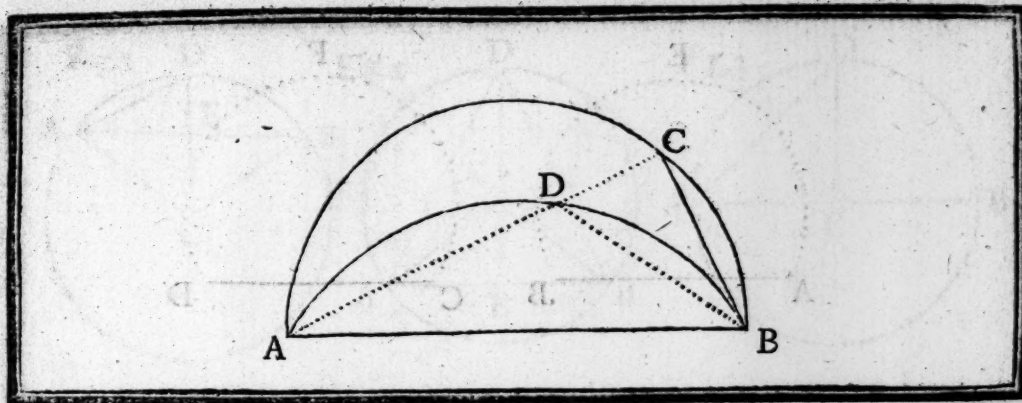
5. The two opposite $\angle$ BAD + BCD of the quadrilateral figure DABC, are also = to 2 $\angle$.

Ax. 1. B. 1.

It may be demonstrated after the same manner, that :

6. The $\angle$ ABC + ADC are = to 2 $\angle$.

Which was to be demonstrated



PROPOSITION XXIII. THEOREM XXI.

UPON the same straight line (AB) & upon the same side of it, there cannot be two similar segments of circles (ADB, ACB,) not coinciding with one another.

Hypothesis.

The segments ADB, ACB, of circles, are upon the same straight line & upon the same side of it.

Thesis.

These segments are dissimilar

DEMONSTRATION.

If not,

The segments ADB, ACB, upon the same chord AB, & upon the same side of it, are similar.

Preparation.

1. Draw any straight line AC, which cuts the segments ADB, ACB, in the points D & C.
2. Draw the straight lines BD, BC.

} *Pos. 1.*

BECAUSE the $\angle$ BDA, BCA, are contained in the similar segments ADB, ACB, (*Hyp. & Prep. 1. & 2.*).

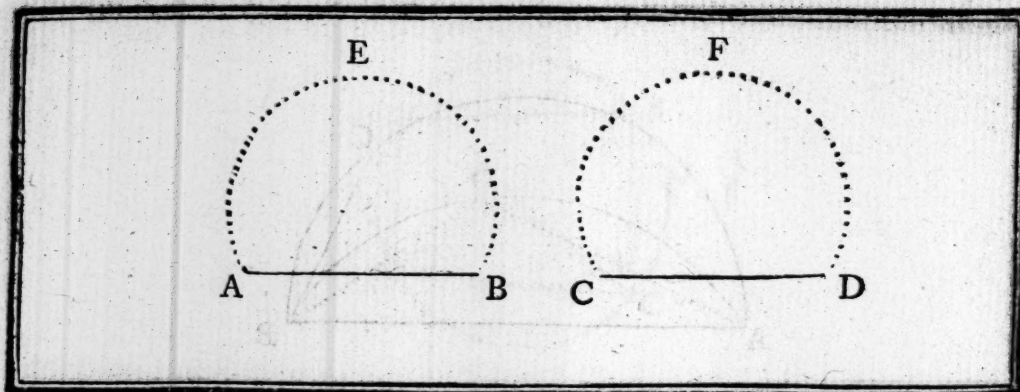
1. These $\angle$ are = to one another.
2. Therefore, the exterior $\angle$ ADB of the $\triangle$ BDC, will be = to its interior opposite one BCD; which is impossible.
3. Consequently, there cannot be two similar segments of $\odot$ ADB, ACB, upon the same side of the same straight line AB, which do not coincide.

Ax. 2. B. 1.

P. 16. B. 1.

Which was to be demonstrated.

Q



PROPOSITION XXIV. THEOREM XXII.

SIMILAR segments of circles (AEB, CFD,) subtended by equal chords (AB, CD,) are equal to one another.

Hypothesis.

- I. The segments of $\odot$ AEB, CFD, are similar.
- II. These segments are subtended by equal chords AB, CD.

Thesis.

The segments AEB, CFD, are = to one another.

DEMONSTRATION.

If not,

The segments AEB, CFD, are unequal.

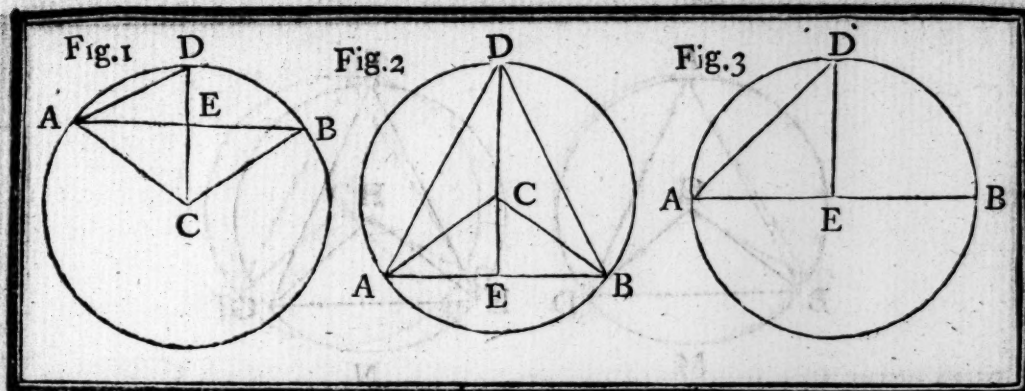
BECAUSE the segment AEB is not = to the segment CFD (Sup.), & the chord AB is = to the chord CD (Hyp. 2.).

1. Upon the same straight line AB or its equal CD, there could be two similar segments of $\odot$, AEB, CFD; which is impossible.
2. Therefore, these segments are = to one another.

P. 23. B. 3.

Which was to be demonstrated.





PROPOSITION. XXV. PROBLEM III.
A Segment of a circle (ADB) being given; to describe the circle of which it is the segment.

Given Sought
 The segment of $\odot$ ADB. The center C of the $\odot$, of which ADB is the segment.
Resolution.

1. Divide the chord AB into two equal parts in the point E. P. 10. B. 1.
2. At the point E in AB, erect the $\perp$ ED. P. 11. B. 1.
3. Draw the straight line AD. Pof. 1.
- And $\angle ADE$ will be $>$, or $<$, or $= \angle DAE$.
- C A S E I. & II.
- If $\angle ADE$ be either $>$ or $< \angle DAE$ (Fig. 1. & 2.).
4. At the point A in DA, make $\angle DAC =$ to $\angle ADE$. P. 23. B. 1.
5. Produce DE to C (Fig. 1.), & draw BC (Fig. 1. & 2.). Pof. 2. & 1.

D E M O N S T R A T I O N.

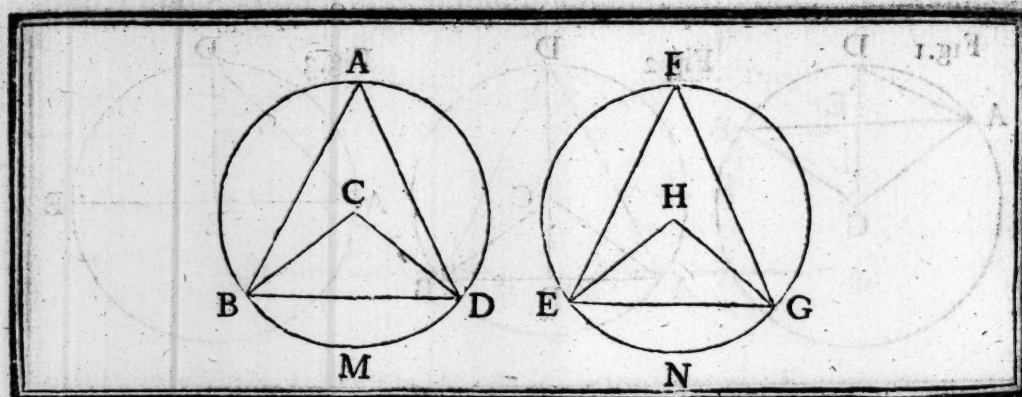
- B** E C A U S E in the $\triangle ADC$ the $\angle DAC$ is $=$ to $\angle ADC$ (Ref. 4.).
1. The side AC is $=$ to the side DC. P. 5. B. 1.
 But in the $\triangle AEC$, $\triangle BEC$, the side AE is $=$ to the side EB, the side EC common to the two $\triangle$, & the $\angle AEC =$ to the $\angle BEC$ (Ref. 2. & Ax. 10. B. 1.).
 2. The base AC will be $=$ to the base BC. P. 4. B. 1.
 3. Consequently, the three straight lines AC, DC, BC, drawn from the point C to the $\odot$ ADB, are $=$ to one another. Ax. 1. B. 1.
 4. Wherefore, the point C is the center of the $\odot$, of which ADB is the segment. P. 9. B. 3.

C A S E III.

- If $\angle ADE$ be $=$ to $\angle DAE$ (Fig. 3.).
- T** H E N the side AE is $=$ to the side ED. P. 5. B. 1.
2. Consequently, AE being $=$ EB (Ref. 1.), the three straight lines AE, ED, EB, drawn from a point E to the $\odot$ ADB, are $=$ to one another. Ax. 1. B. 1.
 3. From whence it follows, that the point E is the center of the $\odot$ of which ADB is the segment. P. 9. B. 3.

Which was to be demonstrated.

Q 2



PROPOSITION XXVI. THEOREM XXIII.

IN equal circles (BADM, EFGN,), equal angles, whether they be at the centers as (C & H) or at the circumferences as (A & F), stand upon equal arches (BMD, ENG,).

Hypothesis.

- I. The $\angle$ C, H, are $\angle$ at the centers, & equal.
- II. The $\angle$ A, F, are $\angle$ at the $\odot$, & equal.
- III. These $\angle$ are contained in the equal $\odot$ BADM, EFGN.

Thesis.

The arches BMD, ENG, upon which these $\angle$ stand, are = to one another.

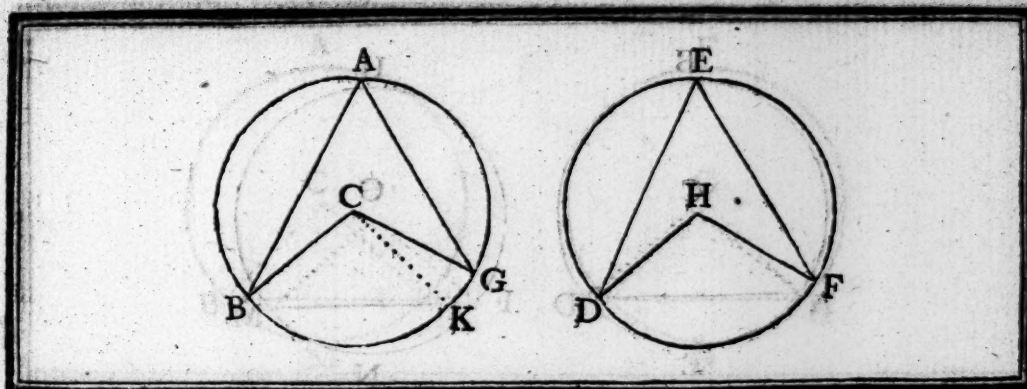
Preparation.

Draw the chords BD, EG.

DEMONSTRATION.

THE two sides CB, CD, of the $\triangle$ BCD being = to the two sides HE, HG, of the $\triangle$ EHG (*Hyp.* 3. & *Ax.* 1. B. 3.), & the $\angle$ C = to the $\angle$ H (*Hyp.* 2.).

1. The base BD will be = to the base EG. P. 4. B. 1.
And because $\angle$ A is = to $\angle$ F (*Hyp.* 1.).
2. The segment BAD is similar to the segment EFG. Ax. 2. B. 3.
3. Wherefore, the base BD being = to the base EG (*Arg.* 1.), these segments will be = to one another. P. 24. B. 3.
Therefore, if the equal segments BAD, EFG, (*Arg.* 3.) be taken away from the equal $\odot$ BADM, EFGN, (*Hyp.* 3.).
4. The remaining arches BMD, ENG, will be also = to one another. Ax. 3. B. 1.
Which was to be demonstrated.



PROPOSITION XXVII. THEOREM XXIV.

IN equal circles (BAG, DEF,) the angles, whether at the centers as (BCG & H) or at the circumferences as (A & E), which stand upon equal arches (BG, DF,) ; are equal to one another.

Hypothesis.

- I. The $\odot$ BAG, DEF, are $=$, as also their arches BG, DF.
- II. The $\angle$ BCG & H at the centers, as also the $\angle$ A & E at the $\odot$, stand upon $=$ arches.

Thesis.

- I. The $\angle$ BCG & H at the centers, are $=$ to one another.
- II. The $\angle$ A & E at the $\odot$, are also $=$ to one another.

DEMONSTRATION.

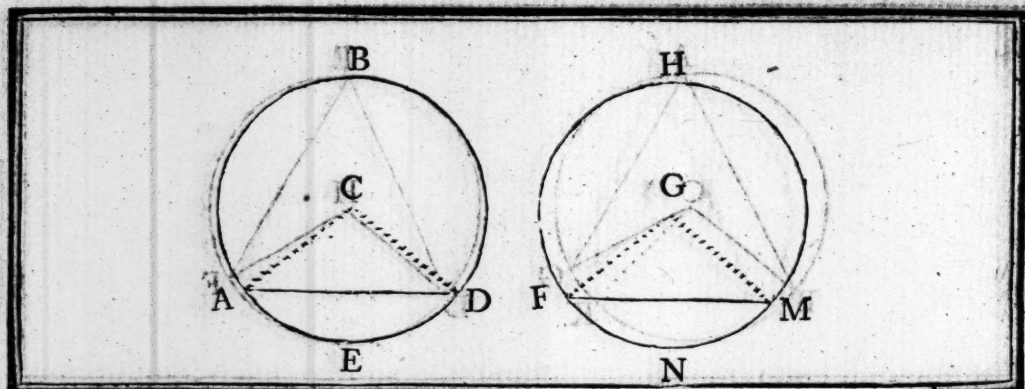
If not,

The $\angle$ BCG & H at the centers will be unequal, & one, as BCG, will be $>$ the other H.

Preparation.

At the point C in the line BC, make the $\angle$ BCK $=$ to $\angle$ H. P. 23. B. 1.

1. **T**HEREFORE the arch BK is $=$ to the arch DF. P. 26. B. 3.
But the arch DF being $=$ to the arch BG (Hyp. 2)
 2. The arch BK will be also $=$ to the arch BG ; which is impossible { Ax. 1. B. 1.
 3. Consequently, the $\angle$ BCG & H at the centers, are $=$ to one another. { Ax. 8. B. 1.
- Which was to be demonstrated. I.
- And these $\angle$ being double of the $\angle$ A & F at the $\odot$ (P. 20. B. 3.).
4. These $\angle$ A & E at the $\odot$, are also $=$ to one another. Ax. 7. B. 1.
- Which was to be demonstrated. II.



PROPOSITION XXVIII. THEOREM XXV.

IN equal circles ($\odot$ ABDE, FHMN,) the equal chords (AD, FM,) subtend equal arches (ABD, FHM or AED, FNM,).

Hypothesis.

- I. The $\odot$ ABDE, FHMN, are equal.
- II. The chords AD, FM, are equal.

Thesis.

The chords AD, FM, subtend equal arches ABD, FHM or AED, FNM.

Preparation.

1. Find the centers C & G of the two $\odot$ ABDE, FHMN.
2. Draw the rays CA, CD, also GF, GM.

P. 1. B. 3.
Pos. 1.

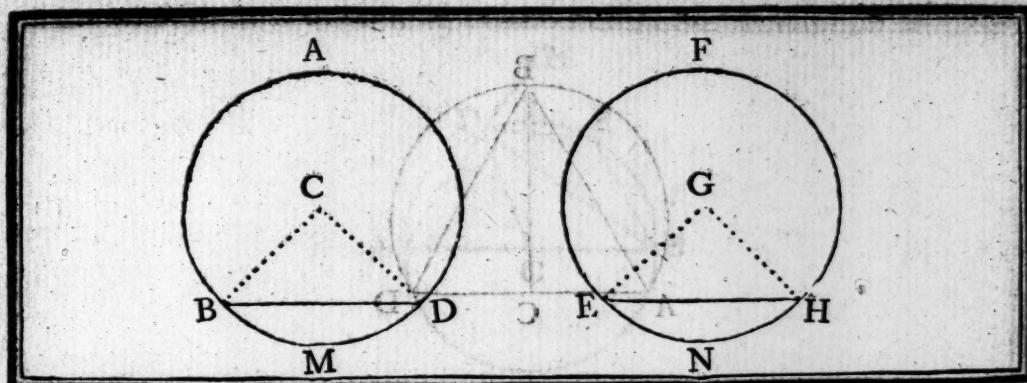
DEMONSTRATION.

BECAUSE the $\odot$ ABDE, FHMN, are equal (Hyp. 1.).

1. The sides CA, CD, & GF, GM, of the $\triangle$ ACD, FGM, are equal. Ax. 1. B. 3.
And the chords AD, FM, being equal (Hyp. 2.).
2. The $\angle$ ACD, FGM, are = to one another. P. 8. B. 1.
3. Consequently, the arches AED, FNM, subtended by the chords AD, FM, will be also = to one another. P. 26. B. 3.
4. And moreover, the whole $\odot$ being equal (Hyp. 1.), the arches ABD, FHM, are also equal. Ax. 3. B. 1.

Which was to be demonstrated.





PROPOSITION XXIX. THEOREM XXVI.

IN equal circles ($\odot$ BADM, EFHN,); equal arches (BMD, ENH,) are subtended by equal chords (BD, EH,).

Hypothesis.

- I. The $\odot$ BADM, EFHN, are equal.
- II. The arches BMD, ENH, are equal.

Thesis.

The chords BD, EH, which subtend these arches, are equal.

Preparation.

1. Find the centers C & G of the two $\odot$ BADM, EFHN.
2. Draw the rays CB, CD, GE, GH.

P. 1. B. 3.
Pos. 1.

DEMONSTRATION.

BECAUSE the $\odot$ BADM, EFHN, are equal (*Hyp.* 1.).

1. The sides CB, CD, & GE, GH, of the $\triangle$ BCD, EGH, are = to one another.

Ax. 1. B. 3.

But the arches BMD, ENH, being also equal (*Hyp.* 2.)

2. The $\angle$ C & G, contained by those equal sides, will be = to one another.

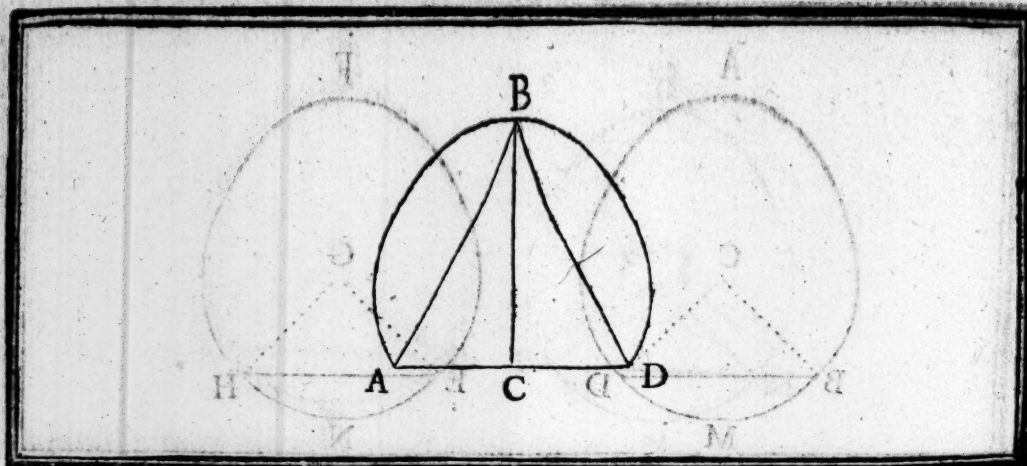
P. 27. B. 3.

3. Consequently, the chord BD is = to the chord EH.

P. 4. B. 1.

Which was to be demonstrated.





PROPOSITION XXX. PROBLEM IV.

TO divide an arch (ABD) into two equal parts (AB, BD,).

Given
The arch ABD.

Sought
The division of the arch ABD
into two equal parts AB, BD.

Resolution.

1. From the point A to the point D, draw the chord AD. Pos. 1.
2. Divide this chord into two equal parts at the point C. P. 10. B. 1.
3. At the point C in the straight line AD, erect the $\perp$ CB, which P. 11. B. 1.
when produced, will divide the arch ABD into two equal
parts at the point B.

Preparation.

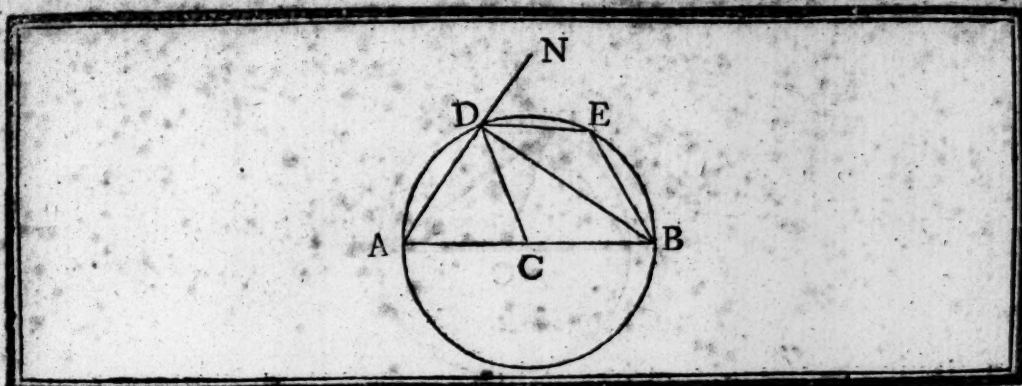
Draw the chords AB, DB. Pos. 1.

BECAUSE the side AC is = to the side CD (Ref. 2), CB common to the two $\triangle$ ABC, DBC, & the $\angle$ ACB = to the $\angle$ DCB (Ax. 10. B. 1. & Ref. 3.).

1. The base AB is = to the base DB. P. 4. B. 1.
2. Consequently, the arches AB & DB, subtended by the equal chords AB, DB, are = to one another, and the whole arch ABD, is divided into two equal parts in B. P. 28. B. 3.

Which was to be done.





PROPOSITION XXXI. THEOREM XXVII.

IN a circle, the angle (ADB) in a semicircle (ADEB), is a right angle; but the angle (DAB) in a segment (DAB) greater than a semicircle, is less than a right angle, & the angle (DEB) in a segment (DEB) less than a semicircle, is greater than a right angle: also the mixtilineal angle (BDA) of the greater segment, is greater than a right angle, & that (BDE) of the lesser segment, is less than a right angle.

CASE I.

Hypothesis.

The $\angle$ ADB is in the semi $\odot$ ADEB.

Thesis.

This $\angle$ ADB is a $\angle$.

Preparation.

1. Draw the ray CD.
2. And produce AD to N.

Pos. 1.

Pos. 2.

DEMONSTRATION.

BECAUSE in the $\triangle$ ADC the side CA is $=$ to the side CD (D. 15. B. 1.).

1. The $\angle$ CDA is $=$ to the $\angle$ CAD.
Again, in the $\triangle$ CDB; the side CD being $=$ to the side CB.
2. The $\angle$ CDB is $=$ to the $\angle$ CBD.
3. Consequently, $\angle$ ADB is $=$ to $\angle$ CAD + CBD.
But $\angle$ NDB is also $=$ to $\angle$ CAD + CBD (P. 32. B. 1.).
4. Wherefore, this $\angle$ NDB is $=$ to $\angle$ ADB.
5. From whence it follows, that $\angle$ ADB is a $\angle$.

P. 5. B. 1.

D. 15. B. 1.

P. 5. B. 1.

Ax. 2. B. 1.

Ax. 1. B. 1.

D. 10. B. 1.

CASE II.

Hypothesis.

The $\angle$ DAB is in the segment DAB $>$ a semi $\odot$.

Thesis.

This $\angle$ DAB is $<$ a $\angle$.

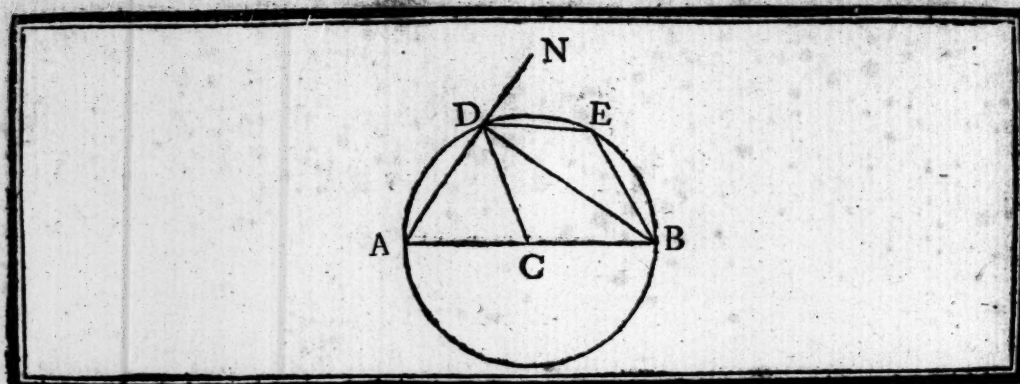
DEMONSTRATION.

BECAUSE in the $\triangle$ ADB, the $\angle$ ADB is a $\angle$ (Case I.).

1. The $\angle$ DAB will be $<$ a $\angle$.

P. 17. B. 1.

R



CASE III.

Hypothesis.

The $\angle$ DEB is in a segment DEB $<$ a semi $\odot$.

Thesis.

This $\angle$ DEB is $>$ a $\angle$.

DEMONSTRATION.

1. **T**HE the opposite $\angle$ DAB + DEB of the quadrilateral figure ADEB are = to 2 $\angle$. P. 22. B. 3.
2. Wherefore, $\angle$ DAB being $<$ a $\angle$ (Case II.), DEB will be necessarily $>$ a $\angle$.

CASE IV.

Hypothesis.

The mixtilineal $\angle$ BDA, BDE, are formed by the straight line BD & the arches DA, DE.

Thesis.

The $\angle$ BDA is $>$ a $\angle$, & the $\angle$ BDE is $<$ a $\angle$.

DEMONSTRATION.

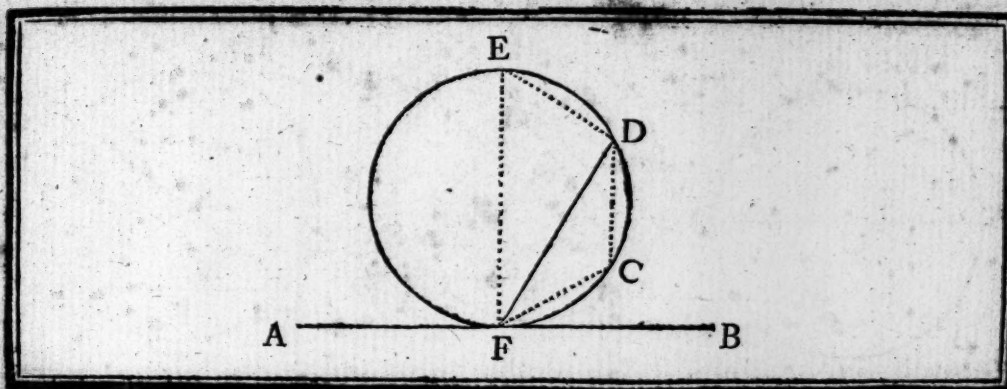
BECAUSE the rectilineal $\angle$ ADB, NDB, are $\angle$ (Case I.).

1. The mixtilineal $\angle$ BDA will be necessarily $>$ a $\angle$, & the mixtilineal $\angle$ BDE $<$ a $\angle$.

Ax. 8. B. 1.

Which was to be demonstrated.





PROPOSITION XXXII. THEOREM XXVIII.

IF a straight line (AB) touches a circle (ECF), & from the point of contact (F) a chord (FD) be drawn; the angles (DFB, DFA,) made by this chord & the tangent, shall be equal to the angles (FED, FCD,) which are in the alternate segments (FED, FCD,) of the circle.

Hypothesis.

- I. BA is a tangent of the $\odot$ ECF.
- II. And FD is a chord of this $\odot$ drawn from the point of contact.

Thesis.

- I. The $\angle$ FED is = to $\angle$ DFB.
- II. The $\angle$ FCD is = to $\angle$ DFA.

Preparation.

1. At the point of contact F in AB, erect the $\perp$ FE. P. 11. B. 1.
2. Take any point C in the arch DF, & draw ED, DC, CF. Pos. 1.

DEMONSTRATION.

BECAUSE the straight line AB touches the $\odot$ ECF (Hyp. 1.), and FE is a $\perp$ erected at the point of contact F in the line AB (Prep. 1.).

1. The straight line FE is a diameter of the $\odot$ ECF. P. 19. B. 3.
2. Consequently, $\angle$ FDE is a $\perp$. P. 31. B. 3.
3. Wherefore, the $\angle$ DEF + DFE are = to a $\perp$. P. 32. B. 1.
But $\angle$ EFB or $\angle$ DFE + $\angle$ DFB being also = to a $\perp$ (Prep. 1.).
4. The $\angle$ DEF + DFE are = to the $\angle$ DFB + DFE. Ax. 1. B. 1.
5. Wherefore, the $\angle$ DEF is = to $\angle$ DFB, or the $\angle$ in the segment DEF is = to the $\angle$ made by the tangent BF & the chord DF. Ax. 3. B. 1.
P. 21. B. 3.

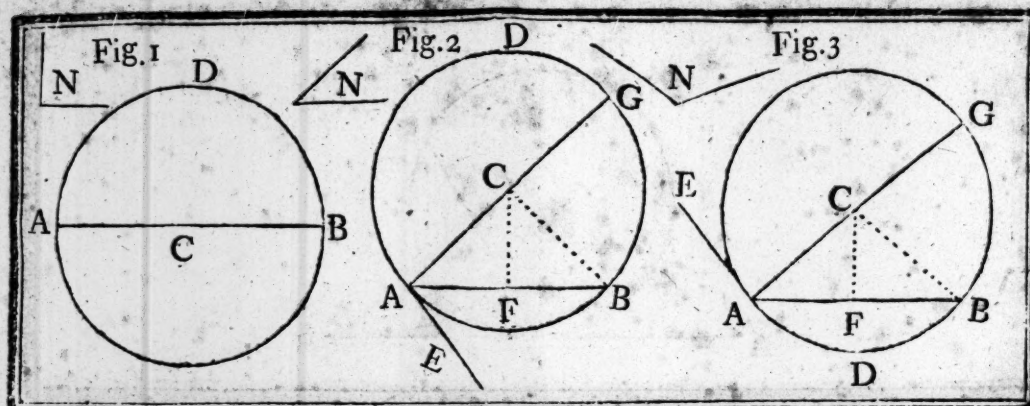
Which was to be demonstrated. I.

The $\angle$ FED + FCD being = to 2 $\perp$ (P. 22. B. 3.), & the adjacent $\angle$ DFB + DFA being also = to 2 $\perp$ (P. 13. B. 1.).

Ax. 1. B. 1.

6. The $\angle$ FED + FCD are = to the $\angle$ DFB + DFA.
7. Wherefore, $\angle$ FED being = to the $\angle$ DFB (Arg. 5.), the $\angle$ FCD is also = to the $\angle$ DFA; or the $\angle$ in the segment FCD is = to the $\angle$ contained by the tangent AF & the chord DF. Ax. 3. B. 1.
P. 21. B. 3.

Which was to be demonstrated. II.



PROPOSITION XXXIII. PROBLEM V.

UPON a given straight line (AB), to describe a segment of a circle (ADB) containing an angle equal to a given rectilineal angle (N).

Given The straight line AB together with $\angle N$. *Sought* The segment ADB described upon AB, containing an $\angle =$ to $\angle N$.

CASE I. If the given $\angle$ is a $\angle$. (Fig. 1.).

IT suffices to describe upon AB a semi $\odot$ ADB.

1. This semi $\odot$ will contain an $\angle =$ to the given right $\angle N$.

Pos. 3.
P. 31. *B.* 3.

CASE II. If the given $\angle$ is acute (Fig. 2.) or obtuse (Fig. 3.)

Resolution.

1. At the point A in AB, make the $\angle BAE =$ to the given $\angle N$. *P.* 23. *B.* 1.
2. At the point A in AE, erect the $\perp$ AG. *P.* 11. *B.* 1.
3. Divide AB into two equal parts in the point F. *P.* 10. *B.* 1.
4. At the point F in AB, erect the $\perp$ FC, which will cut AG in C. *P.* 11. *B.* 1.
5. From the center C at the distance CA, describe the $\odot$ ADG. *Pos.* 3.

Preparation.

Draw the straight line CB.

Pos. 1.

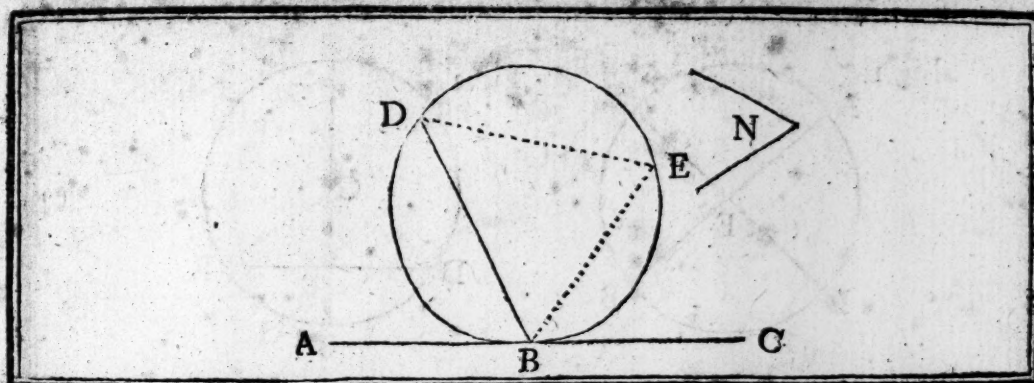
DEMONSTRATION.

BECAUSE in the $\triangle ACF$, $\triangle BCF$, the side AF is $=$ to the side BF (*Ref.* 3.), FC common to the two $\triangle$, & the $\angle AFC =$ the $\angle BFC$ (*Ax.* 10. *B.* 1. & *Ref.* 4.).

1. The base CA is $=$ to the base CB. *P.* 4. *B.* 1.
2. Consequently, the $\odot$ described from the center C at the distance CA, will pass thro' the point B, & ADB is a segment described upon AB. But AE touching the $\odot$ ADB in A (*Ref.* 2. & *P.* 16. *Cor.* *B.* 3.), and AB being a chord drawn from this point of contact A (*Arg.* 2.).
{

D. 15. *B.* 1.
D. 19. *B.* 1.
3. The $\angle$ contained in the alternate segment ADB is $=$ the $\angle BAE$. *P.* 32. *B.* 3.
4. Wherefore, $\angle BAE$ being $=$ to the given $\angle N$ (*Ref.* 1.), the $\angle$ contained in the segment ADB described upon AB, is also $=$ to the given $\angle N$. *Ax.* 1. *B.* 1.

Which was to be done.



PROPOSITION XXXIV. PROBLEM VI.

TO cut off a segment (BED) from a given circle (BDE), which shall contain an angle (DEB) equal to a given rectilineal angle (N).

Given

The $\odot$ BDE, & the rectilineal $\angle$ N.

Sought

The segment BED cut off from this $\odot$, containing an $\angle$ DEB = to the given $\angle$ N.*Resolution.*

1. From any point A to the $\odot$ BDE, draw the tangent ABC. P. 17. B. 3.
2. At the point of contact B in the line AB, make the $\angle$ DBA = to the given $\angle$ N. P. 23. B. 1.

DEMONSTRATION.

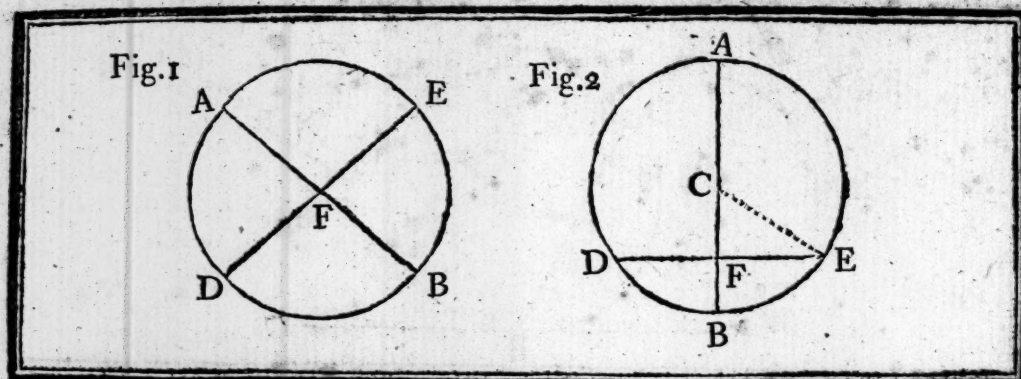
BECAUSE the given $\angle$ N is = to the $\angle$ DBA (*Ref. 2.*), & $\angle$ DEB = to the $\angle$ DBA (*P. 32. B. 3.*).

1. The $\angle$ DEB & N are = to one another.
2. Wherefore, the segment BED is cut off from the $\odot$ BDE, containing an $\angle$ DEB = to the given $\angle$ N.

*Ax. 1. B. 1.**P. 21. B. 3.*

Which was to be done.





PROPOSITION XXXV. THEOREM XXIX.

IF in a circle (DAEB) two chords (AB, DE,) cut one another; the rectangle contained by the segments (AF, FB,) of one of them, is equal to the rectangle contained by the segments (DF, FE,) of the other.

Hypothesis.

- I. AB, DE, are two chords of the same $\odot$ DAEB.
 II. And these chords cut one another in a point F.

Thesis.

The Rgle AF . FB is = to the Rgle DF . FE.

CASE I. If the two chords pass thro' the center F of the $\odot$. Fig. 1.

DEMONSTRATION.

- T**HEN, the straight lines AF, FB, DF, FE, are = to one another.
 2. Consequently, the Rgle AF . FB is = to the Rgle DF . FE.

D. 15. B. 1.
 Ax. 2 B. 2.

CASE II. If one of the chords AB, passes thro' the center & cuts the other DE which does not pass thro' the center at $\perp$ (Fig. 2.).

Preparation.

Draw the ray CE.

Pos. 1.

DEMONSTRATION.

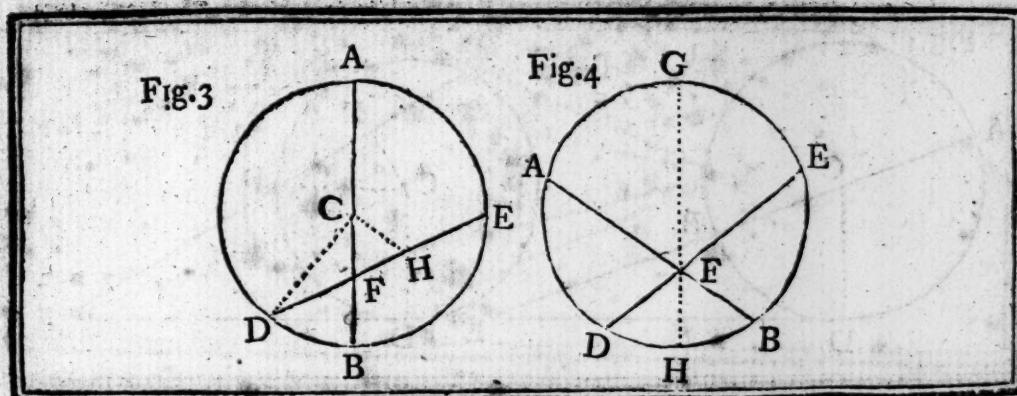
BECAUSE the straight line AB is cut equally in C & unequally in F.

- The Rgle AF . FB + the $\square$ of CF is = to the $\square$ of CB, or is = to the $\square$ of CE.
 But the $\square$ of FE + the $\square$ of CF is also = to the $\square$ of CE (P. 47. B. 1.).
- From whence it follows, that the Rgle AF . FB + the $\square$ of CF is = to the $\square$ of FE + the $\square$ of CF.
- Consequently, the Rgle AF . FB is = to the $\square$ of FE.
 And since DF is = to FE (P. 3. B. 3.), or DF . FE = to the $\square$ of FE (Ax. 2. B. 2.).
- The Rgle AF . FB is also = to the Rgle DF . FE.

{ P. 5. B. 2.
 Ax. 1. B. 1.

Ax. 1. B. 1.
 Ax. 3. B. 1.

Ax. 1. B. 1.



CASE III. If one of the chords AB, passes thro' the center & cuts the other DE which does not pass thro' the center, obliquely (*Fig. 3.*).

Preparation.

1. From the center C, let fall upon DE, the $\perp$ CH.
2. And draw the ray CD.

P. 12. B. 1.
Pos. 1.

DEMONSTRATION.

BECAUSE DH is = to HE (*Prep. 1. & P. 3. B. 3.*).

1. The Rgle DF . FE + the $\square$ of FH is = to the $\square$ of DH.
2. Wherefore, the Rgle DF . FE + $\square$ of FH + $\square$ of CH is = to the $\square$ of DH + $\square$ of CH.
But the $\square$ of FH + $\square$ of CH is = to the $\square$ of CF, & the $\square$ of DH + the $\square$ of CH is = to the $\square$ of CD (*P. 47. B. 1.*).
3. Therefore, the Rgle DF . FE + $\square$ of CF is = to the $\square$ of CD or to the $\square$ of CB.
Moreover, the Rgle AF . FB + $\square$ of CF being = to the same $\square$ of CB (*P. 5. B. 2.*).
4. The Rgle DF . FE + $\square$ of CF is also = to the Rgle AF . FB + $\square$ of CF.
5. Or taking away the common $\square$ of CF, the Rgle DF . FE is = to the Rgle AF . FB.

P. 5. B. 2.

Ax. 2. B. 1.

Ax. 1. B. 1.

Ax. 1. B. 1.

Ax. 3. B. 1.

CASE IV. If neither of the chords AB, DE, passes thro' the center (*Fig. 4.*).

Preparation.

Thro' the point F, draw the diameter GH.

Pos. 1.

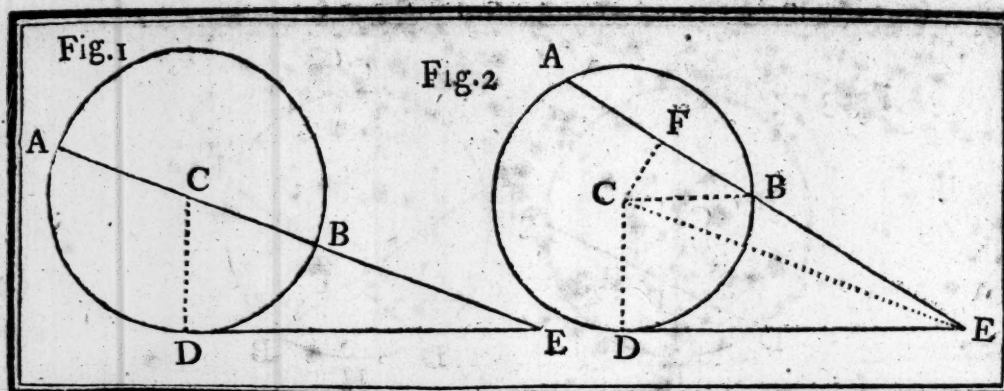
DEMONSTRATION.

BECAUSE each of the Rgles AF . FB & DF . FE is = to the Rgle GF . FH (*Case III.*).

1. These Rgles AF . FB & DF . FE are also = to one another.

Ax. 1. B. 1.

Which was to be demonstrated.



PROPOSITION XXXVI. THEOREM XXX.

IF from any point (E) without a circle (ABD) two straight lines be drawn, one of which (DE) touches the circle, & the other (EA) cuts it; the rectangle contained by the whole secant (AE), & the part of it (EB) without the circle, shall be equal to the square of the tangent (ED).

Hypothesis.

- I. The point E is taken without the $\odot$ ABD.
- II. From this point E, a tangent ED & a secant EA, have been drawn.

Thesis.

The Rgle AE . EB is = to the $\square$ of ED.

CASE I. If the secant AE passes thro' the center (Fig. 1.).

Preparation.

From the point of contact D, Draw the ray CD.

Pos. 1.

DEMONSTRATION.

1. **T**HE ray CD is then $\perp$ to the tangent ED.
And because the straight line AB is bisected in C, & produced to the point E.
2. The Rgle AE . EB + the $\square$ of CB is = to the $\square$ of CE.
Moreover, the $\square$ of CE being also = to the $\square$ of DE + the $\square$ of CD (P. 47. B. 1.), or to the $\square$ of DE + the $\square$ of CB (P. 46. Cor. 3. B. 1.).
3. The Rgle AE . EB + the $\square$ of CB is = to the $\square$ of DE + the $\square$ of CB.
The $\square$ of CB being taken away from both sides.
4. The Rgle AE . EB will be = to the $\square$ of DE.

P. 18. B. 3.

P. 6. B. 2.

Ax. 1. B. 1.

Ax. 3. B. 1.

CASE II. If the secant AE does not pass thro' the center.

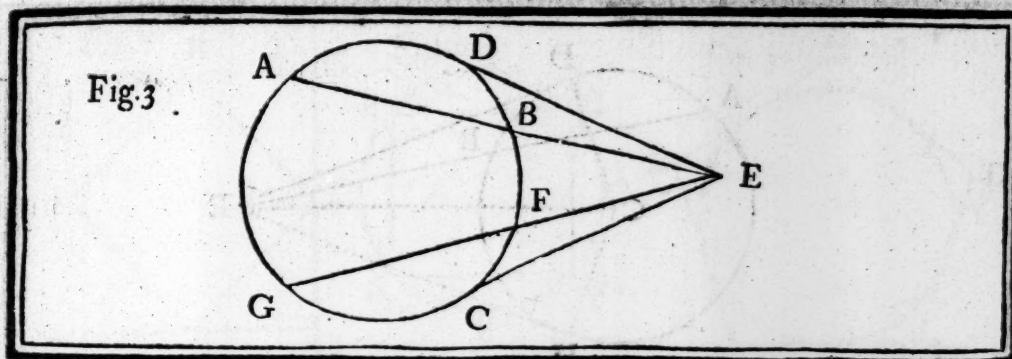
Fig. 2.

Preparation.

1. Let fall from the center C upon AE, the $\perp$ CF.
2. Draw the rays CB, CD, & the straight line CE.

P. 12. B. 1.

Pos. 1.



DEMONSTRATION.

BECAUSE the straight line AB is bisected in F (*Prep. 1. & P. 3. B. 3.*) and produced to the point E.

1. The Rgle AE.EB + $\square$ of FB is = to the $\square$ of FE. *P. 6. B. 2.*
2. Consequently, the Rgle AE.EB + $\square$ of FB + $\square$ of FC is = to the $\square$ of FE + $\square$ of FC, or is = to the $\square$ of CE. *Ax. 2. B. 1.*
 But since the $\square$ of DE + $\square$ of CD is = to the $\square$ of CE, and *P. 47. B. 1.*
 the $\square$ of FB + $\square$ of FC is = to the $\square$ of CB (*P. 47. B. 1.*), or is = to the $\square$ of CD (*D 15. & P. 46. Cor. 3. B. 1.*)
3. The Rgle AE.EB + $\square$ of CD is = to the $\square$ of DE + $\square$ of CD.
4. Consequently, the Rgle AE.EB is = to the $\square$ of DE. *Ax. 3. B. 1.*

Which was to be demonstrated.

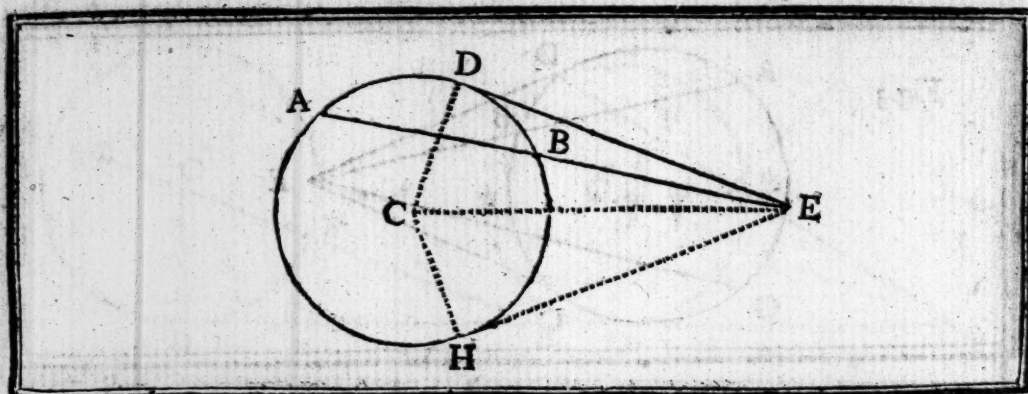
COROLLARY I.

IF (fig. 3.) from any point (E) without a circle (ADBF), there be drawn several straight lines (AE, EG, &c.) cutting it in (B & F, &c.): the rectangles contained by the whole secants (AE, GE), and the parts of them (EB, EF) without the circle, are equal to one another; for drawing from the point (E) the tangent (ED), these rectangles will be equal to the square of the same tangent (ED).

COROLLARY II.

IF from any point (E), without a circle (ADBF), there be drawn to this circle two tangents (ED, EC), they will be equal to one another, because the square of each is equal to the same rectangle (AE.EB).





PROPOSITION XXXVII. THEOREM XXXI.

IF from a point (E), without a circle (ADH), there be drawn two straight lines, one of which (AE) cuts the circle, and the other (ED) meets it; if the rectangle contained by the whole secant (AE) and the part of it without the circle (EB), be equal to the square of the line (ED) which meets it: the line which meets shall touch the circle in D.

Hypothesis.

- I. AE cuts the $\odot$ ADH in B.
- II. ED meets the $\odot$.
- III. The Rgle AE.EB is = to the $\square$ of ED.

Thesis.

The straight line ED touches the $\odot$ ADH in the point D.

Preparation.

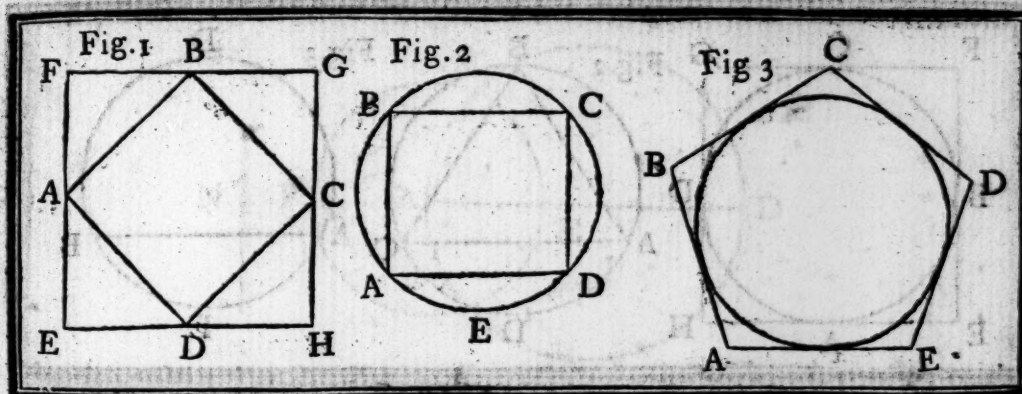
1. From the point E to the $\odot$ ADH draw the tangent EH.
2. Draw the rays CD, CH and the straight line CE.

P. 17. B. 3.

Pos. 1.

DEMONSTRATION.

- B**ECAUSE the Rgle of AE.EB is = to the $\square$ of ED (Hyp. 3.) and the Rgle AE.EB is also = to the $\square$ of EH (Prep. 1 & P. 36. B. 3.)
1. The $\square$ of ED is = to the $\square$ of EH (Ax. 1. B. 1.) or ED = EH. { P. 46. B. 1.
Cor. 3.
 - And moreover, since in the $\triangle CDE$, CHE, the side CD is = to the side CH (D. 15. B. 1), and CE is common to the two $\triangle$.
 2. The $\sphericalangle$ CDE is = to the $\sphericalangle$ CHE. P. 8. B. 1.
 3. Wherefore, $\sphericalangle$ CHE being a $\perp$ (Prep. 1. & P. 18. B. 3), $\sphericalangle$ CDE is also a $\perp$. Ax. 1. B. 1.
P. 16. B. 3.
Cor. 3.
 4. And the straight line ED touches the $\odot$ ADH in the point D.



DEFINITIONS.

I.

A Rectilineal figure (ABCD) is said to be inscribed in another rectilineal figure (EFGH), when all the angles (A, B, C, D) of the inscribed figure, are upon the sides of the figure in which it is inscribed (fig. 1).

II.

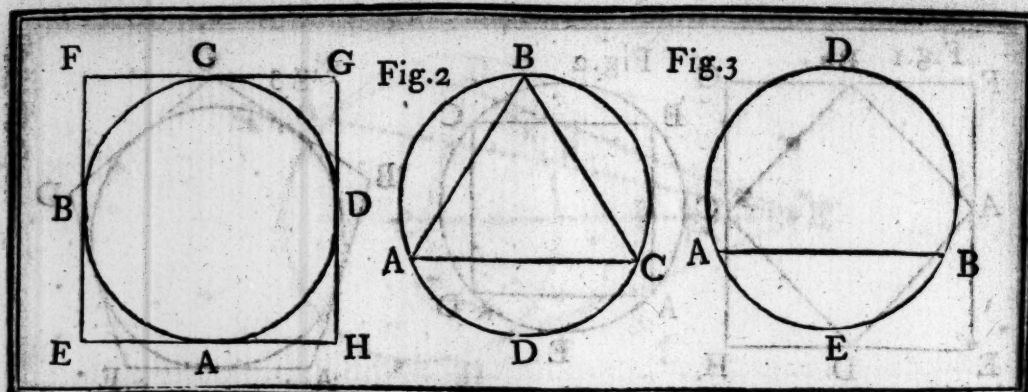
In like manner a rectilineal figure (EFGH) is said to be described about another rectilineal figure (ABCD); when all the sides (EF, FG, GH, HE) of the circumscribed figure pass thro' the angular points (A, B, C, D) of the figure about which it is described, each thro' each (Fig. 1).

III.

A rectilineal figure (ABCD) is said to be inscribed in a circle, when all the angles (A, B, C, D) of the inscribed figure are upon the circumference of the circle (ABCDE) in which it is inscribed (Fig. 2).

IV.

A rectilineal figure (ABCDE) is said to be described about a circle, when each of the sides AB, BC, CD, DE, EA touches the circumference of the circle (Fig. 3).



DEFINITIONS.

V.

A Circle (ABCD) is said to be inscribed in a rectilineal figure (EFGH), when the circumference of the circle touches each of the sides (EF, FG, GH, HE) of the figure in which it is inscribed (Fig. 1).

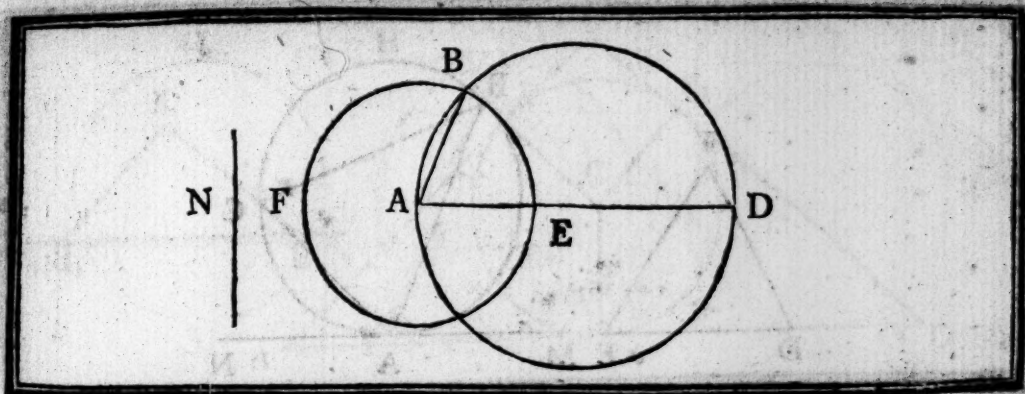
VI.

A circle (ABCD) is described about a rectilineal figure (ABC), when the circumference of the circle passes thro' all the angular points (A, B, C) of the figure about which it is described (Fig. 2).

VII.

A straight line (AB) is said to be placed in a circle (ADBE), when the extremities of it (A & B) are in the circumference of the circle (fig. 3).





PROPOSITION I. PROBLEM I.

IN a given circle (ABD), to place a straight line (AB) equal to a given straight line (N), not greater than the diameter of the circle (ABD).

Given.

A $\odot$ ABD together with the straight line N, not $>$ the diameter of this $\odot$.

Sought.

The straight line AB placed in the $\odot$ ABD $\&=$ to the given straight line N.

Resolution.

Draw the diameter AD of the $\odot$ ABD.

Pos. 1.

CASE I. If AD is $=$ to N.

THERE has been placed in the given $\odot$ ABD a straight line $=$ to the given N.

D. 7. B. 4.

CASE II. If AD is $>$ N.

1. Make AE $=$ to N.

P. 3. B. 1.

2. From the center A at the distance AE describe the $\odot$ EBF, and draw AB.

Pos. 3.

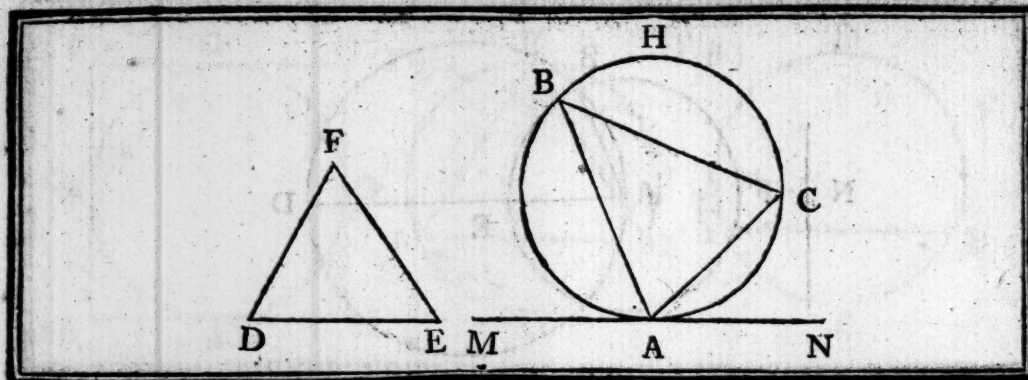
DEMONSTRATION.

BECAUSE AB is $=$ to AE (D. 15. B. 1), and the straight line N is $=$ to AE (Ref. 1.)

1. The straight line AB, placed in the $\odot$ ABD, will be also $=$ to N.

{ Ax. 1. B. 1.
D. 7. B. 1.

Which was to be done.



PROPOSITION II. PROBLEM II.

IN a given circle (ABHC), to inscribe a triangle (ABC) equiangular to a given triangle (DFE).

Given.

A $\odot$ ABHC together with the $\triangle$ DFE.

Sought.

The $\triangle$ ABC inscribed in the $\odot$ ABHC, equiangular to the $\triangle$ DFE.

Resolution.

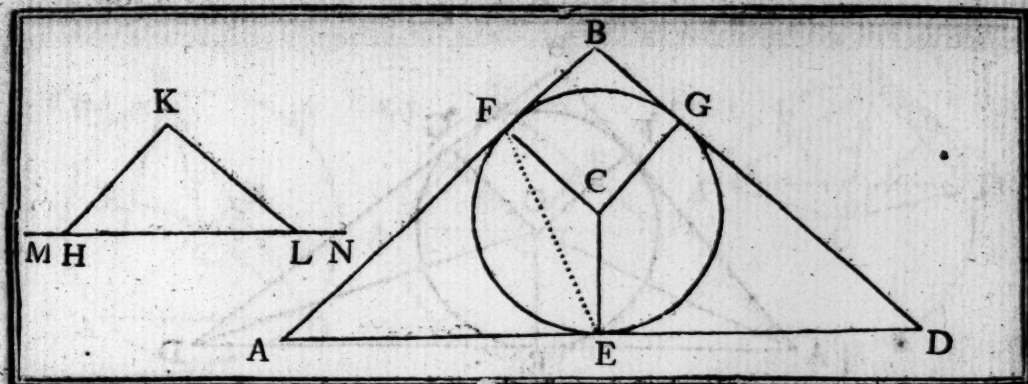
1. From the point M, to the $\odot$ ABHC draw the tangent MN. P. 17. B. 3.
2. At the point of contact A in the line MN make the $\angle$ BAM = to the $\angle$ FED, and the $\angle$ CAN = to the $\angle$ FDE. P. 23. B. 1.
3. Draw BC. Pos. 1.

DEMONSTRATION.

BECAUSE the $\angle$ BCA is = to the $\angle$ BAM (P. 32. B. 3), and the $\angle$ FED is = to the same $\angle$ BAM (Ref. 2); also the $\angle$ CBA is = to the $\angle$ CAN (P. 32. B. 3.) and $\angle$ FDE is = to $\angle$ CAN (Ref. 2).

1. It follows that $\angle$ BCA is = to $\angle$ FED, and $\angle$ CBA = to $\angle$ FDE. Ax. 1. B. 1.
2. Consequently, the third $\angle$ BAC, of the $\triangle$ ABC, is also to the third $\angle$ DFE of the $\triangle$ DFE, and this $\triangle$ ABC is inscribed in the $\odot$ ABHC. P. 32. B. 1.

Which was to be done. D. 3. B. 4.



PROPOSITION III. PROBLEM III.

ABOUT a given circle (EFG) to describe a triangle (ABD), equiangular to a given triangle (HKL).

Given.

The $\odot$ EFG, together with the $\triangle$ HKL.

Sought.

The $\triangle$ ABD described about the $\odot$ EFG, equiangular to the $\triangle$ HKL.

Resolution.

1. Produce the side HL, of the $\triangle$ HKL, both ways. Pos. 2.
2. Find the center C of the $\odot$ EFG, and draw the ray CE. P. 1. B. 3.
3. At the point C in CE, make the $\angle$ ECF = to the $\angle$ KHM, and $\angle$ ECG = to $\angle$ KLN. P. 23. B. 1.
4. Upon CE, CF, CG, erect the $\perp$ AD, AB, DB produced. P. 11. B. 1.

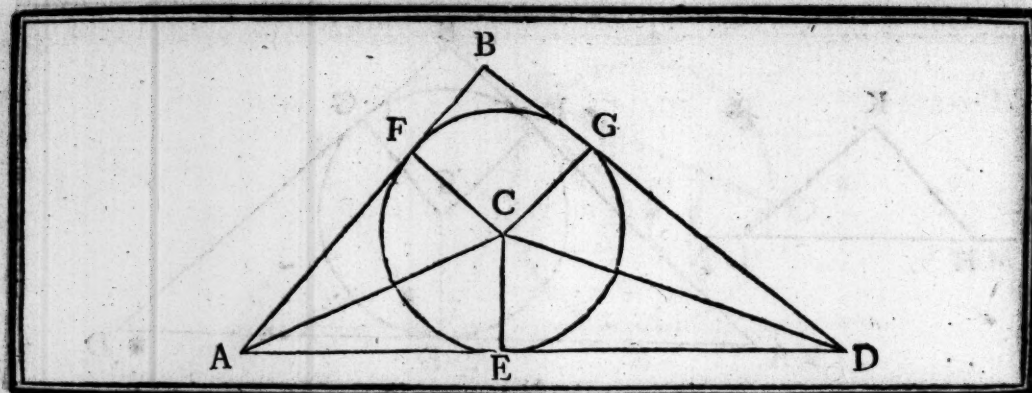
Preparation.

Draw the straight line FE. Pos. 1.

DEMONSTRATION.

- B**ECAUSE the $\angle$ CEA, CFA are $\perp$ (Ref. 4.)
1. $\angle$ FEA + EFA are $< 2 \perp$, & AD, AB meet some where in A. { Ax. 8. B. 1.
Ax. 11. B. 1.
It may be demonstrated after the same manner, that
 2. AD, DB & AB, DB meet somewhere in D & B.
And since AD, AB, DB are $\perp$ at the extremities E, F, G of the rays EF, CF, CG (Ref. 4.)
 3. These straight lines touch the $\odot$ EFG; and the $\triangle$ ABD formed by these straight lines is described about the $\odot$ EFG. { P. 16. B. 3
Cor. D. 4. B. 4
Moreover, the 4 $\angle$ CEA + CFA + ECF + FAE of the quadrilateral figure AFCE being = to 4 $\perp$ (P. 32. B. 1), and the $\angle$ CEA + CFA = to 2 $\perp$ (Ref. 4.)
 4. The $\angle$ ECF + FAE are also = to 2 $\perp$. Ax. 3. B. 1.
 5. Or = to $\angle$ KHM + KHL as being also = to 2 $\perp$. { Ax. 1. B. 1.
P. 13. B. 1.
But $\angle$ ECF being = to $\angle$ KHM (Ref. 3).
 6. The $\angle$ FAE is = to $\angle$ KHL, and $\angle$ GDE = to $\angle$ KLH. Ax. 3. B. 1.
 7. Hence the third $\angle$ FBG of the $\triangle$ ABD is = to the third $\angle$ HKL of $\triangle$ HKL. P. 32. B. 1.
 8. Therefore the $\triangle$ ABD described about the $\odot$ EFG is equiangular to the given $\triangle$ HKL.

Which was to be done.



PROPOSITION IV. PROBLEM IV.
To inscribe a circle (EFG) in a given triangle (ABD).

Given.
 The $\triangle ABD$.

Sought.
 The $\odot$ EFG inscribed in the $\triangle ABD$.

Resolution.

1. Bisect the $\angle BAD$, $\angle BDA$ by the straight lines AC, DC produced until they meet one another in C. P. 9. B. 1.
2. From the point C let fall upon AD the $\perp$ CE. P. 12. B. 1.
3. And from the center C at the distance CE, describe the $\odot$ EFG. P. 3.

Preparation.

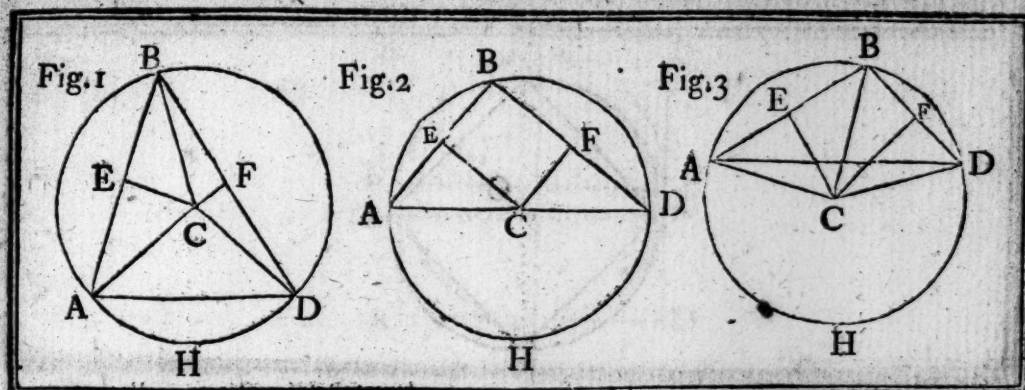
From the point C let fall upon AB & DB the $\perp$ CF, CG. P. 12. B. 1.

DEMONSTRATION.

BECAUSE in the $\triangle AFC$, $\triangle ACE$, the $\angle FAC$ is $=$ to the $\angle CAE$ (Ref. 1), $\angle CFA =$ to $\angle CEA$ (Prep. Ref. 2 & Ax. 10. B. 1); & AC common to the two $\triangle$.

1. The straight line CF is $=$ to CE. P. 26. B. 1.
 In like manner it may be demonstrated, that
2. The straight line CG is $=$ to CE.
3. Consequently, the straight lines CF, CE, CG are $=$ to one another; and the $\odot$ described from the center C at the distance CE will also pass thro' the points F & G. { Ax. 1. B. 1.
 And since the sides AD, AB, DB are $\perp$ at the extremities E, F, G, of the rays CE, CF, CG (Ref. 2 & Prep.). { D. 15. B. 1.
4. These sides will touch the $\odot$ in the points E, F, G. { P. 16. B. 3.
5. Therefore the $\odot$ EFG is inscribed in the $\triangle ABD$. { Cor.

Which was to be done. D. 5. B. 4.



PROPOSITION V. PROBLEM V.

TO describe a circle (ABDH), about a given triangle (ABD).

Given.
The $\triangle ABD$.

Sought.
The $\odot$ ABDH described about
the $\triangle ABD$.

Resolution.

1. Bisect the sides AB, DB in the points E & F. P. 10. B. 1.
2. At the points E & F in AB, DB, erect the $\perp$ EC, FC,
produced until they meet in C. P. 11. B. 1.
3. And whether the point C falls within (fig. 1.) or without
(fig. 3.) or in one of the sides (fig. 2). of the $\triangle ABD$,
from the center C at the distance CA describe the
 $\odot$ ABDH. Pos. 3.

Preparation.

Draw the straight lines CD, CB.

Pos. 1.

DEMONSTRATION.

BECAUSE in the $\triangle AEC$, BEC , the side AE is $=$ to the side EB
(Ref. 1), EC common to the two $\triangle$, & $\angle AEC =$ to $\angle BEC$ (Ref. 2
& Ax. 10. B. 1).

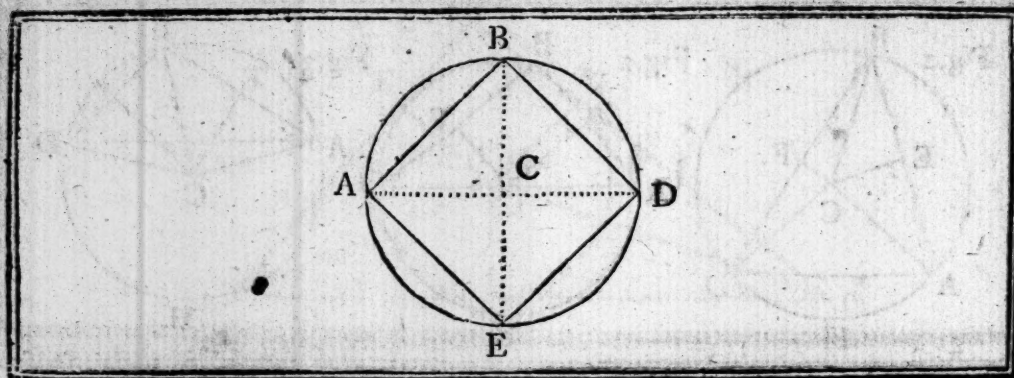
1. The straight line CB is $=$ to CA. P. 4. B. 1.
It may be demonstrated after the same manner, that
2. The straight line CB is $=$ to CD.
3. Consequently, the straight lines CA, CB, CD are $=$ to one another;
and the $\odot$ ABDH described from the center C at the distance $\left\{ \begin{array}{l} \text{Ax. 1. B. 1.} \\ \text{D. 15. B. 1.} \end{array} \right.$
CA, will pass also thro' the points B & D. D. 6. B. 4.
4. Therefore this $\odot$ ABDH is described about the $\triangle ABD$.

Which was to be done.

COROLLARY

IF the triangle ABD be acute angled, the point C falls within it (fig. 1); but if
this triangle be obtuse angled, the point C falls without it (fig. 3); in fine if it be
a right angled triangle, the point C is in one of the sides (fig. 2).

T



PROPOSITION VI. PROBLEM VI.
To inscribe a Square (ABDE), in a given Circle (ABDE).

Given
 The $\odot$ ABDE.

Sought.
 The $\square$ ABDE inscribed in this $\odot$.

Resolution.

1. Draw the Diameters AD, BE, so as to cut each other at $\perp$. *P. 11. B. 1.*
2. Join their Extremities by the straight Lines AB, BD, DE, EA. *Pos. 1.*

DEMONSTRATION.

BECAUSE in the $\triangle ABC$, DBC the side AC is = to the side CD (*Ref. 1. & D. 15. B. 1.*), BC common to the two $\triangle$, & the $\angle BCA =$ to $\angle BCD$ (*Ref. 1. & Ax. 10. B. 1.*).

1. The straight Line AB is = to BD.

P. 4. B. 1.

It may be demonstrated after the same manner, that

2. The straight line BD is = to DE, DE = to EA & EA = to AB.

3. Consequently, the straight lines AB, BD, DE, EA are = to one another, or the quadrilateral figure ABDE is equilateral.

Ax. 1. B. 1.

And because each of the $\angle$ ABD, BDE, DEA, EAB is placed in a semi $\odot$.

4. These $\angle$ are $\perp$, & the equilateral quadrilateral figure ABDE is rectangular.

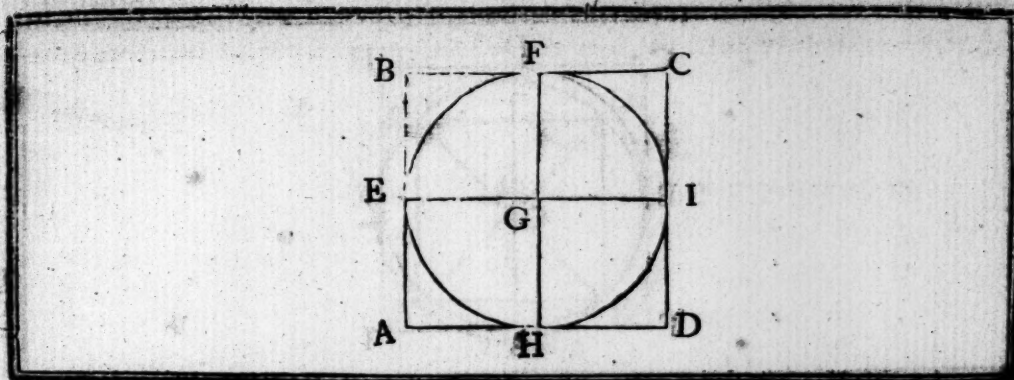
P. 31. B. 3.

5. Wherefore this quadrilateral figure is a square inscribed in the $\odot$ ABDE.

D. 30. B. 1.

D. 3. B. 4.

Which was to be done.



PROPOSITION VII. PROBLEM VII.
TO describe a Square (ABCD) about a given Circle (HEFI).

Given.
 The $\odot$ HEFI.

Sought.
 The $\square$ ABCD described about
 the $\odot$ HEFI.

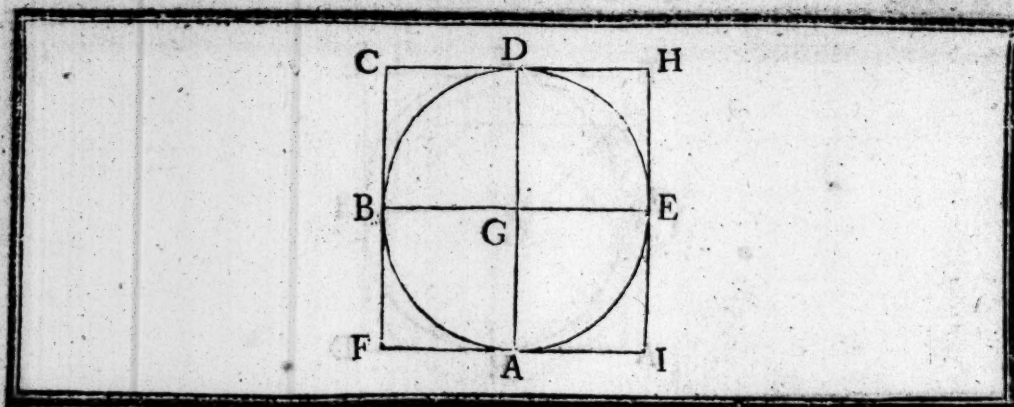
Resolution.

1. Draw the diameters EI, HF so as to cut each other at $\perp$. P.11. B.1.
2. At the Extremities H, E, F, I of those diameters erect the $\perp$ AD, AB, BC, CD. P.11. B.1.

DEMONSTRATION.

1. **T**HE lines DA, AB, BC, CD, are tangents of the $\odot$ HEFI. { P.16. B.3.
Cor.
2. And the straight line AD, is $\perp$ to EI, as also the straight line BC; because the $\angle$ HGE + $\angle$ GHA, & $\angle$ FGE + $\angle$ GFB are $\equiv$ to $2 \perp$ (Ref. 1. & 2). P.28. B.1.
3. Consequently, AD is $\perp$ to BC, likewise AB, HF, DC are $\perp$ to EI. P.30. B.1.
4. Wherefore the quadrilateral figures AI, EC, AF, HC, AC are Pgmes. D.35. B.1.
5. From whence it follows, that the straight lines AD, EI, BC, also AB, HF, DC, are $\equiv$ to one another. P.34. B.1.
6. And since EI is $\equiv$ to HF (D.15. B.1.), the straight lines AD, BC, AB, DC are equal. Ax.1. B.1.
 But $\angle$ EID of the Pgme. AI being a $\perp$ (Ref. 2).
7. The $\angle$ A, which is diagonally opposite to it, is also a $\perp$. P.34. B.1.
 It may be proved after the same manner, that
8. The $\angle$ B, C, D are $\perp$.
9. Consequently, there has been described about the $\odot$ HEFI, a quadrilateral figure ABCD equilateral (Arg. 6.) & rectangular (Arg. 7. & 8); or a square. { D.4. B.1.
D.30. B.1.

Which was to be done.



PROPOSITION VIII. PROBLEM VIII.

TO inscribe a circle (ABDE) in a given square (FGHI).

Given.
The $\square$ FGHI.

Sought.
The $\odot$ ABDE inscribed in
the $\square$ (FGHI).

Resolution.

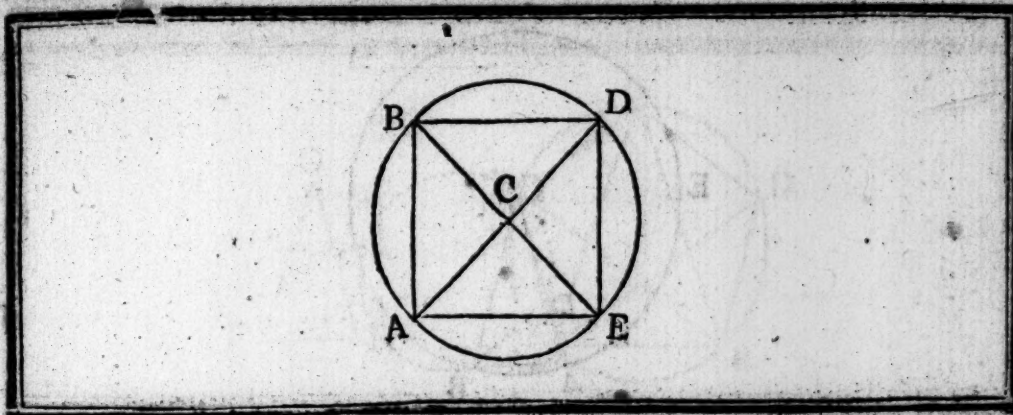
1. Bisect the sides FI, FG of the $\square$ FGHI. P.10. B.1.
2. Thro' the points of section A & B, draw AD Plle. to FG or
IH & BE Plle. to FI or GH. P.31. B.1.
3. From the center C, where AD, BE intersect each other, at
the distance CA describe the $\odot$ ABDE. Pos.3.

DEMONSTRATION.

BECAUSE the figures FE, BH, FD, AH, FC, AE, BD, CH are
Pgmes. (Ref. 2. & D. 35. B. 1).

1. The straight line FA is $=$ to BC & FB $=$ to AC. P.34. B.1.
But the whole lines FI, FG being equal (D. 30. B. 1.) and FA, FB
being the halves of those straight lines (Ref. 1).
2. The straight line FA is $=$ to FB. Ax.7. B.1.
3. Consequently, BC is also $=$ to AC; and likewise AC is $=$ to CE &
BC $=$ to CD. Ax.1. B.1.
4. From whence it follows, that the straight lines AC, BC, CE, CD
are $=$ to one another, and the $\odot$ described from the center { Ax.1. B.1.
C at the distance CA; passes also thro' the points B, D, E. { D.15. B.1.
But the $\angle$ DAF, EBG, ADH, BEI being $\angle$ (P. 34. B. 1.) as being
interior opposite to the $\angle$ GFA, HGB, IHD, FIE (D. 30. B. 1).
5. The straight lines FI, FG, GH, HI are tangents of the { P.16. B.3.
 $\odot$ ABDE. { Cor.
6. Wherefore this $\odot$ is inscribed in the square FGHI. D. 5. B.4

Which was to be done.



PROPOSITION IX. PROBLEM IX.

TO describe a circle (ABDE), about a given square (ABDE).

Given.
The $\square$ ABDE.

Sought.
The $\odot$ ABDE described about
the $\square$ ABDE.

Resolution.

1. Draw the diagonals AD, BE. Pos. 1.
2. From the center C, where the diagonals intersect each other,
and at the distance CA, describe the $\odot$ ABDE. Pos. 3.

DEMONSTRATION.

BECAUSE in $\triangle ABE$, $\triangle EBD$ the side AB is $=$ to the side BD
AE $=$ to ED (*D. 30. B. 1.*), & BE common to the two $\triangle$.

1. The $\sphericalangle$ ABE is $=$ to $\sphericalangle$ EBD, & the whole $\sphericalangle$ ABD is bisected
by BE. P. 8. B. 1.

It may be demonstrated after the same manner, that

2. The $\sphericalangle$ BAE, BDE, AED, are bisected by AD, BE.
But the whole $\sphericalangle$ ABD, BAE being $=$ to one another (*D. 30. B. 1.*).

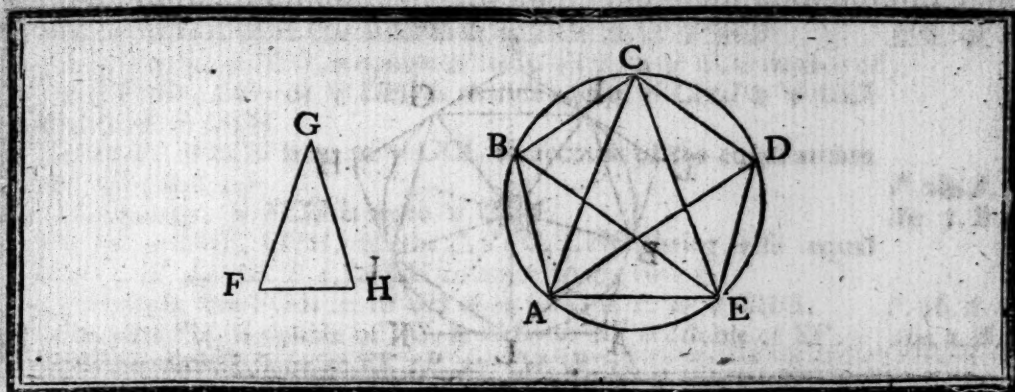
3. Their halves, the $\sphericalangle$ CBA, CAB will be also equal. Ax. 7. B. 1.

4. Consequently, CA is $=$ to CB, likewise CA is $=$ to CE, and CB $=$
to CD. P. 6. B. 1.

5. Hence CA, CB, CE, CD are $=$ to one another, & the $\odot$ described
from the center C at the distance CA, will also pass thro' the
points B, D, E. { Ax. 1. B. 1.
D. 15. B. 1.

6. Wherefore the $\odot$ ABDE is described about the $\square$ ABDE. D. 6. B. 4.

Which was to be done.



PROPOSITION XI. PROBLEM XI.

TO inscribe an equilateral & equiangular pentagon (ABCDE) in a given circle (ACE).

Given.

The $\odot$ ACE.

Sought.

The equilateral & equiangular pentagon ABCDE, inscribed in the $\odot$ ACE.

Resolution.

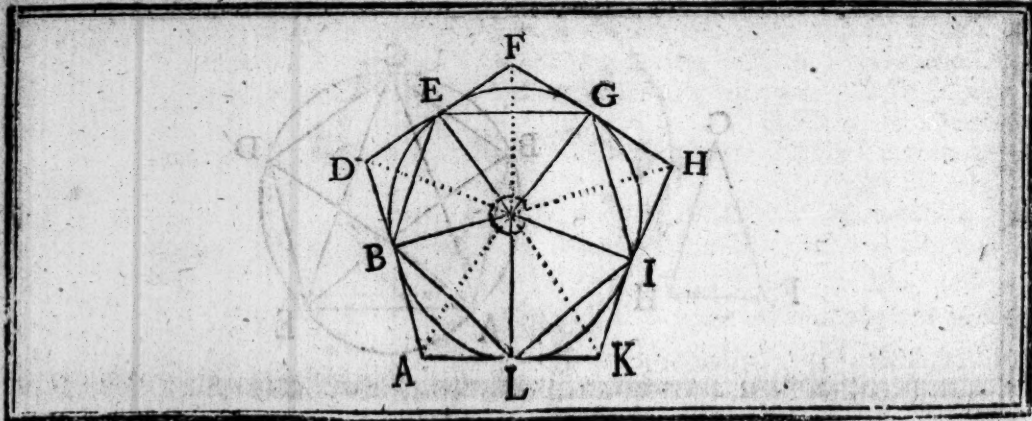
1. Describe the Isosceles $\triangle$ FGH having each of the $\sphericalangle$ at the base FH double of the $\sphericalangle$ at the vertex G. P. 10. B. 4.
2. Inscribe in the $\odot$ ACE a $\triangle$ ACE equiangular to the $\triangle$ FGH. P. 2. B. 4.
3. Bisect the $\sphericalangle$ CAE & CEA at the Base, by the straight lines AD, EB. P. 9. B. 1.
4. Draw the straight lines AB, BC, CD, DE. Pos. 1.

DEMONSTRATION.

BECAUSE each of the $\sphericalangle$ CAE, CEA is double of the $\sphericalangle$ ACE (Ref. 1. & 2.), & these $\sphericalangle$ are bisected (Ref. 3.).

1. The five $\sphericalangle$ VACE, CAD, DAE, BEA, CEB are $\equiv$ to one another. Ax. 7. B. 1.
2. From whence it follows that the arches AE, ED, DC, CB, BA are $\equiv$ to one another, likewise the chords AE, ED, DC, CB, BA. P. 26. B. 3.
But if to the $\equiv$ Arches AE, CD (Arg. 2.), be added the arch ABC. P. 29. B. 3.
3. The whole arch EABC is $\equiv$ to the whole arch ABCD, and $\sphericalangle$ CDE is $\equiv$ to the $\sphericalangle$ DEA. Ax. 2. B. 1.
It may be demonstrated after the same manner, that P. 27. B. 1.
4. Each of the $\sphericalangle$ EAB, ABC, BCD is $\equiv$ to the $\sphericalangle$ CDE or DEA.
5. Wherefore there has been inscribed in the $\odot$ ACE, an equilateral (Arg. 2.) & equiangular (Arg. 4.) pentagon. D. 3. B. 4.

Which was to be done.



PROPOSITION XII. PROBLEM XII.

TO describe an equilateral & equiangular pentagon (ADFHK) about a given circle.

Given.
The $\odot$ LEG.

Sought.
The equilateral pentagon ADFHK
described about the $\odot$ LEG.

Resolution.

1. In the $\odot$ LEG, inscribe an equilateral & equiangular pentagon. *P. 11. B. 4.*
2. To the points B, E, G, I, L, draw the rays CB, CE, CG, CI, CL. *Pos. 1.*
3. At the extremities of these rays erect the $\perp$ produced AD, DF, FH, HK, KA. *P. 11. B. 1.*

Preparation.

Draw the straight lines CA, CD, CF, CH, CK. *Pos. 1.*

DEMONSTRATION.

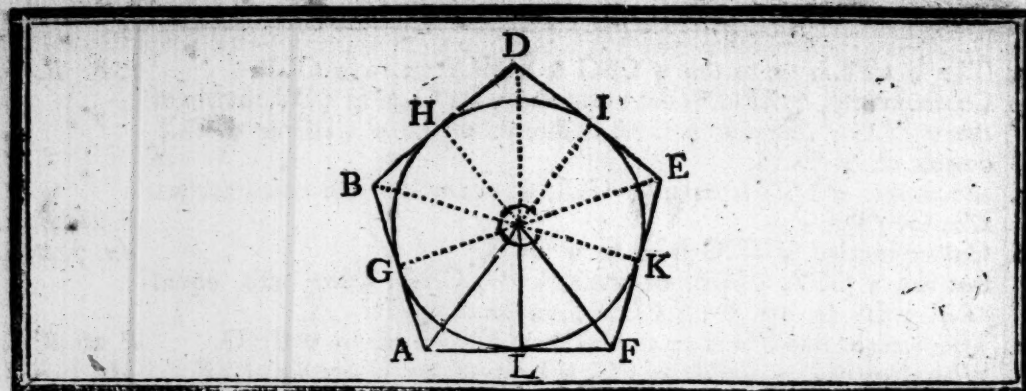
BECAUSE the straight lines AD, DF, FH, HK, KA are $\perp$ at the extremities of the rays CB, CE, CG, CI, CL. (*Ref. 3.*)

1. Those straight lines will touch the $\odot$ in the points B, E, G, I, L. *P. 16. B. 3.*
And the $\angle$ DBE + DEB, FEG + FGE, HGI + HIG, KIL + KLI, *Cor.*
ABL + ALB, taken two by two are $< 2 \perp$. *Ax. 8. B. 1.*
2. Therefore these straight lines AD, DF, FH, HK, KA will meet in the points D, F, H, K, A. *Lem. 1. B. 1.*
But since in the $\triangle$ CEF, CGF the side FE is = to the side FG (*P. 37. Cor. B. 3. & Ref. 3.*), CE = GC. (*D. 13. B. 1.*) & CF common to the two $\triangle$.

3. The $\angle CFE$ is $=$ to the $\angle CFG$ & $\angle ECF =$ to $\angle GCF$. P. 8. B. 1.
4. Consequently, $\angle EFG$, is double of the $\angle CFG$, & $\angle ECG$ double of the $\angle FCG$; likewise $\angle GHI$ is double of the $\angle CHG$ & $\angle GCI$ double of $\angle GCH$.
5. Moreover, $\angle ECG$ is $=$ to $\angle GCI$, on account of the equal arches EG, GI (*Ref. 1.*) P. 28. B. 3.
6. Consequently, $\angle FCG$ is $=$ to $\angle GCH$. Ax. 7. B. 1.
But the $\angle CGF, CGH$, of the $\triangle CFG, CHG$ being also equal (*Ref. 3. & Ax. 10. B. 1.*) & CG common to the two $\triangle$.
7. The straight line FG is $=$ to GH & $\angle CFG$ is $=$ to $\angle CHG$. P. 26. B. 1.
8. Wherefore FH , is double of FG , & likewise DF is double of EF . Ax. 2. B. 1.
And because FG is $=$ to EF (*P. 37. Cor. B. 3.*)
9. The straight line FH is also $=$ to DF , (*Ax. 6. B. 1.*), & likewise the straight lines HK, KA, AD are $=$ to FH , or DF .
Again $\angle EFG$ or DFH being double of the $\angle CFG$, the $\angle GHI$ or FHK double of the $\angle CHG$ and also $\angle CFG =$ to $\angle CHG$. (*Arg. 7.*)
10. The $\angle DFH, FHK$ are $=$ to one another, and likewise the $\angle HKA, KAD, ADF$ are $=$ to DFH or FHK .
11. Consequently there has been described about the $\odot$ LEG a pentagon $ADFFHK$ (*Arg. 1.*) equilateral (*Arg. 9.*), and equiangular (*Arg. 10.*) D. 4. B. 4.

Which was to be done.





PROPOSITION XIII. PROBLEM XIII.
TO inscribe a circle (GHIKL), in a given equilateral and equiangular Pentagon (ABDEF).

Given

The equilateral & equiangular pentagon ABDEF.

Sought

The $\odot$ GHIKL inscribed in this pentagon.

Resolution.

1. Bisect the two $\angle$ BAF, AFE of the pentagon ABDEF by the straight lines produced CA, CF. P. 9. B. 1.
2. From the point of concurrence C let fall upon AF the $\perp$ CL. P. 12. B. 1.
3. From the point C at the distance CL, describe the $\odot$ GHIKL. Pos. 3.

Preparation.

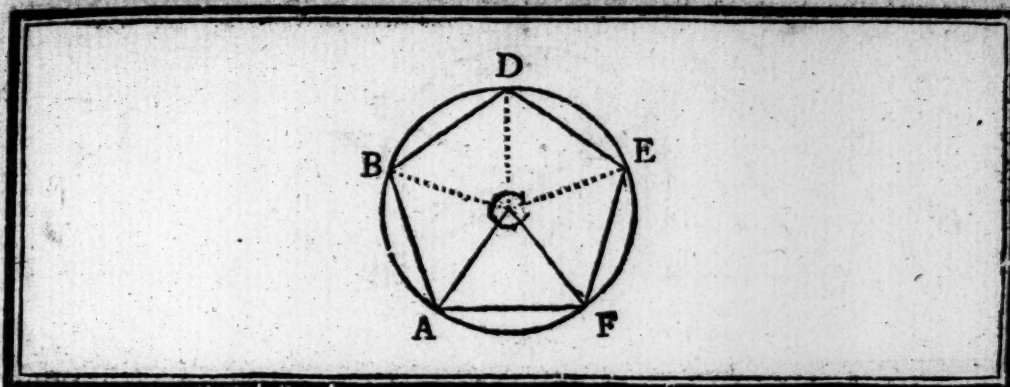
1. Draw the straight lines CB, CD, CE. Pos. 1.
2. From the point C let fall upon AB, BD, DE, EF the $\perp$ CG, CH, CI, CK. P. 12. B. 1.

DEMONSTRATION.

BECAUSE in the $\triangle$ ACF, ACB the side AF is = to the side AB, the side CA common to the two $\triangle$ & $\angle$ CAF = to $\angle$ CAB (*Ref. 1. & given*).

1. It follows that $\angle$ CFA is = to $\angle$ CBA. P. 4. B. 1.
 But $\angle$ AFE being = to $\angle$ DBA and double of $\angle$ CFA (*Ref. 1*).
2. Hence, $\angle$ DBA is also double of the $\angle$ CBA, or $\angle$ CBD = to $\angle$ CBA. *Ax. 6. B. 1.*
 It may be demonstrated after the same manner, that
3. The $\angle$ CDB is = to $\angle$ CDE & $\angle$ CED = to $\angle$ CEF.
 Therefore in the $\triangle$ CBG, CBH, $\angle$ CBG = to $\angle$ CBH (*Arg. 2*),
 $\angle$ CGB = to $\angle$ CHB (*Prep. 2 & Ax. 10. B. 1.*), & CB common to the two $\triangle$. P. 26. B. 1.
4. Consequently, CG is = to CH; likewise CI, CK, CL are = to CH or to CG.
5. Therefore the $\odot$ described from the center C at the distance CL will also pass thro' the points G, H, I, K. D. 15. B. 4.
 And because the straight lines AB, BD, DE, EF, FA are $\perp$ at the extremities of the rays CG, CH, CI, CK, CL (*Prep. 2 & Ref. 2*). D. 15. B. 1.
6. Those straight lines will touch the $\odot$ GHIKL (*P. 16. Cor. B. 3*); and this $\odot$ is inscribed in the pentagon ABDEF. D. 5. B. 4.

Which was to be done.



PROPOSITION XIV. PROBLEM XIV.

TO describe a circle (ADF); about a given equilateral and equiangular pentagon (ABDEF).

Given
The equilateral & equiangular
pentagon.

Sought
The $\odot$ ADF, described about this
pentagon.

Resolution.

1. Bisect the $\angle$ BAF, AFE by the straight lines CA, CF P. 9. B. 1.
produced.
2. From the point C, where those straight lines intersect each
other, at the distance CA describe the $\odot$ ADF. Pos. 3.

Preparation.

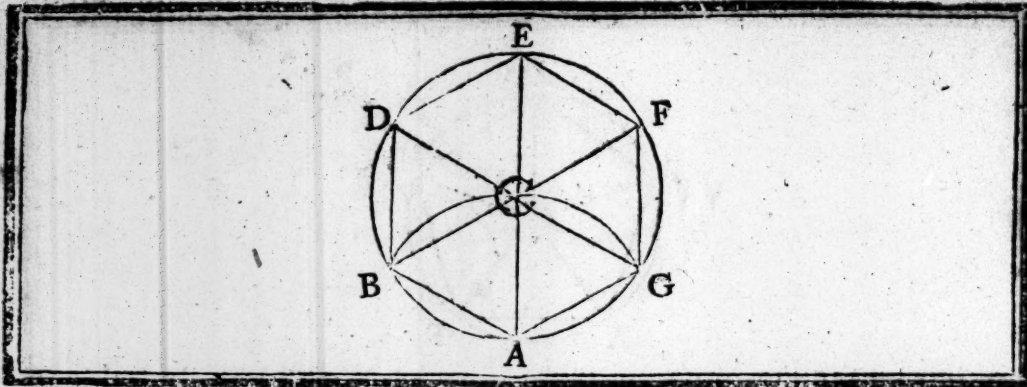
Draw the straight lines CB, CD, CE.

Pos. 1.

DEMONSTRATION.

1. **T**HE straight lines CB, CD, CE bisect the $\angle$ ABD, BDE, DEF. { P. 13. B. 4.
2. And because the $\angle$ BAF is = to the $\angle$ AFE, the $\angle$ CAF will be
also = to the $\angle$ CFA { Cor.
3. Wherefore CA is = to CF. Ax. 7. B. 1.
- It may be demonstrated after the same manner, that P. 6. B. 1.
4. Each of the straight lines CB, CD, CE is = to CA or to CF.
5. From whence it follows, that the $\odot$ described from the center C at
the distance CA will pass thro' the points B, D, E, F. D. 15. B. 1.
6. Consequently the $\odot$ ADF, is described about the given pentagon
ABDEF. D. 6. B. 4.

Which was to be done.



PROPOSITION XV. PROBLEM XV.

TO inscribe an equilateral and equiangular Hexagon (ABDEFG) in a given Circle (BEG).

Given
The $\odot$ BEG.

Sought
The equilateral & equiangular Hexagon
ABDEFG, inscribed in the $\odot$ BEG.

Resolution.

1. Find the center C of the $\odot$ BEG, and draw any diameter AE. *P. 1. B. 3.*
2. From the center A, at the distance AC describe an arch of a $\odot$ BCG. *Pos. 3.*
3. Draw the rays CG, CB produced to D & F. *Pos. 1. & 2.*
4. Draw the straight lines AB, BD, DE, EF, FG, GA. *Pos. 1.*

DEMONSTRATION.

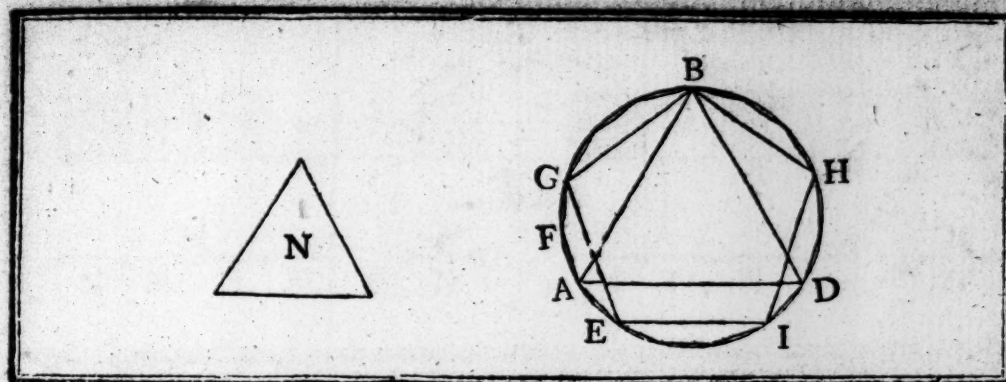
BECAUSE in the $\triangle$ BCA, the side BC is = to the side AC, & AB is also = to AC (*Ref. 3. & D. 15. B. 1.*).

1. This $\triangle$ is equilateral & equiangular. *{ D. 24. B. 1. P. 5. B. 1. }*
2. Wherefore, $\angle$ BCA is = to the third part of 2 $\angle$. & likewise $\angle$ ACG is also = to the third part of 2 $\angle$. *P. 32. B. 1.*
- But the $\angle$ BCA + $\angle$ ACG + $\angle$ GCF being = to 2 $\angle$. (*P. 13. B. 1.*),
3. The $\angle$ GCF is also = to the third part of 2 $\angle$; & the $\angle$ BCA, ACG, GCF are = to one another. *Ax. 1. B. 1.*
4. Consequently, the $\angle$ FCE, ECD, DCB which are = to them as being their vertical opposite ones, are also = to one another. *P. 15. B. 1.*
5. Hence, the arches BA, AG, GF, FE, ED, DB are = to one another, as likewise the chords BA, AG, GF, FE, ED, DB. *{ P. 26. B. 3. P. 29. B. 3. }*
6. Therefore the Hexagon ABDEFG inscribed in the $\odot$ BEG is equilateral.

Moreover the arch BA being = to the arch ED (*Arg. 5.*); if the common arch AGFE be added to both.

7. The arch BAGFE will be = to the arch AGFED. *Ax. 2. B. 1.*
8. From whence it follows, that $\angle$ EDB is = to $\angle$ DBA, and likewise each of the $\angle$ FED, GFE, AGF is = to the $\angle$ EDB, or to the $\angle$ DBA. *P. 27. B. 3.*
9. Therefore the equilateral Hexagon ABDEFG, inscribed in the $\odot$ BEG is also equiangular. *D. 3. B. 4.*

Which was to be done.



PROPOSITION XVI. PROBLEM XVI.

TO inscribe an equilateral and equiangular quindecagon (EAFG &c.) in a given circle (EBI).

Given
The $\odot$ EBI

Sought
The equilateral & equiangular
quindecagon EAFG &c.

Resolution.

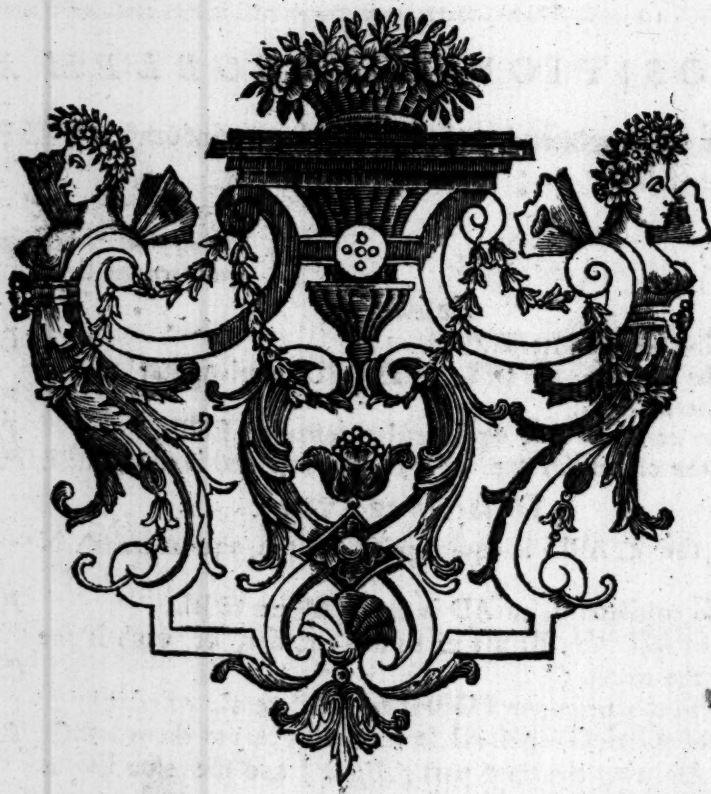
1. Describe an equilateral $\triangle N$. P. 1. B. 1.
2. Inscribe in the $\odot$ EBI, a $\triangle ABD$, equiangular to the equilateral $\triangle N$. P. 2. B. 4.
3. And an equilateral & equiangular pentagon EGBHI. P. 11. B. 4.
4. Draw the chord EA & place it 15 times around in the $\odot$ EBI. P. 1. B. 4.

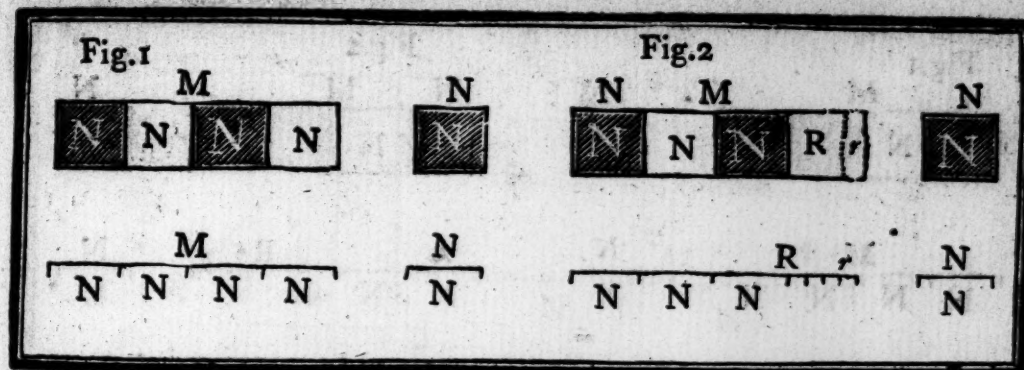
DEMONSTRATION.

BECAUSE the $\triangle ABD$ is equiangular to the equilateral $\triangle N$ (Ref. 2).

1. This $\triangle$ is also equilateral, or AD is = to AB = to BD. P. 6. B. 1.
2. And the arches AD, AB, BD are = to one another, or each is the third part of the whole $\odot$. P. 28. B. 3.
Again, because the pentagon EGBHI is equilateral, (Ref. 3).
3. Each of the arches EG, GB, BH, HI, IE is the fifth part of the whole $\odot$. P. 28. B. 3.
But the arch AB being the third part (Arg. 2) and the arch EG or GB the fifth part of the $\odot$ (Arg. 3).
4. There may be placed in the arch AB five sides of the quindecagon, and in each of the arches EG, GB three sides of the quindecagon, or in the arch EGB six sides of the quindecagon.
5. Consequently one of these sides may be placed in the arch AE, and the equilateral quindecagon EAFG &c. will be inscribed in the $\odot$ EBI. D. 3. B. 4.
Moreover, since each of its $\sphericalangle$ FAE is contained in an arch FHE which is = to thirteen parts of the fifteen, into which the circumference is divided.
6. These $\sphericalangle$ will be = to one another. P. 27. B. 3.
7. Therefore there has been inscribed in the $\odot$ EBI, an equilateral & equiangular quindecagon EAFG.

Which was to be done.





DEFINITIONS.

I.

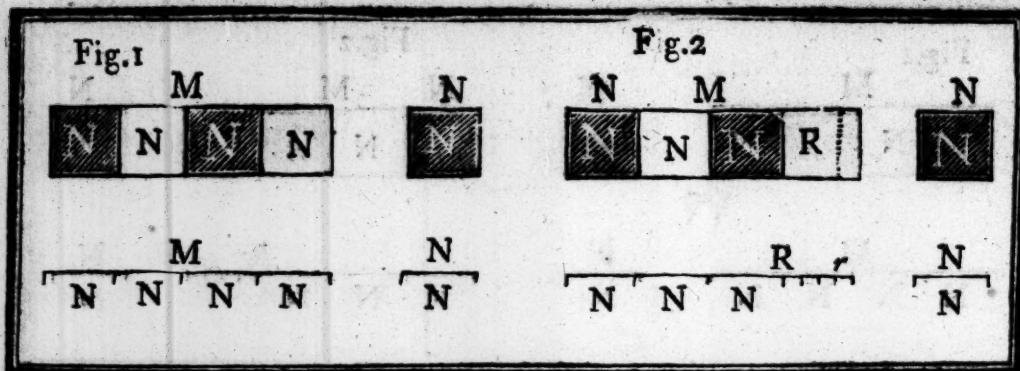
A Less magnitude is said to be a *part* of a greater magnitude, when the less measures the greater.

§. 1. By the expression of measuring a magnitude Euclid means to be contained in it a certain number of times without a remainder, that is a less magnitude *N* (fig. 1.) measures a greater *M*, when the magnitude *N* is contained in *M* without a remainder twice, thrice, four times, and in general, any number of times whatsoever, or which comes to the same, when the less magnitude *N* repeated twice, thrice four times, and in general any number of times produces a whole, equal to the greater *M*.

§. 2. Those parts which measure a whole without a remainder, are called aliquot parts, and such as are not contained in a whole exactly, but are measured by some other determined quantity which measures also the whole, are called aliquant parts.

Thus the numbers 2, 3, 4, 6 are so many aliquot parts of the number 12 considered as a whole; as each of the numbers 2, 3, 4, 6 is found repeated in 12 a certain number of times without a remainder. But the numbers 5, 7, 9 &c. are aliquant parts of the same whole 12; as they do not measure 12 but with a remainder: although they are all measured by unity as well as 12; which often happens in other numbers different from unity, as in the number 9 which is commensurable to 12 by the number 3, as also by unity.

Likewise the magnitude *N* (fig. 2.) is an aliquant part of the magnitude *M* ($= N + N + N + R$ &c), if *N* measures *M* leaving a remainder *R*, and this remainder *R* be such, that it measures *N* or at least that one of its determined parts as *r* measures this remainder *R*, as also the magnitude *N* & consequently the whole *M*.



DEFINITIONS.

§. 3. **I**N general numbers are said to be commensurable to each other which may result from unity or one of its aliquot parts repeated a determined number of times : or what amounts to the same that which is measured by unity or one of its aliquot parts.

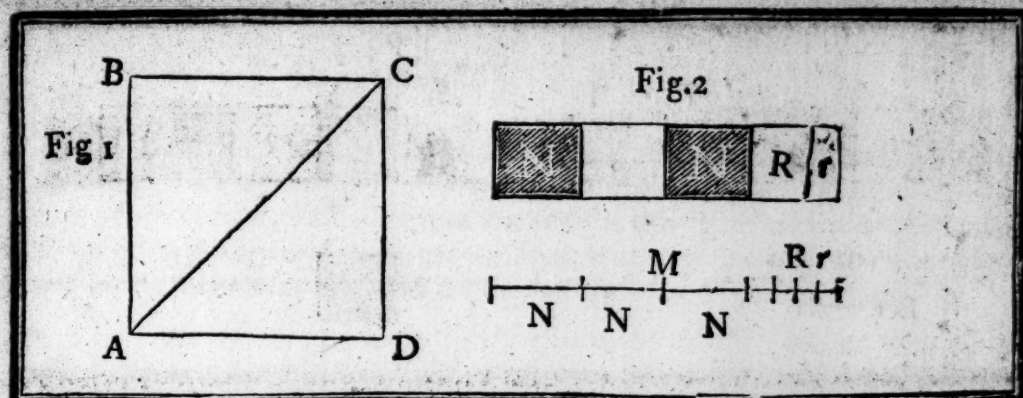
Thus the numbers 6, 9, 17, and the fractions $\frac{2}{3}$, $\frac{5}{7}$ are commensurable numbers ; because the first may be conceived to result from the determined and successive addition of unity ; and the last from that of the fractions $\frac{2}{3}$ & $\frac{5}{7}$ aliquot parts of unity.

§. 4. According to this definition, a commensurable quantity, is that which results from the determined repetition of any determined quantity. A quantity is therefore commensurable, when it contains one of its parts, as often as a determined number contains unity.

§. 5. Commensurability is therefore something relative. The magnitudes M and N are commensurable, as having a common and determined measure τ which can be taken for unity, and measure them both exactly ; or, as those two magnitudes may arise from the determined repetition of the same quantity R, be it what it will.

§. 6. It follows from this notion of commensurable numbers, that they are all multiples of each other, or aliquot parts, or aliquant parts. For if the quantities M and N, are commensurable, N measures M, or M measures N, or some other determined number τ measures them both. In the first case, the number M, is a multiple of N, in the second case M, is an aliquot part of N, and in the third, the lesser of the two is an aliquant part of the least. The same is true with respect to rational magnitudes in general.

§. 7. The number which cannot result from a determined repetition of unity or of one of its aliquot parts is called, irrational or incommensurable, with respect to unity. And in general, magnitudes which cannot result from the determined repetition of the same determined quantity considered as unity, are, incommensurable to one another, or irrational.



DEFINITIONS.

THUS the side (AD or DC) of the square (ABCD) is incommensurable to its diagonal (AC), or how much one contains of the other is inassignable (Fig. 1).
 §. 8. From whence it follows, that if two magnitudes M and N, are incommensurable to each other, M cannot be a multiple of N; nor an aliquot part, nor in fine an aliquant part of this same N, for if it was, the magnitudes M and N could be measured by the same determined magnitude, which is repugnant to the notion of incommensurability (Fig. 2)

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less.

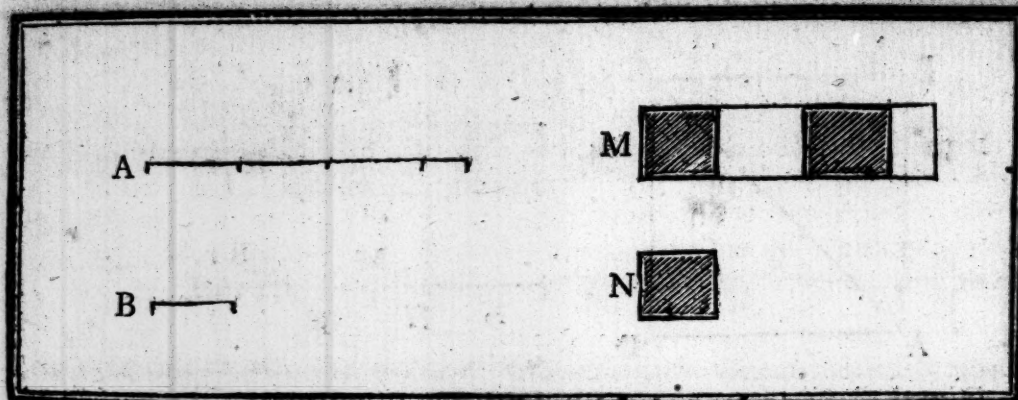
Thus, the number 12 is said to be a multiple, of the number 4, because 4 measures 12 without a remainder.

To the term of multiple corresponds that of submultiple, which signifies, that a less magnitude is an aliquot part of a greater; thus 4 is a submultiple of 12, as 12 is a multiple of 4.

III.

Ratio, is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.

This definition is imperfect, and is commonly believed to be none of Euclid's, but the addition of some unskilful editor; for though the idea of ratio includes a certain relation of the quantities of two homogeneous magnitudes, yet this general character is not sufficient; because the quantities of two magnitudes are susceptible of several sorts of relations different from that of ratio. Thus, when in a circle the square of the perpendicular let fall from the circumference on the diameter, is represented as constantly equal to the difference of the squares of the ray, and of the portion of the ray intercepted between the center and the perpendicular, without doubt, this perpendicular is considered as bearing a certain relation to this portion of the ray, but it is manifest that this relation is not a ratio, since the quantities are compared only by the means of the ray which is a third homogeneous magnitude different from the quantities compared.



DEFINITIONS.

IV.

MMAGNITUDES are said to have a ratio to one another ; when the less can be multiplied so as to exceed the other.

§. 1. *The lines A & B have a ratio to one another, because the line B, for example, taken three times and a half, is equal to the line A, and taken four times exceeds it. The Rgles M & N have also a ratio to one another, because the Rgle N taken three times and a half, is = to Rgle M, and repeated oftner exceeds it.*

But the line B, and the Rgle M have no ratio to one another, because the line B repeated ever so often, can never produce a magnitude which would equal or exceed the Rgle M. Therefore, only magnitudes of the same kind can have a ratio to one another, as numbers to numbers, lines to lines, surfaces to surfaces, and solids to solids.

§. 2. *In consequence of this definition, a finite magnitude and an infinite one, have no ratio to one another, though they be supposed of the same kind. For a magnitude conceived infinite, is conceived without bounds, consequently a finite magnitude repeated ever so often (provided the number of repetitions be determined) can never become equal or exceeds an infinite magnitude.*

§. 3. *A ratio is commensurable, when the terms of the ratio M & N are commensurable to each other, & a ratio is said to be incommensurable when the terms of the ratio are incommensurable.*

§. 4. *The antecedent of the ratio of M to N, is the first of the two terms which are compared, and the other is called its consequent.*

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth.

DEFINITIONS.

If the multiple of the first, be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth, or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

§. 1. *The ratio of the number 2 to the number 6, is the same as that of the number 8 to the number 24, for if the two antecedents 2 & 8 be multiplied by the same number M, and the two consequents 6 & 24 by another number N; the multiple 2 M of the first antecedent cannot be = or > or < the multiple 6 N of its consequent, unless the multiple of the second antecedent 8 M, be at the same time = or > or < the multiple 24 N of its consequent, for it is evident that*

If 2 M be = to 6 N,

2 M + 2 M + 2 M + 2 M is also = 6 N + 6 N + 6 N + 6 N, that is, 8 M = 24 N.

Likewise,

If 2 M be > 6 N, then

2 M + 2 M + 2 M + 2 M is also > 6 N + 6 N + 6 N + 6 N, that is, 8 M > 24 N.

And in fine,

If 2 M be < 6 N, then

2 M + 2 M + 2 M + 2 M is also < 6 N + 6 N + 6 N + 6 N, that is, 8 M < 24 N.

§. 2. *On the contrary, the numbers 2, 3 & 7, 8 are not in the same ratio; for if the antecedents be multiplied by 3, and the consequents by 2, there will result the four multiples 6, 6, 21, 16, where the multiple 6 of the 1st antecedent is equal to the multiple 6 of its consequent, whilst 21 multiple of the 11. antecedent is greater than 16 multiple of its consequent.*

§. 3. *Incommensurable magnitudes can never have their equimultiples equal, otherwise they would be commensurable to one another, wherefore incommensurables are shewn to be proportional only from the joint excess or defect of their equimultiples; whereas commensurable magnitudes being capable of a joint equality, and inequality of their equimultiples, are shewn to be proportional from the joint equality or excess of their equimultiples, hence it is that the signs in this definition by which proportionality is discovered, are applicable to all kinds of magnitude whatsoever.*

§. 4. *What is true with respect to the correspondence of multiples, is also true with respect to that of submultiples. But it is probable that Euclid preferred the use of multiples to that of submultiples, because he could not prescribe to take submultiples without first shewing how to divide magnitude into equal parts, whilst the formation of multiples required no such principle. This Geometer had a right to assume for granted, that the double triple, or any multiple of a magnitude could be taken, but was under the necessity of shewing by the*

Resolution of a problem, how to take away an aliquot part from a given line, and the resolution of this problem supposing the doctrine of similitude, could not be given but in the IX. Proposition of the VI. Book.

VI.

Magnitudes which have the same ratio, are called proportionals.

When four magnitudes A, B, C, D are proportional, it is usually expressed thus, $A : B = C : D$ and in words, the first is to the second as the third to the fourth.

VII.

When of the equimultiples of four magnitudes (taken as in the 5th definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth, and on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

§. 1. *Such are the ratios $3 : 2$ & $11 : 9$ for if the antecedents be multiplied by 9, and the consequents by 13, there will result $27 : 26$; $99 : 117$.*

$$\begin{array}{r} 3 : 2 ; 11 : 9 \\ 9 \ 13 \quad 9 \ 13 \\ \hline \end{array}$$

$$27 : 26 ; 99 : 117$$

Where the correspondence of the multiples does not hold, the first antecedent 27 being greater than its consequent 26 whilst the second antecedent 99 is less than its consequent 117.

§. 2. *To discover by inspection the inequality of two ratios $A : B$ & $C : D$ by this character of the non correspondence of multiples, it suffices to chuse for multiples, the two terms of one of the two ratios, for Ex. $C : D$, and to multiply the antecedents A & C by the consequent D of this ratio; and the two consequents B & D by the antecedent C of the same ratio, in this manner.*

$$\begin{array}{r} A : B ; C : D \\ D \ C ; D : C \\ \hline \end{array}$$

$$\begin{array}{r} 3 : 5 ; 7 : 9 \\ 9 \ 7 \quad 9 \ 7 \\ \hline \end{array}$$

$$AD : BC ; CD : D.C$$

$$27 : 35 ; 63 ; 63$$

Which being done, the two products C.D & D.C will be found equal, whilst the two others A.D & B.C are unequal, and in particular, if the multiple of one of the antecedents be greater than that of its consequent, whilst the multiple of the other is equal to its, then the terms of the lesser ratio have been chosen for multipliers. On the contrary, if the multiple of one of the antecedents be less than that of its consequent, whilst the multiple of the other is equal to its, then the terms of the greater ratio have been chosen for multipliers.

VIII.

Analogy or proportion, is the similitude of ratios.

As a sign and character of proportionals has been already given (in Def. 5.) this is a superfluous definition, a remark of some scholiast jumbled into the text which interrupts the coherence of Euclid's genuine definitions.

IX.

Proportion consists in three terms at least.

§. 1. *Proportion consisting in the equality of two ratios, and each ratio having two terms, in a proportion there are four terms, of which the first and fourth are called the extremes, and the second and third the means, those four terms are considered as only three, when the consequent of the first ratio at the same time holds the place of the antecedent of the second ratio: it is for this reason, that proportions are distinguished into discrete, and continued. A proportion is discrete when the two means are unequal, and it is called continued when these same terms are equal, thus this proportion $2 : 4 = 5 : 10$ is discrete because the two mean terms 4 & 5 are unequal, on the contrary, the proportion $2 : 4 = 4 : 8$ is a continued proportion on account of the equality of the mean terms 4 & 4.*

§. 2. *A series of magnitudes in continued proportion, forms a geometrical progression, such are the numbers 1, 2, 4, 8, 16, 32, 64, &c.*

X.

When three magnitudes are proportional the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on quadruplicate, &c. increasing the denomination still by unity in any number of proportionals.

XII.

In proportionals, the antecedent terms are called Homologous to one another, as also the consequents to one another.

XIII.

Proportion is said to be *alternate* when the antecedent of the first ratio is compared with the antecedent of the second, and the consequent of the first ratio with the consequent of the second.

*If $A : B = C : D$ } then by alternation. $\{ A : C = B : D$
 $4 : 5 = 16 : 20$ } $\{ 4 : 16 = 5 : 20$*

When the proportion is disposed after this manner, it is said to be done by permutation or alternately, permutando or alternando.

XIV.

But when the consequents are changed into antecedents, and the antecedents into consequents in the same order, it is said that the comparison of the terms is made by *inversion* or *invertendo*.

$$\left. \begin{array}{l} A : B = C : D \\ 3 : 9 = 4 : 12 \end{array} \right\} \text{therefore invertendo.} \left\{ \begin{array}{l} B : A = D : C \\ 9 : 3 = 12 : 4 \end{array} \right.$$

XV.

But the comparison is made by *composition* or *componendo*, when the sum of the consequents and antecedents is compared with their respective consequents.

$$\left. \begin{array}{l} A : B = C : D \\ 3 : 9 = 4 : 12 \end{array} \right\} \begin{array}{l} \text{therefore} \\ \text{componendo} \end{array} \left\{ \begin{array}{l} A + B : B = C + D : D \\ 3 + 9 : 9 = 4 + 12 : 12 \end{array} \right.$$

XVI.

The comparison is made by *division* of ratio, or *dividendo* when the excess of the antecedent above its consequent, is compared with its consequent.

$$\left. \begin{array}{l} \text{If } A : B = C : D \\ 9 : 3 = 12 : 4 \end{array} \right\} \text{dividendo.} \left\{ \begin{array}{l} A - B : B = C - D : D \\ 9 - 3 : 3 = 12 - 4 : 4 \end{array} \right.$$

XVII.

The comparison is made by *the conversion* of ratio, or *convertendo*, when the antecedent is compared to the excess of the antecedent above its consequent.

$$\left. \begin{array}{l} \text{If } A : B = C : D \\ 9 : 3 = 12 : 4 \end{array} \right\} \begin{array}{l} \text{therefore} \\ \text{convertendo.} \end{array} \left\{ \begin{array}{l} A : A - B = C : C - D \\ 9 : 9 - 3 = 12 : 12 - 4 \end{array} \right.$$

XVIII.

A conclusion is drawn from *equality of ratio* or *ex æquo*, when comparing two series of magnitudes of the same number, such that the ratios of the first be equal to the ratios of the second, each to each, (whether the comparison be made in the same order or in an inverted one), it is concluded that the extremes of the two series are in proportion.

The sense of this definition is as follows, if A, B, C, D be a series of four magnitudes, and a, b, c, d a series of four other magnitudes, such that

$$\left. \begin{array}{l} A : B = a : b \\ B : C = b : c \\ C : D = c : d \end{array} \right\} \text{or in an inverted order.} \left\{ \begin{array}{l} A : B = c : d \\ B : C = b : c \\ C : D = a : b \end{array} \right.$$

In the one or the other case it is allowed to infer ex æquo, when the ratio of the extreames $A : D$ of the I. series is equal to the ratio of the extreames $a : d$ of the II. series; or that $A : D = a : d$.

I. A, B, C, D	15, 3, 45, 9
II. a, b, c, d	10, 2, 30, 6
$A : D = a : d$	$15 : 9 = 10 : 6$

XIX.

The equality of ratio is called ordinate ratio, when the ratio of the first series are equal to the ratios of the second series each to each in the same direct order.

For Example let $A : B = a : b$
 $B : C = b : c$
 $C : D = c : d$

Here the ratios are equal each to each in the same direct order, because the first magnitude is to the second of the first rank, as the first to the second of the other rank, and as the second is to the third of the first rank, so is the second to the third of the other, and so on in order. If therefore it is inferred that the extreames are proportional, or that $A : D = a : d$, the inference is said to be made from direct equality, or ex æquo ordinate.

XX.

On the contrary, equality of ratio is called inverted or perturbate analogy, in the second case, that is when the ratios of the first series are equal to those of the second series each to each, taking those last in an inverted order.

§. 1. Let the two series of magnitudes be.

$$\left. \begin{array}{l} A, B, C, D \\ a, b, c, d \end{array} \right\} \text{where it is supposed } \left\{ \begin{array}{l} A : B = c : d \\ B : C = b : c \\ C : D = a : b \end{array} \right.$$

Here the ratios of the I. series are equal to the ratios of the II. series each to each, but in an inverted order, that is the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank, and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank, and so in a cross order.

If therefore it be inferred that $A : D = a : d$.

This inference is said to be made ex æquo perturbate.

§. 2. Beginners may easily distinguish the case of direct equality from that of perturbate equality, if they remember that when two terms are common to two proportions, and that they occupy indifferently either the first and third, or the second and fourth place, that it is always the case of direct equality; For Example.

$$\begin{array}{ccc} A : B = a : b & B : A = b : a & A : B = a : b \\ B : C = b : c & \text{or } B : C = b : c & \text{or } C : B = c : b \\ \hline A : C = a : c & A : C = a : c & A : B = a : c \end{array}$$

Here are always two proportions which have in common the two terms B & b occupying the first and third, or the second and fourth places; the two other terms A & C are proportional to the two others a & c taking them in the same order.

§. 3. On the contrary when the two terms which are common to the two proportions, are either the means or the extremes, it is the case of perturbate equality, for example

$$\begin{array}{ccc} \text{If } A : B = b : c & B : A = c : b & A : B = b : c \\ B : C = a : b & \text{or } B : C = a : b & \text{or } C : B = b : a \\ \hline A : C = a : c & A : C = a : c & A : C = a : c \end{array}$$

In those three cases the terms B & b which are common to the two proportions, are either the extremes or the means; consequently the other terms are in proportion, so that the two terms, which arise from the same proportion A & C or a & c remain extremes or means.

These are the denominations given to the different ways of concluding by analogy, Euclid now proceeds to demonstrate that they are just.



POSTULATES.

I.

LET it be granted, that any magnitude may be doubled, tripled, quadrupled, or in general, that any multiple of it may be taken.

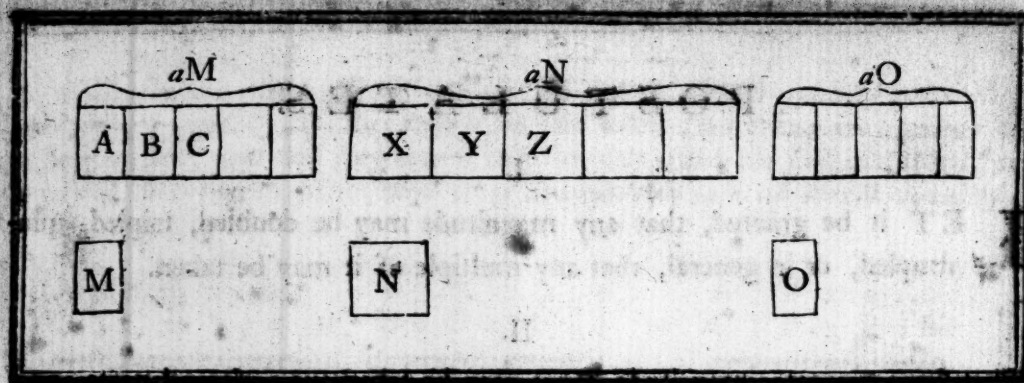
II.

That from a greater magnitude, there may be taken one or several parts equal to a less magnitude of the same kind.

ABBREVIATIONS.

Mgn.	Magnitude.
Mult.	Multiple.
Equimult.	Equimultiple.





PROPOSITION I. THEOREM I.

IF any number of magnitudes (aM , aN , aO &c) be equimultiples of as many (M , N , O &c) each of each, the sum ($aM + aN + aO$ &c) of all the first is the same multiple of the sum ($M + N + O$ &c) of all the second, as any one of the first (aM) is of its part (M).

Hypothesis.

aM } are M each
 aN } equimultiples N of
 aO } of O each

Thesis.

$aM + aN + aO$ is the same multiple of $M + N + O$ that aM is of M , or aN of N &c.

Preparation.

The mgn. aM being the same multiple of M , that aN is of N (*Hyp.*), as many magnitudes A, B, C , &c. as can be taken out of aM each equal to M , so many X, Y, Z , &c. can be taken out of aN , each equal to N .

Let then A } be each
 B } equal to M &
 C } X } each
 Y } equal to
 Z } N

Pos. 2. B. 5.

DEMONSTRATION.

BECAUSE aM is the same multiple of M , that aN is of N (*Hyp.*),

1. As many magnitudes X, Y, Z , &c. as are in aN each equal to N , so many A, B, C , &c. are there in aM each equal to M .

But

$A = M$ & $X = N$ (*Prep.*),

2. Therefore

$A + X = M + N$

Ax. 2. B. 1.

Likewise

$B = M$ & $Y = N$ (*Prep.*),

3. It follows that

$B + Y = M + N$

Ax. 2. B. 1.

Again, because

$C = M$ & $Z = N$ (*Prep.*),

4. It follows that

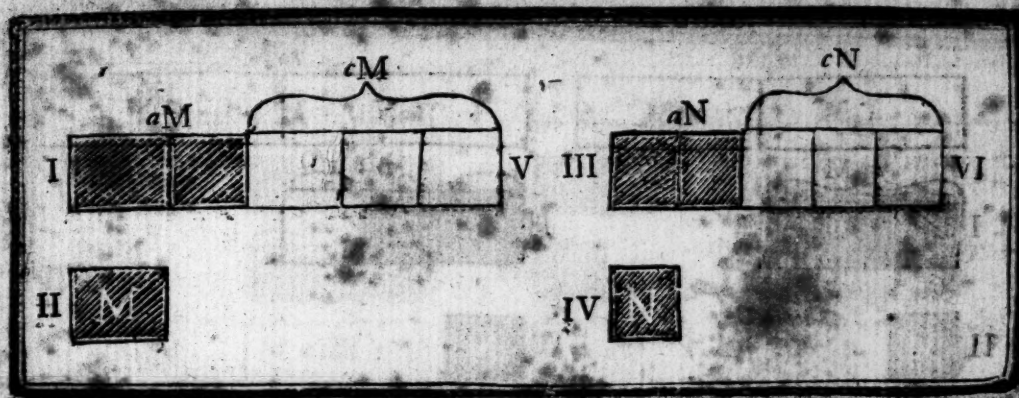
$C + Z = M + N$

Ax. 2. B. 1.

Consequently there is in aM as many Magnitudes $= M$, as there are in $aM + aN = M + N$.

5. From whence it follows that $aM + aN$ is the same multiple of $M + N$, that aM is of M , or that aN is of N , & likewise $aM + aN + aO$ is the same multiple of $M + N + O$, that aM is of M or aN of N , &c.

Which was to be demonstrated



PROPOSITION II. THEOREM II.

IF the first magnitude (aM) be the same multiple of the second (M), that the third (aN) is of the fourth (N); & the fifth (cM) the same multiple of the second (M), that the sixth (cN) is of the fourth (N); then shall the first together with the fifth ($aM + cM$) be the same multiple of the second (M), that the third together with the sixth ($aN + cN$) is of the fourth (N).

Hypothesis.

aM } likewise $\left\{ \begin{array}{l} cM \\ cN \end{array} \right.$ are & equimultiples of M each & of N each

Thesis.

$aM + cM$ is the same multiple of M , that $aN + cN$ is of N .

DEMONSTRATION.

BECAUSE aM is the same multiple of M , that aN is of N (*Hyp.*),

1. There are as many magnitudes in $aM =$ to M as there are in $aN =$ to N .

In like manner, because cM is the same multiple of M , that cN is of N (*Hyp.*),

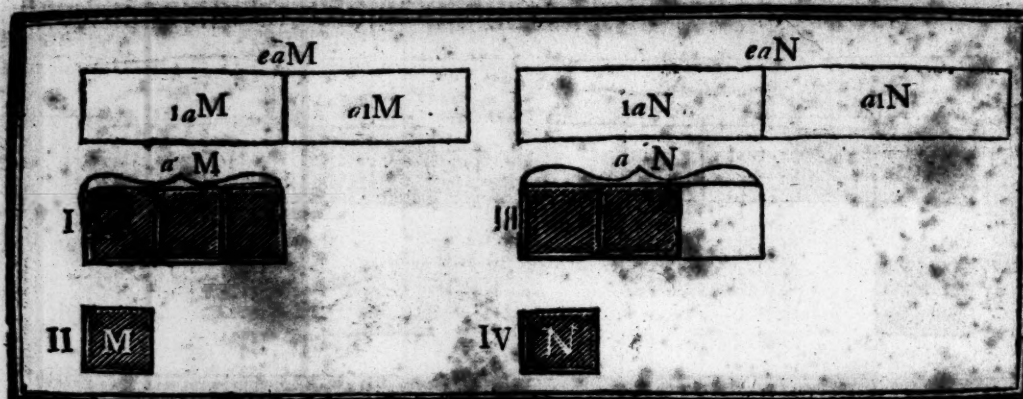
2. There are as many magnitudes in $cM =$ to M as there are in $cN =$ to N .

3. Consequently, as many as are in the whole $aM + cM$ equal to M , so many are there in the whole $aN + cN =$ to N .

4. Therefore $aM + cM$ is the same multiple of M that $aN + cN$ is of N .

Ax. 2. B. 1.

Which was to be demonstrated.



PROPOSITION III. THEOREM III.

IF the first magnitude ($a M$) be the same multiple of the second M , that the third ($a N$) is of the fourth ($a N$), and if of the first ($a M$) and third ($a N$) there be taken equimultiples ($e a M$, $e a N$); these ($e a M$, $e a N$) shall be equimultiples, the one of the second (M) and the other of the fourth (N).

Hypothesis.

- I. $a M$ } are two { M each
 $\&$ } equimultiples { $\&$ of
 $a N$ } of { N each
- II. $e a M$ } are two { $a M$ each
 $\&$ } equimultiples { $\&$ of
 $e a N$ } of { $a N$ each

Thesis.

$e a M$ is the same multiple of M that $e a N$ is of N .

Preparation.

Divide $e a M$ into its parts $1 a M$, $a 1 M$, &c. each $= a M$,
 And $e a N$ into its parts $1 a N$, $a 1 N$, &c. each $= a N$.

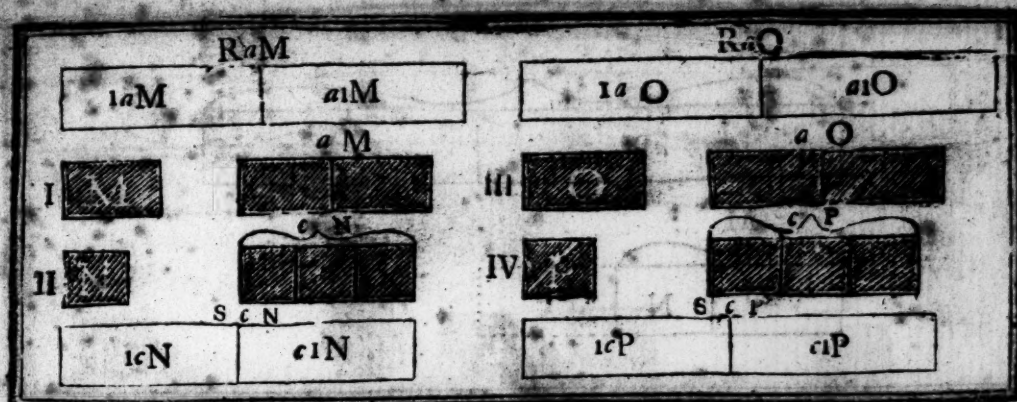
DEMONSTRATION.

BECAUSE $e a M$ is the same multiple of $a M$, that $e a N$ is of $a N$ (Hyp. 2.).

- There are as many magnitudes in $e a M =$ to $a M$ as there are in $e a N =$ to $a N$.
- Therefore the number of parts $1 a M$, $a 1 M$, &c. in $e a M$, is = to the number of parts $1 a N$, $a 1 N$, &c. in $e a N$.
 But because $a M$ is the same multiple of M , that $a N$ is of N , and that $1 a M = a M$, $1 a N = a N$.
- The magnitude $1 a M$ is the same multiple of M , that $1 a N$ is of N .
- In like manner $a 1 M$ is the same multiple of M , that $a 1 N$ is of N .
 Since then I mgn. $1 a M$ is the same multiple of the II mgn. M .
 that the III mgn. $1 a N$ is of the IV mgn. N
 & that the V mgn. $a 1 M$ is the same multiple of the II mgn. M
 that the VI mgn. $a 1 N$ is of the IV mgn. N .
- It follows that the magnitude $e a M$, composed of the I & V mgn. $1 a M + a 1 M$, is the same multiple of the II mgn. M , that the mgn. $e a N$, composed of the III & VI mgn. $1 a N + a 1 N$ is of the IV mgn. N .

Which was to be demonstrated

P. 2. B. 5.



PROPOSITION IV. THEOREM IV.

IF four magnitudes (M, N, O, P) are proportional; then any equimultiples (aM, aO) of the first (M) and third (O), shall have the same ratio to any equimultiples (cN, cP) of the second (N) and fourth (P).

Hypothesis.

$$I. M : N = O : P.$$

Thesis.

$$aM : cN = aO : cP.$$

$$II. \left\{ \begin{array}{l} aM \\ \& \\ aO \end{array} \right\} \begin{array}{l} \text{are} \\ \text{equimult.} \\ \text{of} \end{array} \left\{ \begin{array}{l} M \\ \& \text{ also } \\ O \end{array} \right\} \begin{array}{l} \text{are} \\ \text{equimult.} \\ \text{of} \end{array} \left\{ \begin{array}{l} cN \\ \& \\ cP \end{array} \right\} \begin{array}{l} \text{are} \\ \text{equimult.} \\ \text{of} \end{array} \left\{ \begin{array}{l} N \\ \& \\ P \end{array} \right\}$$

Preparation.

1. Take of aM & of aO any equimult. RaM, RaO
2. Likewise of cN & of cP any equimult. ScN, ScP

Pof. 1. B. 5.

DEMONSTRATION.

BECAUSE aM is the same mult. of M , that aO is of O (Hyp. 2), & the mgns. RaM, RaO are equimult. of the mgns. aM, aO (Prep. 1).

1. The magnitude RaM is the same multiple of M , that the magnitude RaO is of O .

P. 3. B. 5.

2. In like manner, the magnitude ScN is the same multiple of N that ScP is of P .

And as $M : N = O : P$ (Hyp. 1.) & RaM, RaO are any equimultiples of the I term M and of the III O ; and ScN, ScP any equimultiples of the II term N and of the IV P (Arg. 1 & 2).

3. If RaM be $>$, $=$ or $<$ ScN , RaO will be $>$, $=$ or $<$ ScP . D. 5. B. 5.

But the magnitudes RaM & RaO are any equimultiples of the magnitudes aM & aO , and the magnitudes ScN, ScP are any equimultiples of the magnitudes cN & cP (Prep. 1 & 2).

4. Consequently, the ratio, of aM to cN is $=$ to the ratio of aO to cP ; or $aM : cN = aO : cP$

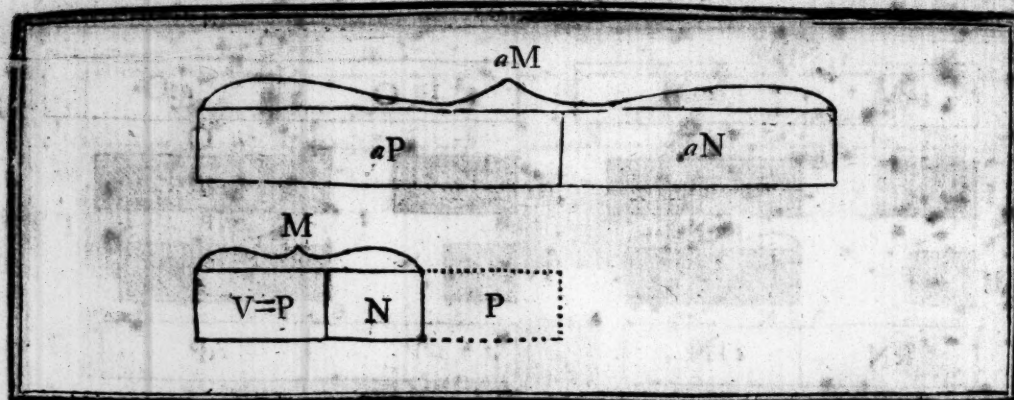
D. 5. B. 5.

Which was to be demonstrated.

COROLLARY.

IT is manifest that if ScN be $>$, $=$ or $<$ RaM ; likewise ScP will be $>$, $=$ or $<$ RaO (Arg. 3.); hence $cN : aM = cP : aO$ (D. 5. B. 5.).

Therefore, if four magnitudes be proportional, they are also by inversion or invertendo.



PROPOSITION V. THEOREM V.

IF a magnitude (aM) be the same multiple of another (M), which a magnitude (aN) taken from the first, is of a magnitude (N) taken from the other, the remainder (aP) shall be the same multiple of the remainder (V), that the whole (aM), is of the whole (M).

Hypothesis.

- I. $\left\{ \begin{array}{l} \text{The mgn. } aM \text{ \& } M \text{ are two wholes} \\ \text{The mgn. } aN \text{ \& } N \text{ their parts taken away} \\ \text{And the mgn. } aP \text{ \& } V \text{ the remainders} \end{array} \right.$
- II. $\left\{ \begin{array}{l} aM \text{ is the same multiple of } M \\ \text{that } aN \text{ is of } N. \end{array} \right.$

Thesis.

aP is the same multiple of V , that aM is of M .

Preparation.

Take a magnitude P such, that aP may be the same multiple of P , that aN is of N , or aM of M .

Pos. 2. B. 5.

DEMONSTRATION.

BECAUSE aN is the same multiple of N , that aP is of P (Prep.),

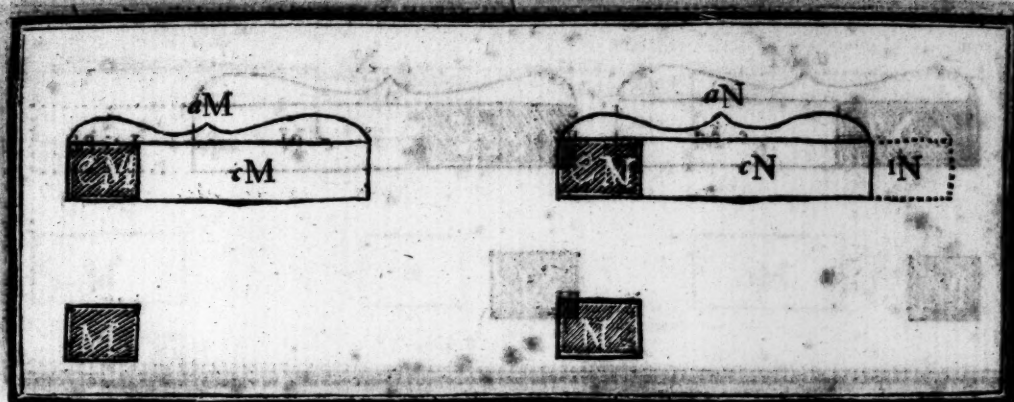
1. The sum $aN + aP$, or aM , of the first, is the same multiple of the sum $N + P$ of the last, that aN is of N .
But aM is the same multiple of M , or of $N + V$, that aN is of N (Hyp. 2).
2. Consequently, the mgn. aM is equimultiple of the mgn. $N + P$, & $N + V$.
3. And of course $N + P = N + V$.
Taking away the common mgn. N .
4. It follows that the mgn. P is $=$ to the mgn. V .
5. Consequently, aP being the same multiple of P , that aM is of M (Prep.), aP is also the same multiple of V , that aM is of M ,

P. 1. B. 5.

Ax. 7. B. 1.

Ax. 3. B. 1.

Which was to be demonstrated.



PROPOSITION VI. THEOREM VI.

IF two magnitudes (aM , aN) be equimultiples of two others (M & N) & if equimultiples (cM & cN) of these, be taken from the first two, the remainders (eM & eN) are either equal to these others (M & N), or equimultiples of them.

Hypothesis.

- I. $\begin{cases} aM \text{ \& } aN \text{ are two wholes} \\ cM \text{ \& } cN \text{ their parts taken away} \\ eM \text{ \& } eN \text{ the remainders} \end{cases}$

- II. $\begin{cases} aM & cM \\ \text{ \& also \& } & \\ aN & cN \end{cases}$ are equimultiples of $\begin{cases} M \\ \text{ \& } \\ N \end{cases}$

Thesis.

- I. If $eM = M$, eN will be $= N$.
 II. If eM be multiple of M , eN will be equimultiple of N .

CASE I. If eM be $= M$.

Preparation.

Let $1N = N$.

Pof. 2. B. 5.

DEMONSTRATION.

BECAUSE tM is the same multiple of M , that cN is of N (Hyp. 2.), & that $eM = M$. (Sup. 1.), & $1N = N$ (Prep.),

1. The mgn. $tM + eM$, or aM , will be the same multiple of M that $cN + 1N$ is of N .

But aM being the same multiple of M , that aN or $eN + cN$ is of N (Hyp. 2.)

2. The two mgns. $cN + 1N$ & $eN + cN$ are equimultiples of the same mgn. N .

3. Wherefore the mgn. $cN + 1N = eN + cN$.

Ax. 6. B. 1.

Taking away the common mgn. cN ,

4. It follows that $1N$ is $= eN$.

Ax. 3. B. 1.

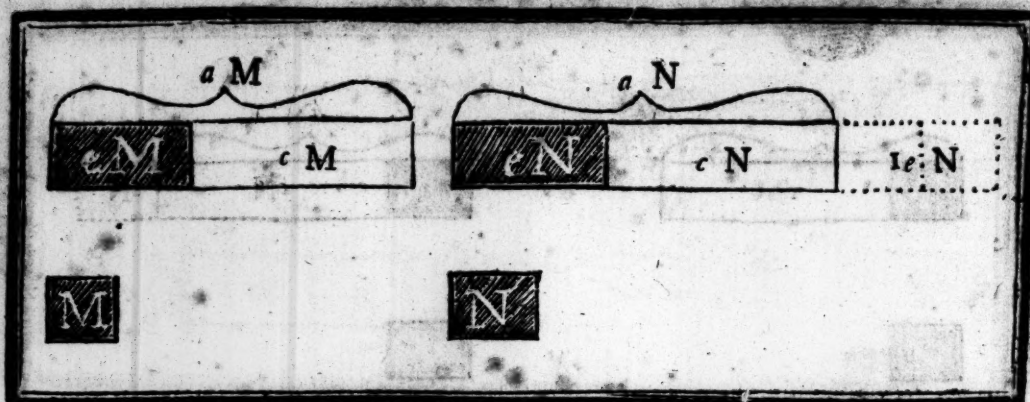
But $1N$ is $= N$ (Prep.);

5. Consequently, eN is $= N$.

Ax. 1. B. 1.

6. Therefore if eM be $= M$, eN is $= N$.

Which was to be demonstrated. 1.



CASE II. If $e M$ be multiple of M .

Preparation.

Take $1 e N$ the same multiple of N , that $e M$ is of M .

Pos. 1. B. 5.

DEMONSTRATION.

BECAUSE $e M$ is the same multiple of M , that $1 e N$ is of N (*Prep.*), & that $c M$ is the same multiple of M , that $c N$ is of N (*Hyp. 2*).

1. The magnitude $e M + c M$ or $a M$, will be the same multiple of M , that $1 e N + c N$ is of N .

P. 2. B. 5.

But $a M$ being the same multiple of M that $a N$ or $e N + c N$ is of N (*Hyp. 2*).

2. Therefore, the two mgn. $1 e N + c N$ & $e N + c N$ are equimultiples of the same mgn. N .

Ax. 6. B. 1.

3. Consequently, $1 e N + c N$ is $= e N + c N$
Taking away the common mgn. $c N$

4. It follows that the mgn. $1 e N$ is $= e N$

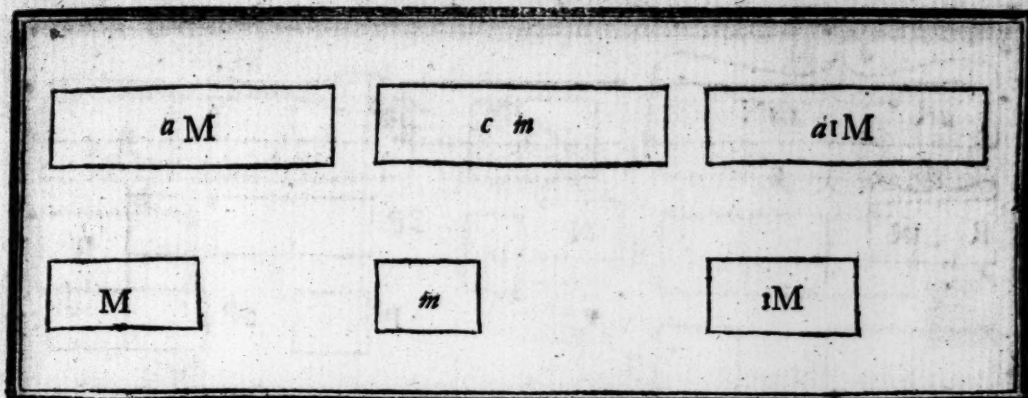
Ax. 3. B. 1.

But $1 e N$ is the same multiple of N that $e M$ is of M (*Prep.*).

5. Therefore, if $e M$ be an equimultiple of M , $e N$ will be an equimultiple of N

Which was to be demonstrated. *11.*





PROPOSITION VII. THEOREM VII.

EQUAL magnitudes (M & IM), have the same ratio to the same magnitude (m), and the same (m); has the same ratio to equal magnitudes (M & IM).

Hypothesis.

M & IM are two equal mgn's;
& m is a third.

Thesis.

I. $M : m = IM : m$
II. $m : M = m : IM$.

Preparation.

1. Take of M & of IM any equimultiples aM & aIM .
2. And of m any multiple whatever cm .

} *Pos. 1. B. 5*

DEMONSTRATION.

BECAUSE aM & aIM are equimultiples of M & of IM (*Prep. 1*), & $M = IM$ (*Hyp.*).

1. The mgn. aM is $= aIM$.
2. Therefore, if aM be $>$, $=$, or $< cm$; aIM will likewise be $>$, $=$, or $< cm$.

Ax. 6. B. 1.

But aM & aIM are equimultiples of the I term M . and of the III term IM , as cm and cm are of the II term m and of the IV term m ,

3. Consequently $M : m = IM : m$.

D. 5. B. 5.

Which was to be demonstrated. 1.

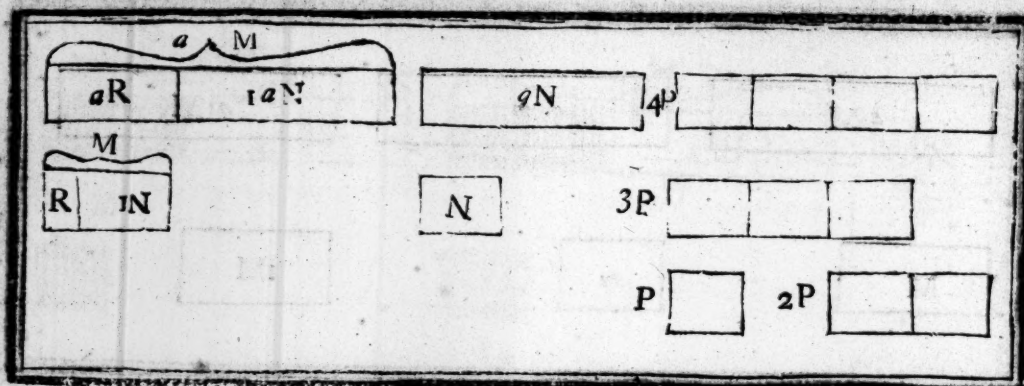
And because $aM = aIM$ (*Arg. 1*);

4. It also follows that, if cm be $>$, $=$, or $< aM$, likewise cm will be $>$, $=$, or $< aIM$.
5. Therefore $m : M = m : IM$.

D. 5. B. 5.

Which was to be demonstrated. 11.

Z



PROPOSITION VIII. THEOREM VIII.
OF unequal magnitudes (M & N), the greater (M) has a greater ratio to the same (P), than the less (N) has; and the same magnitude (P) has a greater ratio to the less (N), than it has to the greater (M).

Hypothesis.

I. $M > N$.

II. P is any magnitude.

Thesis.

I. $M : P > N : P$

II. $P : N > P : M$

I. Preparation.

1. Take from the greater M a part $1N$ = to the less N, and the remainder R will be either $<$, or $>$ or infine $= N$; Suppose first this remainder to be $< N$.
2. Take aR a multiple of this remainder $> P$;
3. Take $1aN$ & aN the same mult. of $1N$ & N that aR is of R. *Pos. 1. B. 1.*
4. Take the mgn. $2P$ double of P; the mgn. $3P$ triple of P and so on until the multiple of P be that which first becomes greater than aN , and let $4P$ be that multiple.

DEMONSTRATION.

BECAUSE $4P$ is the multiple of P which first becomes $> aN$ (*Prep. 4.*),

1. The next preceding mult. $3P$ is not $> aN$, or aN is not $< 3P$.
 Moreover aR and $1aN$ being equimultiples of R & of $1N$ (*Prep. 3.*),

2. The mgn. $aR + 1aN$, or aM is the same multiple of $R + 1N$ or M, that aR is of R.

P. 1. B. 5.

Or that aN is of N (*Prep. 3.*).

3. Therefore aM and aN are equimultiples of M and of N.

Moreover, aN and $1aN$ being equimultiples of the = mgn. N and $1N$ (*Prep. 3 & 1.*).

4. The mgn. aN is $= 1aN$

Ax. 6. B. 1.

But aN is not $< 3P$ (*Arg. 1.*).

5. Consequently, $1aN$ is not $< 3P$

But aR is $> P$ (*Prep. 2.*).

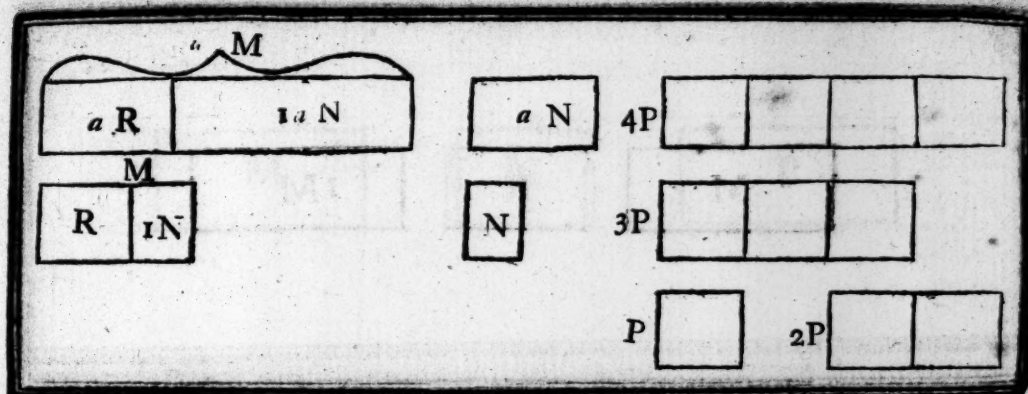
6. Therefore, by adding, $aR + 1aN$ or $aM > 4P$.

Since then aM is $> 4P$, and $aN < 4P$ (*Prep. 4.*), and aM , aN are equimultiples of the antecedents M and N and $4P$, $4P$ equimultiples of the consequents P and P (*Arg. 3 & Prep. 4.*).

It follows that $M : P > N : P$

D. 7. B. 5.

Which was to be demonstrated. 1.



- Moreover, since aN is supposed $< 4P$ (*Prep. 4*), & $aM > 4P$ (*Arg. 6*).
 8. It is evident that the mgn. $4P$ is $> aN$, & the same mgn. $4P < aM$.
 But $4P$ and $4P$ being equimultiples of the antecedents P and P ,
 and aN , aM equimultiples of the consequents N and M ,
 9. It follows that $P : N > P : M$.

D. 7. B. 5.

Which was to be demonstrated. 11.

II. Preparation.

If R be supposed $> 1N$ or N .

5. Take $1aN$ a multiple of $1N > P$.
6. Take aR & aN the same multiples of R & of N that $1aN$ is of $1N$. *Pos. 1. B. 5.*
7. Let $4P$ be the first multiple of $P > aR$; consequently the next preceding multiple $3P$ will not be $> aR$, or aR will not be $< 3P$.

DEMONSTRATION.

- IT may be proved as before (*Arg. 1. 2 & 3*), that
 1. The mgns. aM and aN are equimultiples of the mgns. M & N .
 Moreover, aR & aN being equimultiples of R & of N (*Prep. 6*),
 and R being $> N$ (*Sup.*),
 2. It follows that aR is $> aN$
 But aR not being $< 3P$ (*Prep. 7*),
 And the mgn. $1aN$ being $> P$ (*Prep. 5*),
 3. Then by adding, $aR + 1aN$, or $aM > 4P$.
 But aR being $< 4P$ (*Prep. 7*), & this same aR being $> aN$ (*Arg. 2*),
 4. Much more then aN is $< 4P$.
 But aM & aN are equimultiples of the antecedents M & N (*Arg. 1*)
 and $4P$, $4P$ equimultiples of the consequents P & P , & moreover
 $aM > 4P$ & $aN < 4P$ (*Arg. 3 & 4*).
 5. Consequently $M : P > N : P$.

D. 7. B. 5.

Which was to be demonstrated. 1.

Moreover, without changing the Preparation, it may be demonstrated
 as in the precedent case (*Arg. 8 & 9*), that

6. The ratio of $P : N$ is $>$ the ratio of $P : M$.

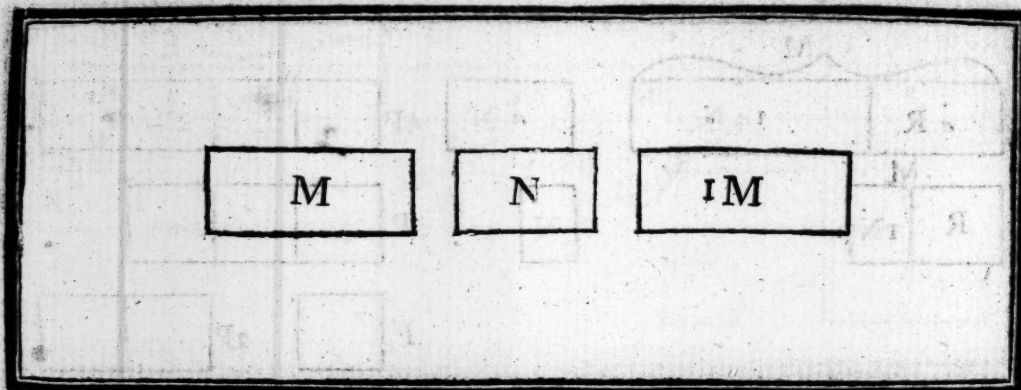
Which was to be demonstrated. 11.

III.

And applying the same preparation and same reasoning to the last
 case when $R = 1N$,

7. The demonstration will be completed as in the two precedent cases.

Which was to be demonstrated. 1 & 11.



PROPOSITION IX. THEOREM IX.

MAGNITUDES (M & $1M$) which have the same ratio to the same magnitude (N) : are equal to one another. And those (M & $1M$) to which the same magnitude (N) has the same ratio, are equal to one another.

Hypothesis.

$$M : N = 1M : N.$$

Thesis.

$$\text{The mgn. } M = 1M.$$

DEMONSTRATION.

I.

If not, the two mgns. M & $1M$ are unequal.

1. **T**HEN the two mgns. M & $1M$ have not the same ratio to the same mgn. N

P. 8. B. 5.

But they have the same ratio to this same mgn. N (*Hyp.*) ;

2. Therefore the mgn. M is $=$ to the mgn. $1M$.

Hypothesis.

$$N : M = N : 1M.$$

Thesis.

$$\text{The mgn. } M = 1M.$$

DEMONSTRATION.

II.

If not, the two mgns. M & $1M$ are unequal.

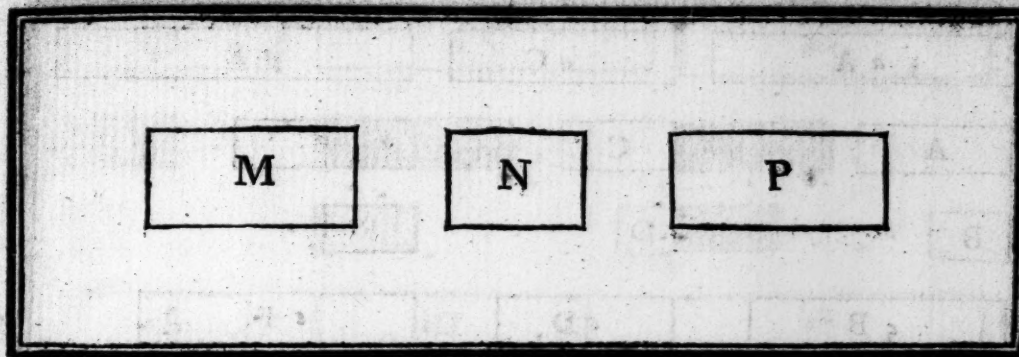
1. **T**HEN the same mgn. N has not the same ratio to the two mgns. M & $1M$.

P. 8. B. 5.

But it has the same ratio to those two mgns. (*Hyp.*)

2. Therefore the mgn. M is $=$ to the mgn. $1M$.

Which was to be demonstrated.



PROPOSITION X. THEOREM X.

THAT magnitude (M) which has a greater ratio than another (P) has unto the same magnitude (N) is the greater of the two, and that magnitude (P) to which the same (N) has a greater ratio than it has unto another magnitude (M) is the lesser of the two.

Hypothesis.

 $M : N$ is $> P : N$.

Thesis.

The mgn. M is $> P$.

DEMONSTRATION.

I

If not; M is $= P$, or $< P$.CASE I. If M be $= P$.

1. **T**HEN the mgns. M & P have the same ratio to the same mgn. N. *P. 7. B. 5;*
But they have not the same ratio to the same mgn. N (*Hyp.*);
2. Therefore the mgn. M is not $=$ to the mgn. P.

CASE II. If M be $< P$.

3. **T**HE ratio M : N would be $<$ the ratio P : N (*Hyp.*); *P. 8. B. 5;*
But the ratio M : N is not $<$ the ratio P : N (*Hyp.*);
4. Therefore the mgn. M is not $<$ the mgn. P.
But neither is the mgn. M $= P$ (*Arg. 2.*);
5. It remains then that M be $> P$.

Hypothesis.

 $N : P > N : M$.

Thesis.

The mgn. P is $< M$.

DEMONSTRATION.

II.

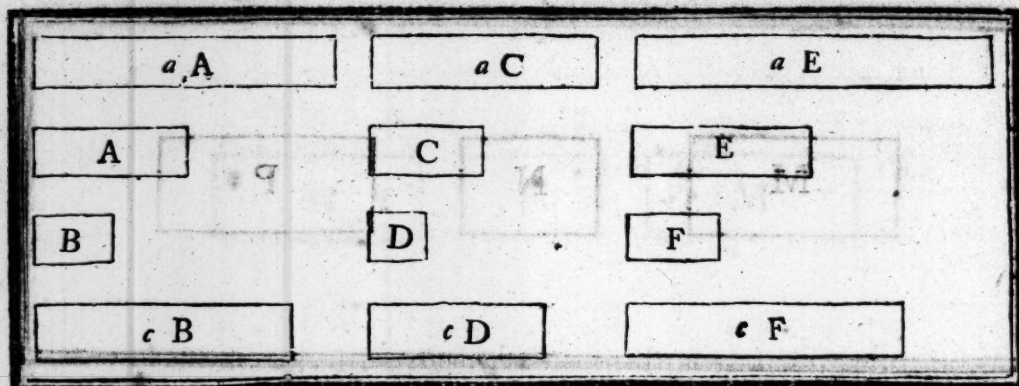
If not, P is $=$ or $> M$.CASE I. If P be $= M$.

1. **T**HE ratio N : M would be $=$ to the ratio of N : P. *P. 7. B. 5.*
2. Which being contrary to the Hypothesis, P cannot be $= M$.

CASE II. If P be $> M$.

3. **T**HE ratio N : M would be $>$ the ratio N : P. *P. 8. B. 5.*
4. Which being also contrary to the Hypothesis, P cannot be $> M$.
But neither is P $= M$. (*Arg. 2.*);
5. Therefore P is $< M$.

Which was to be demonstrated.



PROPOSITION XI. THEOREM XI.

RATIOS ($A : B$ & $E : F$) that are equal to a same third ratio ($C : D$), are equal to one another.

Hypothesis.

Thesis.

The ratios $\left\{ \begin{array}{l} A : B \\ \& \\ E : F \end{array} \right.$ are $=$ to the same ratio $C : D$.

 $A : B = E : F$.

Preparation.

1. Take any equimultiples $a A$, $a C$, $a E$ of the three antecedents A , C , E .
2. And any equimultiples $c B$, $c D$, $c F$ of the three consequents B , D , F .

Pos. 1. B. 5.

DEMONSTRATION.

BECAUSE $A : B = C : D$ (Hyp.),

1. If the multiple $a A$ be $>$, $=$ or $<$ the multiple $c B$, the equimultiple $a C$ is likewise $>$, $=$ or $<$ the equimultiple $c D$.

D. 5. B. 5.

In like manner since $C : D = E : F$ (Hyp.)

2. If the multiple $a C$ be $>$, $=$ or $<$ the multiple $c D$, the equimultiple $a E$ will be likewise $>$, $=$ or $<$ the equimultiple $c F$.

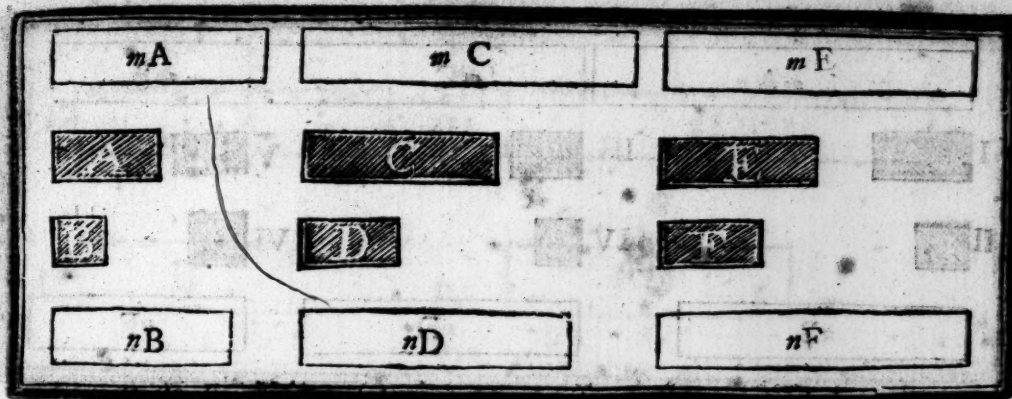
3. Consequently if the multiple $a A$ be $>$, $=$ or $<$, the multiple $c B$; the equimultiple $a E$ is likewise $>$, $=$ or $<$ the equimultiple $c F$.

4. Consequently, $A : B = E : F$.

D. 5. B. 5.

Which was to be demonstrated.





PROPOSITION XII. THEOREM XII.

IF any number of magnitudes (A, B, C, D, E, F , &c) be proportionals. The sum of all the antecedents ($A + C + E$ &c) is to the sum of all the consequents ($B + D + F$ &c), as one of the antecedents is to its consequent.

Hypothesis.

Thesis.

The mgs. A, B, C, D, E, F are proportionals
or $A : B = C : D = E : F$ &c.

$$A + C + E : B + D + F = A : B.$$

Preparation.

1. Take of the antecedents A, C, E the equimultiples $m A, m C, m E$
 2. And of the consequents B, D, F the equimultiples $n B, n D, n F$
- } *Pos. 1. B. 5.*

DEMONSTRATION.

SINCE then $A : B = C : D = E : F$ (*Hyp.*);

1. If $m A$ be $>$, $=$ or $<$ $n B$, like wise $m C$ is $>$, $=$ or $<$ $n D$; & $m E$ is $>$, $=$ or $<$ $n F$

D. 5. B. 5.

Therefore adding on both sides the mgs. $>$, $=$, or $<$.

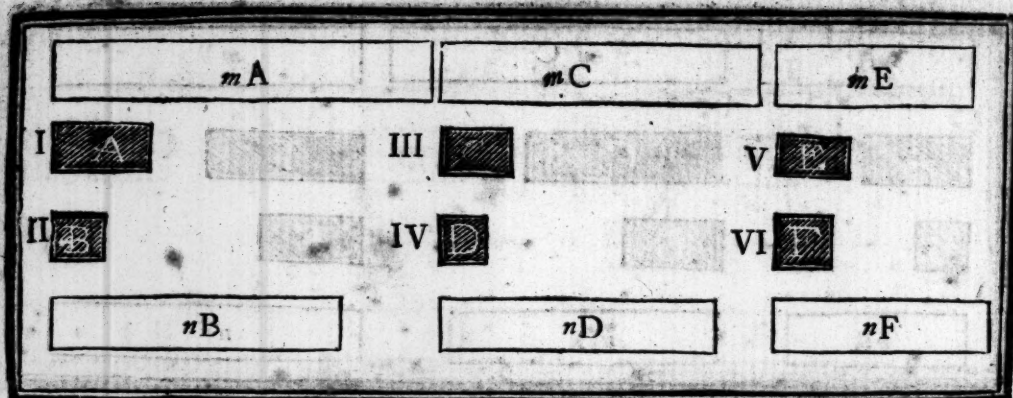
2. The mgs. $m A + m C + m E$ will be constantly $>$, $=$, or $<$ the mgs. $n B + n D + n F$ according as $m A$ is $>$, $=$, or $<$ $n B$. But the mgs. $m A + m C + m E$ & $m A$ are equimultiples of the mgs. $A + C + E$ & A (*Prep. 1 & P. 1. B. 5.*); also the mgs. $n B + n D + n F$ & $n B$ are equimultiples of the mgs. $B + D + F$ & B (*Prep. 2 & P. 1. B. 5.*);

3. Consequently $A + C + E : B + D + F = A : B$

D. 5. B. 5.

Which was to be demonstrated.





PROPOSITION XIII. THEOREM XIII.

IF the first magnitude (A) has to the second (B), the same ratio, which the third (C) has to the fourth (D); but the third (C) to the fourth (D) a greater ratio than the fifth (E) to the sixth (F): the first (A) shall have to the second (B) a greater ratio than the fifth (E) has to the sixth (F).

Hypothesis.

- I. $A : B = C : D$.
 II. $C : D > E : F$.

Thesis.

$$A : B > E : F.$$

Preparation.

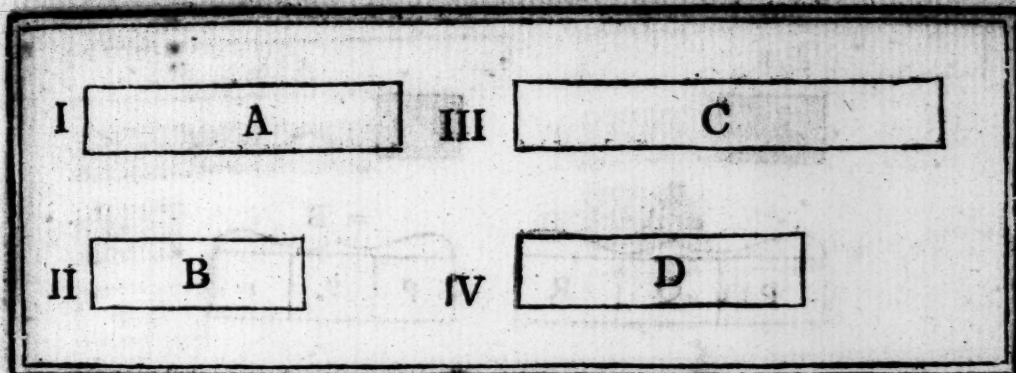
1. The ratio of $C : D$ being $>$ the ratio of $E : F$ (*Hyp. 2*) there may be taken of the antecedents C & E , the equimult. mC & mE ; and likewise of the consequents D & F the equimult. nD & nF , such, that mC is $>$ nD , but mE is not $>$ nF ; } *Pos. 1. B. 5.*
} *D. 7. B. 5.*
2. Take mA the same multiple of A that mC is of C ,
3. And nB the same multiple of B that nD is of D . } *Pos. 1. B. 5.*

DEMONSTRATION.

SINCE then $A : B = C : D$ (*Hyp. 1.*), and that mA , mC are equimultiples of the antecedents, & nB , nD equimultiples of the consequents (*Prep. 2 & 3*).

1. The mgn. mA will be $>$, $=$ or $<$ nB ; according as mC is $>$, $=$ or $<$ nD . } *D. 5. B. 5.*
2. Therefore mA is also $>$ nB .
 But mE is not $>$ nF (*Prep. 1*), & the mgn. mA & mE are equimultiples of the antecedents A & E , & nB , nF equimultiples of the consequents B & F (*Prep. 1 & 2*).
3. Consequently the ratio $A : B$, is $>$ than the ratio $E : F$. } *D. 7. B. 5.*

Which was to be demonstrated.



PROPOSITION XIV. THEOREM XIV.

IF four magnitudes (A, B, C, D) be proportionals, then if the first (A) be greater, equal, or less, than the third (C), the second (B) shall be greater, equal, or less, than the fourth (D).

Hypothesis.

- I. $A : B = C : D$
- II. A is $>$, $=$ or $<$ C.

Thesis.

According as A is $>$, $=$ or $<$ C.
B will be $>$, $=$ or $<$ D.

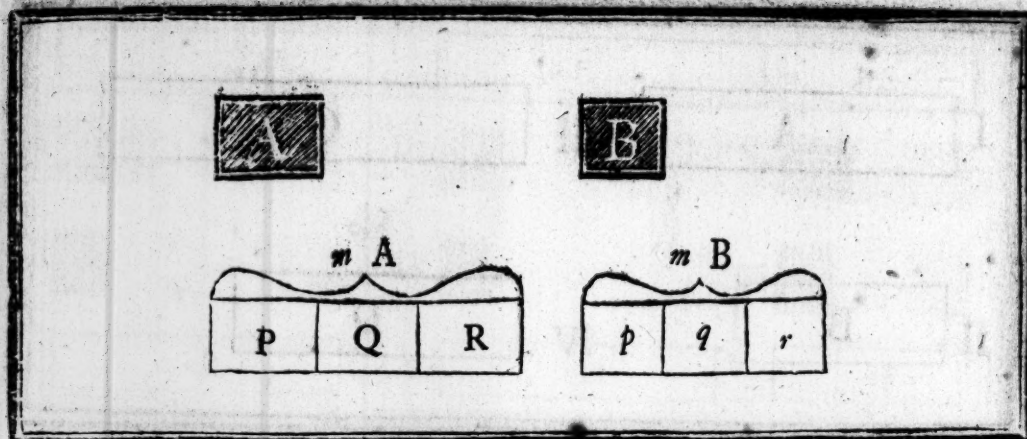
CASE I. If A be $>$ C.

DEMONSTRATION.

1. **T**HEN the ratio of A : B is $>$ the ratio C : B. P. 8. B. 5.
But $A : B = C : D$ (Hyp. 1).
2. Therefore the ratio of C : D is $>$ the ratio C : B. P. 13. B. 5.
3. From whence it follows, that D is $<$ B or B $>$ D. P. 10. B. 5.
It may be demonstrated after the same manner, if $A = C$, that B will be $=$ D; & if A be $<$ C, that B will be $<$ D.
4. Consequently, according as A is $>$, $=$ or $<$ C, B will be $>$, $=$ or $<$ D.

Which was to be demonstrated.





PROPOSITION XV. THEOREM XV.

MAGNITUDES (A & B) have the same ratio to one another which their equimultiples ($m A$ & $m B$) have.

Hypothesis.

The mgn. $m A$ & $m B$ are equimult.
of the mgn. A & B.

Thesis.

$$A : B = m A : m B.$$

Preparation.

1. Divide $m A$ into its parts P, Q, R each = A.
2. And $m B$ into its parts p, q, r each = B.

} *Pos. 2. B. 5.*

DEMONSTRATION.

BECAUSE the mgn. $m A$, $m B$ are equimultiples of the mgn. A & B (*Hyp.*).

1. The number of parts P, Q, R &c. is = to the number of parts p, q, r &c.

And P being = Q = R (*Prep. 1*), & $p = q = r$ (*Prep. 2*),

2. The mgn. $P : p = Q : q = R : r$ &c.

3. Wherefore $P + Q + R$, or $m A : p + q + r$ or $m B = P : p$.
But since $P = A$ & $p = B$ (*Prep. 1 & 2*),

4. The mgn. $P : p = A : B$.

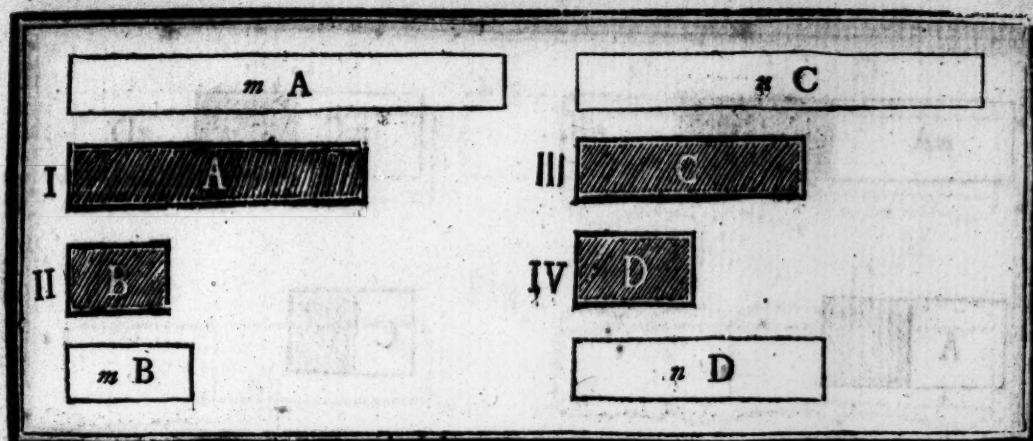
5. Consequently $A : B = m A : m B$.

{ *P. 7. B. 5.*
P. 11. B. 5.
P. 12. B. 5.

P. 7. B. 5.
P. 11. B. 5.

Which was to be demonstrated.





PROPOSITION XVI. THEOREM XVI.

IF four magnitudes (A, B, C, D) of the same kind be proportionals, they shall also be proportionals when taken alternately.

Hypothesis.

$$A : B = C : D.$$

Thesis.

$$A : C = B : D.$$

Preparation.

1. Take of the terms A & B of the first ratio, any equimult. $m A$, & $m B$.
 2. Take of the terms C & D of the second ratio any equimult. $n C$, & $n D$.
- } *Pos. 1. B. 5.*

DEMONSTRATION.

BECAUSE $m A$ & $m B$ are equimult. of the mgn. A & B (*Prep. 1*),

1. Then $A : B = m A : m B$. *P. 15. B. 5.*
 - But $A : B = C : D$ (*Hyp.*).
 2. Therefore $C : D = m A : m B$. *P. 11. B. 5.*
 3. Likewise $C : D = n C : n D$. *P. 15. B. 5.*
 4. Consequently $m A : m B = n C : n D$. *P. 11. B. 5.*
 5. Wherefore, if $m A$ be $>$, $=$ or $<$ $n C$, $m B$ will be $>$, $=$ or $<$ $n D$. *P. 14. B. 5.*
- But $m A$ & $m B$ being equimult. of the terms A & B considered as antecedents (*Prep. 1*), & $n C$, $n D$ equimult. of the terms C & D considered as consequents (*Prep. 2*),

6. Consequently $A : C = B : D$. *D. 5. B. 5.*

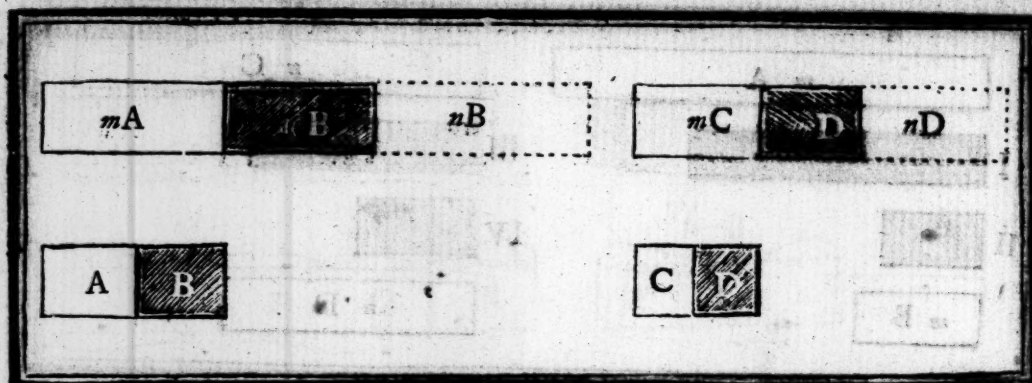
Which was to be demonstrated.

COROLLARY.

IT follows from this proposition that if four mgn. are proportionals, according as the first is greater, equal or less than the second, the third is likewise greater, equal, or less than the fourth.

For since $A : B = C : D$ (*Hyp.*),

1. Then $A : C = B : D$. *P. 16. B. 5.*
2. Therefore, according as A is $>$, $=$ or $<$ B, C will be likewise $>$, $=$ or $<$ D. *P. 14. B. 5.*



PROPOSITION XVII. THEOREM XVII.

IF two magnitudes together ($A + B$) have to one of them (B), the same ratio which two others ($C + D$) have to one of these (D), the remaining one (A) of the first two ($A + B$) shall have to the other (B), the same ratio which the remaining one (C) of the last two ($C + D$) has to the other of these (D).

Hypothesis.

$$A + B : B = C + D : D$$

Thesis.

$$A : B = C : D.$$

Preparation.

1. Take of the mgn. A, B, C, D any equimult. mA, mB, mC, mD .

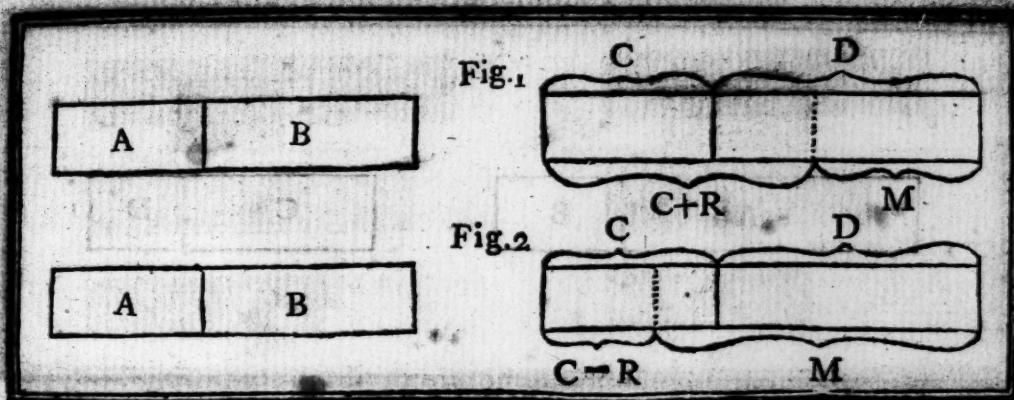
2. And of the mgn. B & D any equimult. nB, nD .

Pos. 1. B. 5.

DEMONSTRATION.

1. **T**HEN the whole mgn. $mA + mB$ will be the same mult. of the mgn. $A + B$, that mA is of A , or mC of C . *P. 1. B. 5.*
2. In like manner, the whole mgn. $mC + mD$ is the same mult. of the mgn. $C + D$, that mC is of C . *P. 1. B. 5.*
3. Consequently, $mA + mB$ is the same mult. of $A + B$, that $mC + mD$ is of $C + D$.
4. Also the mgn. $mB + nB, mD + nD$ are equimult. of the mgn. B & D . But $A + B : B = C + D : D$ (*Hyp.*), & $mA + mB, mC + mD$ are equimult. of the antecedents $A + B$ & $C + D$ (*Arg. 3*); also $mB + nB, mD + nD$ are equimult. of the consequents B & D (*Arg. 4*).
5. Consequently, if $mA + mB$ be $>$, $=$ or $<$ $mB + nB, mC + mD$ is also $>$, $=$ or $<$ $mD + nD$. *D. 5. B. 5.*
But if $mA + mB$ be $>$, $=$ or $<$ $mB + nB$; taking away the common part mB .
6. The remainder mA will be $>$, $=$ or $<$ the remainder nB . In like manner, if $mC + mD$ be $>$, $=$ or $<$ $mD + nD$; taking away the common part mD .
7. The remainder mC will be $>$, $=$ or $<$ the remainder nD .
8. Wherefore, if mA be $>$, $=$, or $<$ nB ; mC will be likewise $>$, $=$ or $<$ nD .
But mA & mC are equimult. of A & of C considered as antecedents (*Prep. 1*); & nB, nD equimult. of B & D considered as consequents (*Prep. 2*).
9. Consequently, $A : B = C : D$. *D. 5. B. 5.*

Which was to be demonstrated.



PROPOSITION XVIII. THEOREM XVIII.

IF four magnitudes (A, B, C, D) be proportionals, the first and second together ($A+B$) shall be to the second (B) as the third and fourth together ($C+D$) to the fourth (D).

Hypothesis.

$$A : B = C : D.$$

Thesis.

$$A + B : B = C + D : D.$$

DEMONSTRATION.

If not, $A+B : B = C+D : \text{another mgn, } M < \text{ or } > D$.

CASE I. Let $M < D$, or $M + R = D$ (Fig. 1).

SINCE then $A+B : B = C+D : M$, or $A+B : B = C+M+R : M$

1. Dividendo $A : B = C+R : M$ P. 17. B. 5.

But $A : B = C : D$ (Hyp.);

2. Hence, $C+R : M = C : D$ P. 11. B. 5.

But $C+R$ is $> C$ (Ax. 8. B. 1);

3. Therefore M is $> D$, & the supposition of $M < D$, is impossible P. 14. B. 5.

CASE II. Let $M > D$, or $M = D + R$ (Fig. 2).

BECAUSE $A+B : B = C+D : M$, or $A+B : B = C+D : D+R$

4. Dividendo $A : B = C-R : D+R$ P. 17. B. 5.

But $A : B = C : D$ (Hyp.);

5. Hence, $C-R : M = C : D$ P. 11. B. 5.

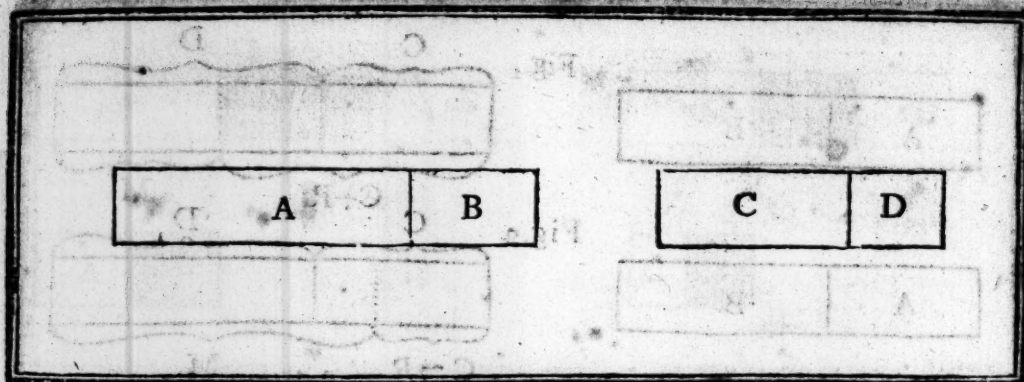
But $C-R$ is $< C$ (Ax. 8. B. 1);

6. Therefore M is $< D$, & the supposition of $M > D$, is impossible. P. 14. B. 5.

Since then M is neither $< D$ (Arg. 3) nor $> D$ (Arg. 7),

7. It follows that $M = D$ & $A+B : B = C+D : D$.

Which was to be demonstrated.



PROPOSITION XIX. THEOREM XIX.

IF a whole magnitude $(A+B)$ be to a whole $(C+D)$, as a magnitude (A) taken from the first is to a magnitude (C) taken from the other, the remainder (B) shall be to the remainder (D) , as the whole $(A+B)$ is to the whole $(C+D)$.

Hypothesis.
 $A + B : C + D = A : C$

Thesis.
 $B : D = A + B : C + D$

DEMONSTRATION.

BECAUSE

- | | | | |
|-------------------------|-------------------|--------------------------|---------------------|
| | $A + B : C + D =$ | $A : C.$ (<i>Hyp.</i>) | |
| 1. Therefore Alternando | $A + B :$ | $A = C + D : C.$ | <i>P. 16. B. 5.</i> |
| 2. Then Dividendo | $B :$ | $A = D : C.$ | <i>P. 17. B. 5.</i> |
| 3. Alternando again | $B :$ | $D = A : C.$ | <i>P. 16. B. 5.</i> |
| But since | $A + B : C + D =$ | $A : C.$ (<i>Hyp.</i>) | |
| 4. It follows that | $B :$ | $D = A + B : C + D.$ | <i>P. 11. B. 5.</i> |

Which was to be demonstrated.

COROLLARY.

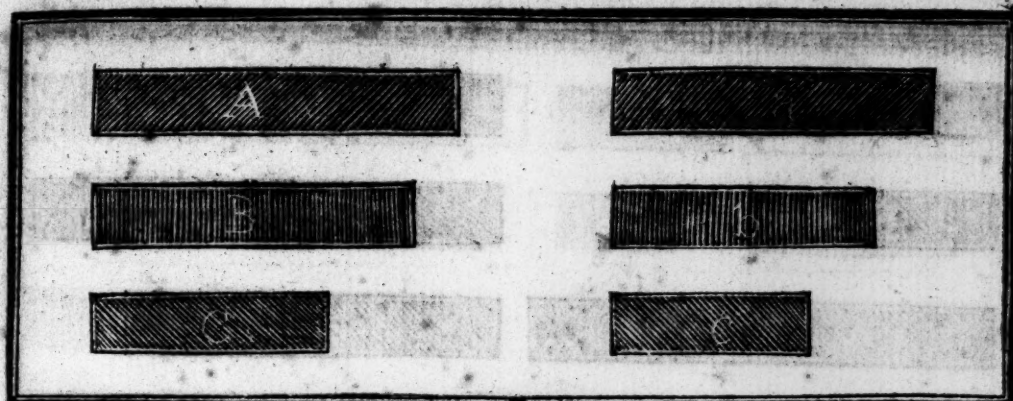
IF magnitudes taken jointly be proportionals, that is if $A + B : A = C + D : C$, it may be inferred by conversion that $A + B : B = C + D : D$ (*D. 17. B. 5.*).

For $A + B : C + D = A : C$ (*Hyp. & P. 16.*).

Wherefore $A + B : B + D = B : D$ (*P. 19.*).

Consequently $A + B : B = C + D : D$ (*P. 16.*).





PROPOSITION XX. THEOREM XX.

IF there be three magnitudes (A, B, C) and other three (*a*, *b*, *c*) which taken two and two in a direct order, have the same ratio; if the first (A) be greater than the third (C), the fourth (*a*) shall be greater than the sixth (*c*) and if equal, equal, and if less, less.

Hypothesis.

- I. $A : B = a : b$
 II. $B : C = b : c$

Thesis.

According as A is $>$, $=$ or $<$ C.
a is also $>$, $=$ or $<$ *c*.

DEMONSTRATION.

CASE I. Let A be $>$ C.

BECAUSE A is $>$ C.

1. The ratio $A : B$ is $>$ $C : B$.

But $A : B = a : b$ (Hyp. 1).

And $C : B = c : b$ (Hyp. 2 & P. 4 Cor. B. 5).

P. 8. B. 5.

2. Therefore, the ratio $a : b$ is $>$ $c : b$.

P. 13. B. 5.

3. Consequently, *a* is also $>$ *c*.

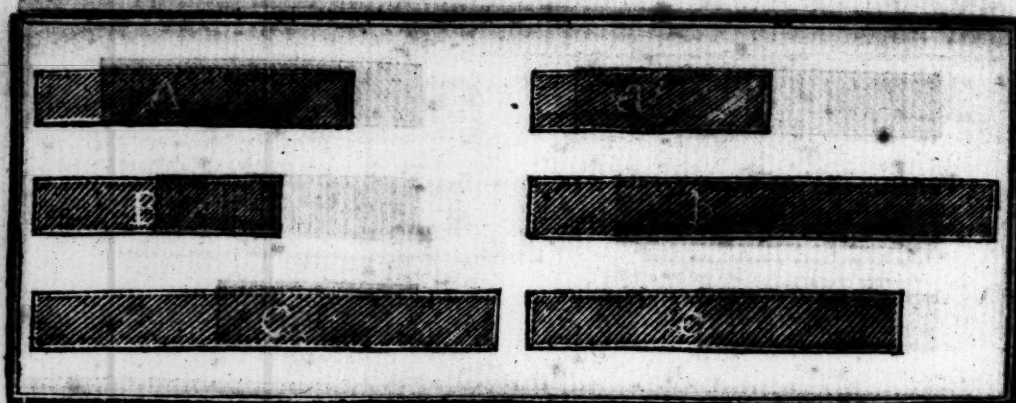
P. 10. B. 5.

4. It may be proved after the same manner, that if A be $=$ C, *a* shall be $=$ *c*, & if A be $<$ C, *a* shall be $<$ *c*.

5. Consequently, according as A is $>$, $=$ or $<$ C, *a* will be also $>$, $=$ or $<$ *c*.

Which was to be demonstrated.





PROPOSITION XXI. THEOREM XXI.

IF there be three magnitudes (A, B, C), and other three (a, b, c), which have the same ratio taken two and two, but in a cross order; if the first magnitude (A) be greater than the third (C), the fourth (a) shall be greater than the sixth (c), and if equal, equal; and if less, less.

Hypothesis.

- I. $A : B = b : c$
 II. $B : C = a : b$

Thesis.

According as A is $>$, $=$ or $<$ C.
 a is also $>$, $=$ or $<$ c.

CASE I. Let A be $>$ C.

DEMONSTRATION.

BECAUSEA is $>$ C

1. The ratio of $A : B > C : B$

P. 8. B. 5.

But $A : B = b : c$ (Hyp. 1).

& invertendo $C : B = b : a$ (Hyp. 2. & P. 4. Cor. B. 5.).

2. Consequently the ratio $b : c > b : a$

P. 13. B. 5.

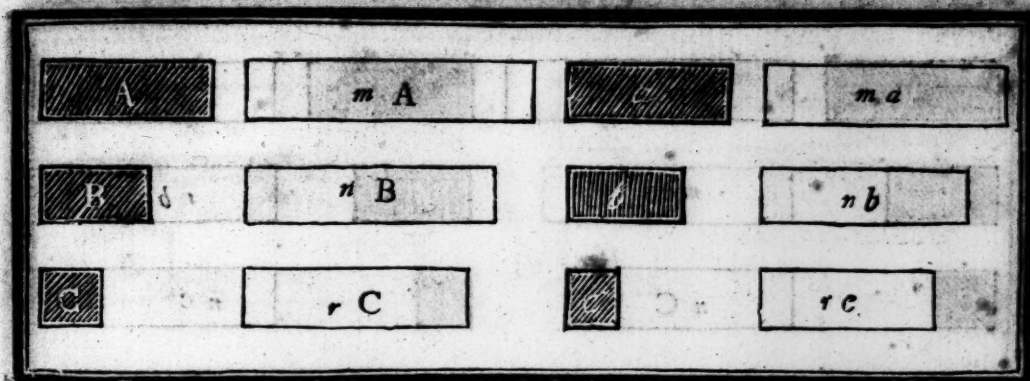
3. Therefore c is $<$ a, or $a >$ c

P. 10. B. 5.

4. It may be demonstrated after the same manner, if A be $=$ B, also a shall be $=$ c; and if A be $<$ C, a shall be $<$ c

5. Consequently, according as A is $>$, $=$ or $<$ C, a shall be $>$, $=$ or $<$ c.

Which was to be demonstrated.



PROPOSITION XXII. THEOREM XXII.

IF there be any number of magnitudes ($A, B, C, \&c.$) and as many others ($a, b, c, \&c.$), which taken two and two in order have the same ratio, the first shall have to the last of the first magnitudes, the same ratio which the first of the others has to the last, by equality of direct ratio, or *ex aequo ordinate*.

Hypothesis.

- I. $A : B = a : b$
 II. $B : C = b : c$

Thesis.

$$A : C = a : c.$$

Preparation.

1. Take of A & a any equimult. $m A$ & $m a$
2. And of B & b any equimult. $n B$ & $n b$
3. And of C & c any equimult. $r C$ & $r c$.

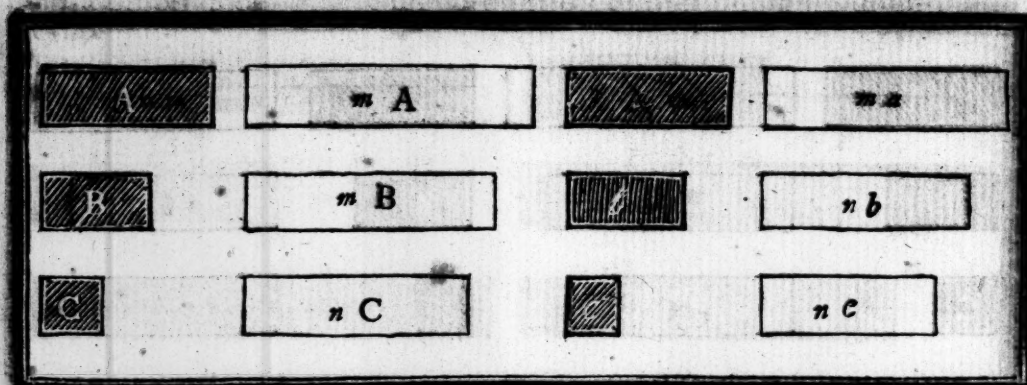
} *Pos. 1. B. 5.*

DEMONSTRATION.

- BECAUSE $A : B = a : b$ (*Hyp. 1*).
 1. It follows that $m A : n B = m a : n b$ *P. 4. B. 5.*
 And because $B : C = b : c$ (*Hyp. 2*).
 2. It follows that $n B : r C = n b : r c$ *P. 4. B. 5.*
 3. Therefore, $m A, n B, r C$ & $m a, n b, r c$ form two series of magnitudes which taken two by two in order have the same ratio.
 4. Wherefore, by equality of ratio, according as the first $m A$ of the first series is $>$, $=$ or $<$ the third $r C$, the first $m a$ of the other series will be $>$, $=$ or $<$ the third $r c$. *P. 20. B. 5.*
 5. Consequently, $A : C = a : c$. *D. 5. B. 5.*

Which was to be demonstrated.

B b



PROPOSITION XXIII. THEOREM XXIII.

IF there be any number of magnitudes ($A, B, C, \&c.$) and as many others ($a, b, c, \&c.$) which taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last, by equality of perturbate ratio or *ex æquo perturbate*.

Hypothesis.

- I. $A : B = b : c.$
- II. $B : C = a : b.$

Thesis.

$$A : C = a : c.$$

Preparation.

1. Take of A, B, a , any equimult. $m A, m B, m a.$
2. And of C, b, c , any equimult. $n C, n b, n c.$

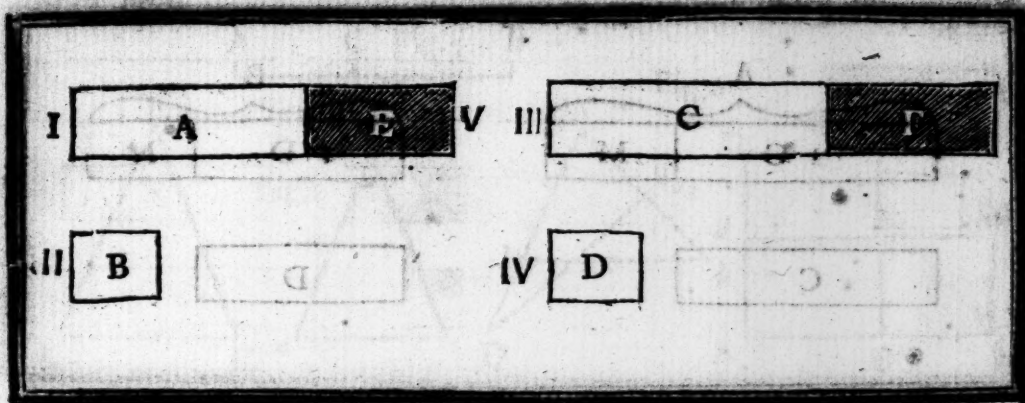
} *Pos. 1. B. 5.*

DEMONSTRATION.

BECAUSE $m A$ & $m B$ are equimult. of A & B (*Prep. 1*).

1. It follows that $A : B = m A : m B.$ *P. 15. B. 5.*
2. And $b : c = n b : n c.$
- But $A : B = b : c.$ (*Hyp. 1*).
3. Therefore, $m A : m B = n b : n c.$ *P. 11. B. 5.*
- And because $B : C = a : b.$ (*Hyp. 2*).
4. It follows that $m B : n C = m a : n b.$ *P. 4. B. 5.*
5. Wherefore, $m A, m B, n C$, & $m a, m b, n c$ form two series of mgn. which taken two and two in a cross order have the same ratio.
6. Consequently, by equality of ratio, according as the first $m A$ of the first series is $>$, $=$ or $<$ the third $n C$, the first $m a$ of the other series will be $>$, $=$ or $<$ the third $n c.$ *P. 21. B. 5.*
7. For which reason $A : C = a : c.$ *D. 5. B. 5.*

Which was to be demonstrated.



PROPOSITION XXIV. THEOREM XXIV.

IF four magnitudes (A, B, C, D) be proportionals and that a fifth (E) has to the second (B) the same ratio which a sixth (F) has to the fourth (D), the first and fifth together (A + E) shall have to the second (B), the same ratio which the third and sixth together (C + F) have to the fourth (D).

Hypothesis.

- I. $A : B = C : D$
 II. $E : B = F : D$

Thesis.

$$A + E : B = C + F : D$$

DEMONSTRATION.

BECAUSE

1. It follows invertendo

And because

2. Ex æquo ordinate

3. Componendo

But since

4. It follows,

$$\text{Ex æquo ordinate } A + E : B = C + F : D$$

$$E : B = F : D \text{ (Hyp. 2).}$$

$$B : E = D : F$$

$$A : B = C : D \text{ (Hyp. 1).}$$

$$A : E = C : F$$

$$A + E : E = C + F : F$$

$$E : B = F : D \text{ (Hyp. 2).}$$

{ P. 4. B. 5.
 Cor.

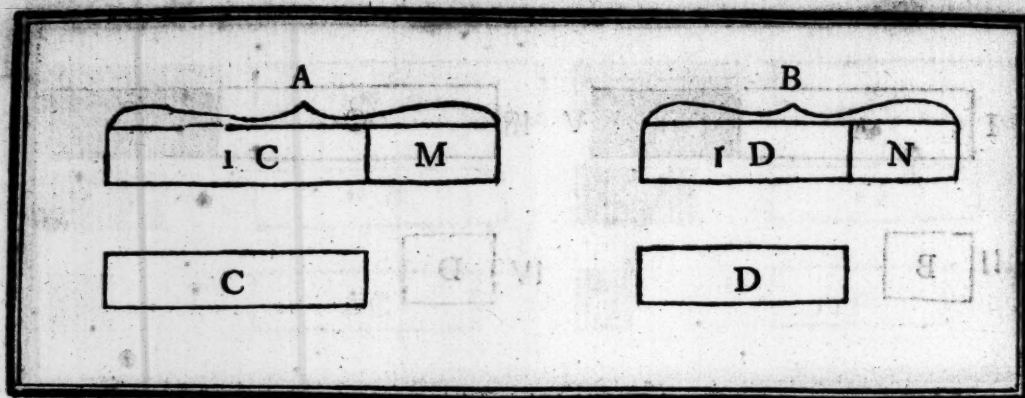
P. 22. B. 5.

P. 18. B. 3.

P. 22. B. 5.

Which was to be demonstrated.





PROPOSITION XXV. THEOREM XXV.

IF four magnitudes (A, B, C, D) are proportionals, the greatest (A) and least of them (D) together, are greater than the other two (B & C) together.

Hypothesis.

- I. $A : B = C : D$
 II. A is the greatest term, & Consequently (*)
 D the least.

Thesis.

$$A + D > B + C.$$

Preparation.

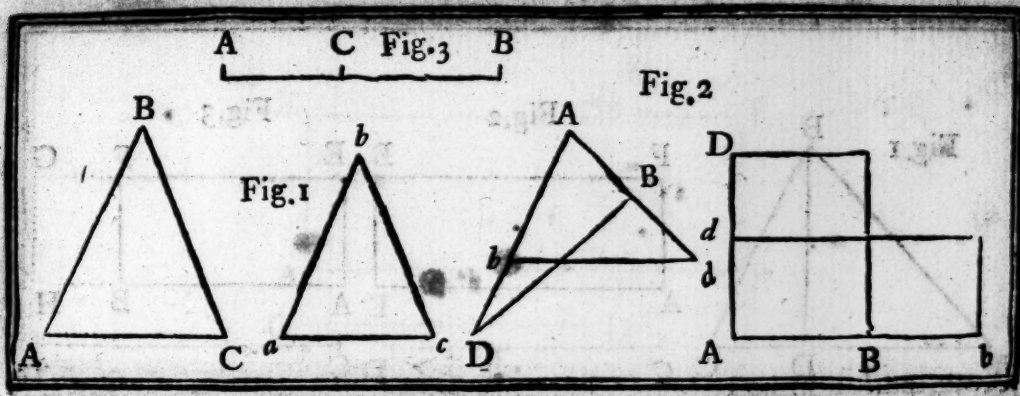
Take $1C = C$ & $1D = D$.

DEMONSTRATION.

- B**ECAUSE $A : B = C : D$ (Hyp. 1.) & $C = 1C$ & $D = 1D$ (Prep.).
1. It follows that $A : B = 1C : 1D$ P. 7. B. 5.
 2. Wherefore $A : B = M : N$ P. 19. B. 5.
 - But the mgn. A being $> B$ (Hyp. 2.).
 3. The mgn. M is also $> N$ { P. 16. B. 5.
 - Moreover, because $C = 1C$ & $D = 1D$ (Prep. 1 & 2). { Cor.
 4. It follows that $1C + D = 1D + C$
 - And since M is $> N$ (Arg. 3).
 5. It follows that $1C + D + M > 1D + C + N$, that is $A + D$ is $> B + C$. Ax. 4. B. 1.

Which was to be demonstrated.

(*) EUCLID supposes the consequence of this Hypothesis sufficiently evident from the foregoing truths; for since $A : B : C : D$ (Hyp. 1.), & $A > C$ (Hyp. 2.), B is $> D$ (P. 14. B. 5.). Likewise A being $> B$ (Hyp. 2.) C is $> D$ (P. 16. Cor. B. 5.), Consequently D is the least of the IV terms.



DEFINITIONS.

I.

SIMILAR rectilineal figures (Fig. 1.) are those (ABC, abc), which have their several Angles (A, B, C , and a, b, c) equal, each to each, and the sides (AB, AC, BC , and ab, ac, bc) about the equal angles, proportionals (that is $AB : AC = ab : ac$, also $AB : BC = ab : bc$, and $AC : BC = ac : bc$).

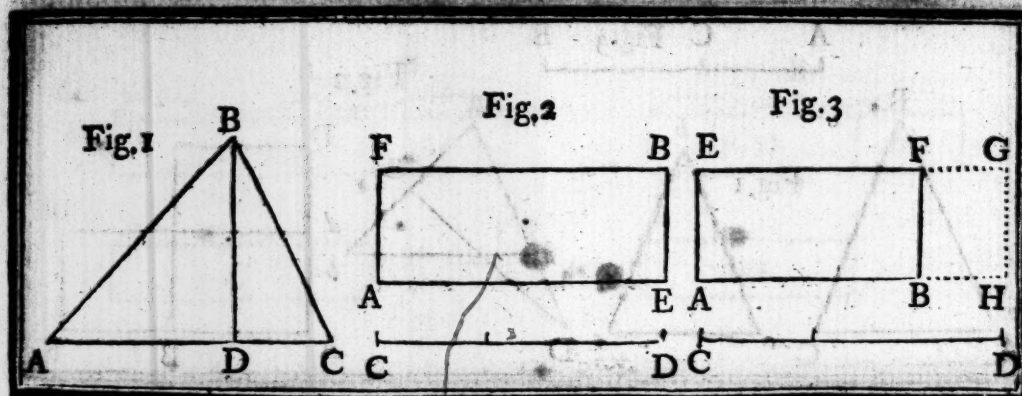
II.

THE Figures (DAB, dAb) are reciprocal (Fig. 2.), when the antecedents (AD, Ab) and the consequents (Ad, AB) of the ratios, are in each of the figures, (that is $AD : Ad = Ab : AB$).

Or the figures (DAB, dAb) are reciprocal; when the two sides (AD, AB and Ad, Ab), in each of those figures, about the same angle (A), or equal angles, are the extremes or means of the same proportion, that is, a side (AD) in the first figure is to a side (Ad) of the other, as the remaining side (Ab) of this other is to the remaining side (AB) of the first.

III.

A Straight line (AB) is said to be cut in mean and extream ratio, (Fig. 3.) when the whole (AB), is to the greater segment (BC), as the greater segment, is to the less (AC).



DEFINITIONS.

IV.

THE altitude of any figure (A B C) (Fig. 1.), is the perpendicular (BD) let fall from the vertex (B) upon the base (A C).

IT follows from this Definition, that if two figures placed upon the same straight line, have the same altitude, they are between the same parallels; because from the nature of parallels the perpendiculars let fall from one to the other are always equal.

V.

A Ratio (A B . B C . C D : D E . E F . F G) is compounded of several others (A B : D E + B C : E F + C D : F G) when its terms result from the multiplication of the terms of those compounding ratios.

VI.

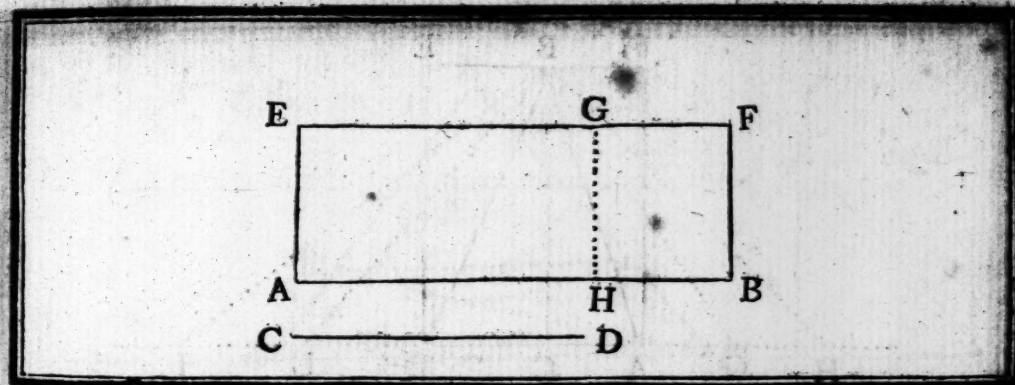
A Parallelogram (A B) (Fig. 2.) is said to be applied to a straight line (C D), when it has for its base or for its side this proposed straight line (C D).

VII.

A Deficient parallelogram (A F), (Fig. 3.) is that whose base (A B) is less than the proposed line (C D) to which it is said to be applied.

VIII.

BUT the deficiency of a deficient parallelogram (A F), (Fig. 1.) is a parallelogram (B G) contained by the remainder of the proposed straight line (C D) and the other side (B F) of the deficient parallelogram.



DEFINITIONS.

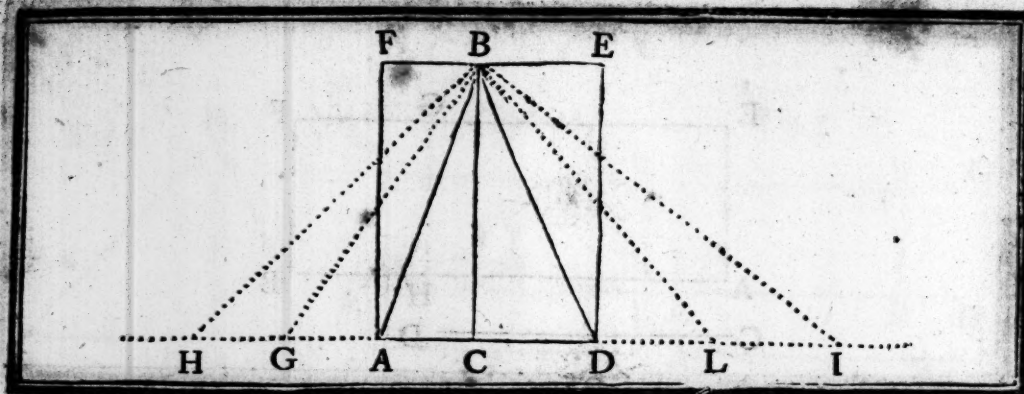
IX.

AN exceeding parallelogram (AF) is that, whose base (AB) is greater than the proposed line (CD), to which it is said to be applied.

X.

AND the excess of an exceeding parallelogram (AF) is a parallelogram (HF) contained by the excess of the base (AB) above the proposed straight line (CD) and the other side (BF) of the exceeding parallelogram,





PROPOSITION I. THEOREM I.

TRIANGLES (ABC, CBD), and parallelograms (CF, CE), of the same altitude, are one to another as their bases (AC, CD).

Hypothesis.

The $\triangle ABC$, CBD, & pgms.
CF, CE, have the same altitude

Thesis.

I. The $\triangle ABC : \triangle CBD = AC : CD$.
II. The pgm. CF : pgm. CE = AC : CD.

Preparation.

1. Produce AD indefinitely to H & I.
2. Take $AG = AC = GH$, also $DL = CD = LI$.
3. Draw BG, BH, BL, BI.

Pos. 2. B. 1.

P. 3. B. 1.

Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE the $\triangle ABC$, GBA, HBG, are upon equal bases AC, AG, GH, (Prep. 2), & between the same ples. HI, FE, (Hyp. & D. 35. B. 1. & Rem. D. 4. B. 6.).

1. Those $\triangle$ are = to one another. P. 38. B. 1.
2. From whence it follows, that the $\triangle HBC$, & the base HC, are equimult. of the $\triangle ABC$, & of the base AC.
It may be demonstrated after the same manner, that
3. The $\triangle CBI$, & the base CI, are equimult. of the $\triangle CBD$, & of the base CD.
4. Consequently, the mgn. HBC & HC, are equimult. of the mgn. ABC & AC (Arg. 2), & the mgn. CBI & CI are equimult. of the mgn. CBD & CD, (Arg. 3.).
But if the $\triangle HBC$, be $>$, = or $<$ the $\triangle CBI$, the base HC is also $>$, = or $<$ the base CI, (P. 38. B. 1.).

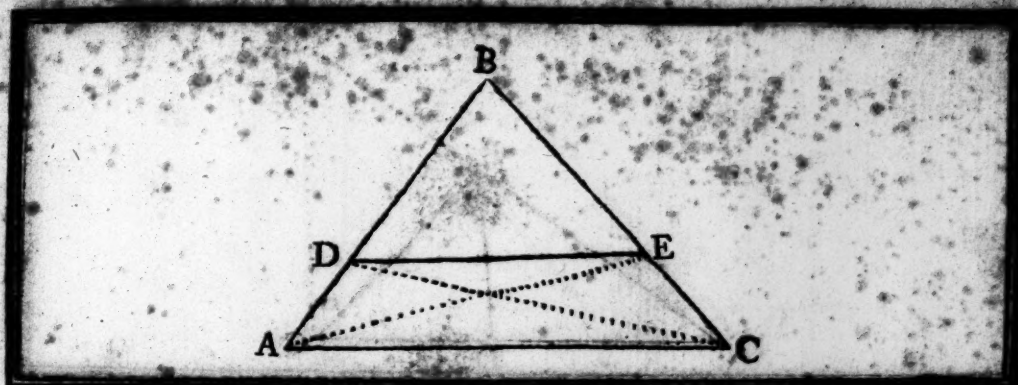
5. Consequently, the $\triangle ABC : \triangle CBD = AC : CD$. D. 5. B. 5.

Which was to be demonstrated. I.

But the $\triangle ACB$, CBD, being the halves of the pgms. CF, CE, (P. 41. B. 1.)

5. It follows, that $\triangle ACB : \triangle CBD = \text{pgm. CF} : \text{pgm. CE}$. P. 15. B. 5.
6. Wherefore the pgm. CF : pgm. CE = AC : CD. P. 11. B. 5.

Which was to be demonstrated. II.



PROPOSITION II. THEOREM II.

IF a straight line (DE) be drawn parallel to one of the sides (AC) of a triangle (ABC): it shall cut the other sides (AB, BC) proportionally, (that is $AD : DB = CE : EB$); and if the sides (AB, BC) be cut proportionally, the straight line (DE) which joins the points of section shall be parallel to the remaining side (AC) of the triangle.

Hypothesis.

The straight line DE is p^{lle}. to AC.

Thesis.

 $AD : DB = CE : EB$.

Preparation.

Draw the straight lines AE, CD.

P^{of}. I. B. I.

I. DEMONSTRATION.

BECAUSE DE is p^{lle}. to AC (Hyp.).
 1. The $\triangle DAE$ is $= \triangle ECD$. P. 37. B. I.
 2. Consequently, $\triangle DAE : \triangle DBE = \triangle ECD : \triangle DBE$. P. 7. B. 5.
 But the $\triangle DAE : \triangle DBE = AD : DB$.
 & the $\triangle ECD : \triangle DBE = CE : EB$ (P. 1. B. 6.)
 3. Therefore $AD : DB = CE : EB$. P. 11. B. 5.

Which was to be demonstrated.

Hypothesis.

 $AD : DB = CE : EB$.

Thesis.

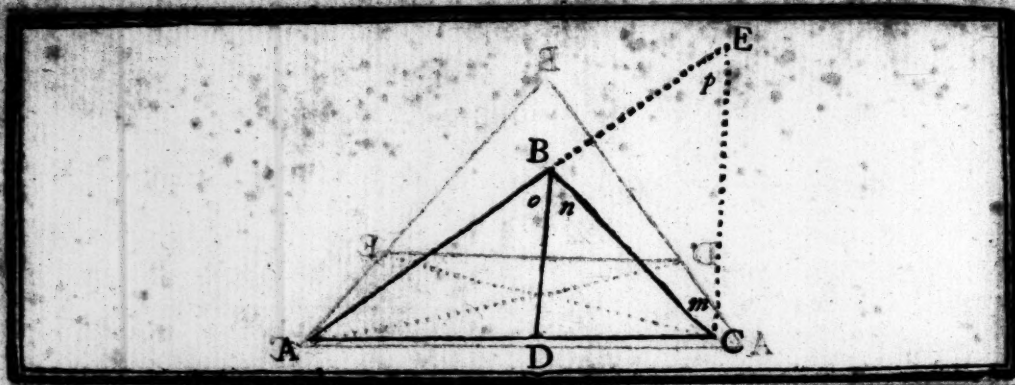
The straight line DE is p^{lle}. to AC:

II. DEMONSTRATION.

BECAUSE the $\triangle DAE$, $\triangle DBE$ are between the same p^{lles},
 as also the $\triangle ECD$, $\triangle DBE$.
 1. It follows that $\triangle DAE : \triangle DBE = AD : DB$.
 & the $\triangle ECD : \triangle DBE = CE : EB$. } P. 1. B. 6.
 But $AD : DB = CE : EB$ (Hyp.).
 2. Therefore the $\triangle DAE : \triangle DBE = \triangle ECD : \triangle DBE$. P. 11. B. 5.
 3. Wherefore the $\triangle DAE$ is $= \triangle ECD$. P. 9. B. 5.
 4. Consequently, the straight line DE is p^{lle}. to AC. P. 39. B. I.

Which was to be demonstrated.

C c



PROPOSITION III. THEOREM III.

IF the angle (B) of a triangle (ABC) be divided into two equal angles by a straight line (BD) which cuts the base in (D), the segments of the base (AD, DC) shall have the same ratio which the other sides (AB, BC) of the triangle have to one another; and if the segments of the base (AD, DC) have the same ratio which the other sides (AB, BC) of the triangle have to one another, the straight line (BD) drawn from the vertex (B) to the point of section (D) divides the vertical angle (ABC) into two equal angles.

Hypothesis.

The straight line BD divides the $\angle ABC$ into two equal parts, or $\angle o = \angle n$.

Thesis.

$AD : DC = AB : BC$.

Preparation.

1. Thro' the point C draw CE p^{lle}. to DB.
2. Produce AB until it meets CE in E.

I. DEMONSTRATION.

BECAUSE the straight lines DB, CE are p^{lle}. (Prep. 1).

1. It follows that $AD : DC = AB : BE$.
2. And that $\angle n = \angle m$, & $\angle o = \angle p$.
But, $\angle o$ being $=$ to $\angle n$ (Hyp.).
3. The $\angle m$ is also $=$ to $\angle p$, & $BC =$ to BE .
4. Wherefore $AD : DC = AB : BC$.

Which was to be demonstrated.

Hypothesis.

$AD : DC = AB : BC$.

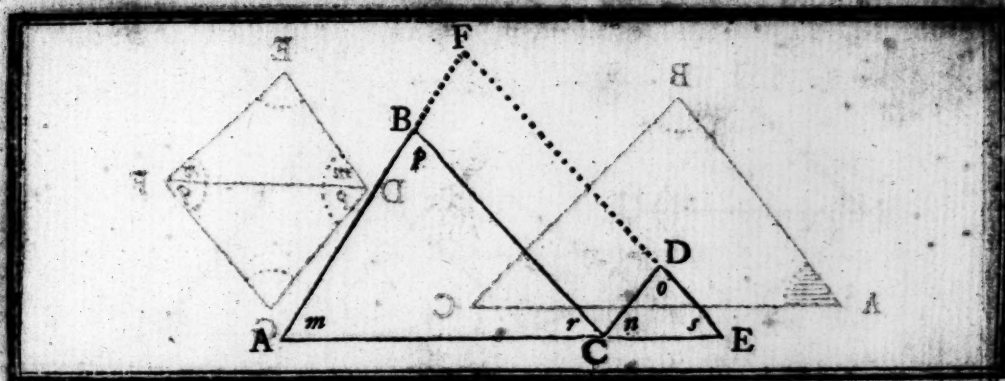
Thesis.

BD bisects $\angle ABC$ or $\angle o = \angle n$.

II. DEMONSTRATION.

BECAUSE the straight lines DB, CE are p^{lle}. (Prep. 1).

1. It follows that $AD : DC = AB : BE$.
- But $AD : DC = AB : BC$ (Hyp.).
2. Wherefore $AB : BE = AB : BC$.
3. Consequently, $BE = BC$, & $\angle m = \angle p$.
But $\angle m$ is also $=$ to $\angle n$, & $\angle p = \angle o$ (P. 29. B. 1).
4. Consequently, $\angle n$ is $=$ to $\angle o$, or BD bisects $\angle ABC$.



PROPOSITION IV. THEOREM IV.
THE sides (AC, AB & CE, CD, &c) about the equal angles (m & n , &c) of equiangular triangles (ABC, CDE) are proportionals; and those sides (AB, CD, &c) which are opposite to the equal angles (r & s , &c) are homologous sides; that is, are the antecedents or consequents of the ratios.

Hypothesis.

The $\triangle ABC$, CDE are equiangular,

or $\angle m = \angle n$, $\angle r = \angle s$,
 $\angle p = \angle o$.

Thesis.

I. $\begin{cases} AB : AC = CD : CE \\ AC : BC = CE : DE \\ AB : BC = CD : DE \end{cases}$

II. The sides $\begin{cases} AC, CE \\ BC, DE \end{cases}$ opposite to the equal $\angle$ are homologous.

Preparation.

1. Place the $\triangle ABC$, CDE, so that the bases AC, CE may be in the same straight line.
2. Produce the sides AB, DE indefinitely to F.

Pos. 2. B.1.

DEMONSTRATION.

BECAUSE the $\angle m + r$ of $\triangle ABC$ are $< 2\angle$ (P.17. B.1.) & $\angle r = \angle s$ (Hyp.).

1. The $\angle m + s$ are also $< 2\angle$, & AB, DE meet somewhere in F. *Lem. B.1.*

2. But $\angle m$ being $=$ to $\angle n$ & $\angle r =$ to $\angle s$ (Hyp.).

3. The straight lines AF, CD also BC, FE are p^lle. *P.28. B.1.*

4. And the quadrilateral figure CF is a p^grm. *D.35. B.1.*

5. Consequently, BC, FD; also CD, BF are $=$ to one another. *P.34. B.1.*

6. But BC being p^lle. to the side FE of the $\triangle FAE$ (Arg. 2).

7. Therefore, $AB : BF = AC : CE$. *P. 2. B.6.*

8. Or alternando $AB : AC = BF : CE$. *P.16. B.5.*

9. Or $AB : AC = CD : CE$, CD being $=$ to BF. (Arg. 4). *P. 7. B.5.*

Likewise CD being p^lle. to the side AF of the $\triangle FEA$.

10. It may be proved in the same manner, that $AC : BC = CE : DE$.

11. Consequently, $AB : BC = CD : DE$. *P.22. B.1.*

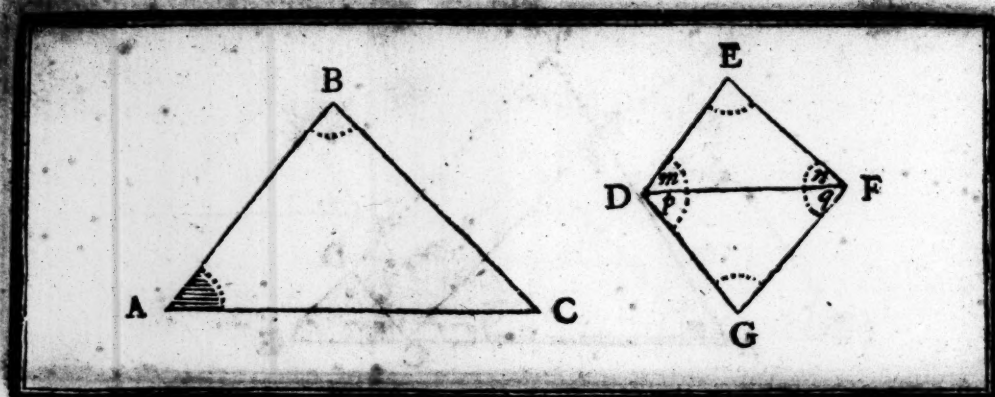
Which was to be demonstrated. 1.

But the sides AB, CD, also AC, CE & BC, DE are opposite to the equal $\angle r$ & s , p & o , m & n .

10. Consequently, the sides AB, CD; AC, CE; BC, DE opposite to the equal $\angle$ are homologous. *D.12. B.5.*

Which was to be demonstrated. 11.

Cor. Therefore equiangular triangles are also similar (D.1. B.6.)



PROPOSITION V. THEOREM V.

IF the sides of two triangles (ABC, DEF) be proportionals, those triangles shall be equiangular, and have their equal angles (A & m , C & n , &c) opposite to the homologous sides (BC, EF & AB, DE , &c).

Hypothesis.

The $\triangle ABC, DEF$ have their sides proportionals, that is,

$$\begin{cases} AB : AC = DE : DF \\ AB : BC = DE : EF \\ AC : BC = DF : EF \end{cases}$$

II. The sides BC, EF, AB, DE, AC, DF are homologous.

Thesis.

I. The $\triangle ABC, DEF$ are equiangular.

II. The $\angle$ opposite to the homologous sides are $=$; or $\angle A = \angle m, \angle C = \angle n$ & $\angle B = \angle E$.

Preparation.

1. At D in DF make $\angle p = \angle A$ & at F , $\angle q = \angle C$. P. 23. B. 1.
2. Produce the sides DG, FG until they meet in G . Lem. B. 1.

DEMONSTRATION.

BECAUSE in the equiangular $\triangle ABC, DGF$ (Prep. 1. & P. 32. B. 1), $\angle C = \angle q$ & $\angle B = \angle G$.

1. $AB : AC = DG : DF$, & $AB : AC = DE : DF$. (Hyp. 1). P. 4. B. 6.

2. Therefore, $DG : DF = DE : DF$. & DG is $=$ to DE . P. 11. B. 5.

3. It may be proved after the same manner, that $GF = EF$. P. 9. B. 5.

Since then in the two $\triangle DEF, DGF$, the sides $DE, EF =$ the sides DG, GF (Arg. 2. & 3), & the base DF is common to the two $\triangle$.

4. The $\angle n$ & m are $=$ to the $\angle q$ & p each to each. P. 8. B. 1.

5. And the $\triangle DEF, DGF$ are equiangular. P. 8. B. 1.

But the $\triangle DGF$, is equiangular to the $\triangle ABC$ (Prep. 1. & P. 32. B. 1),

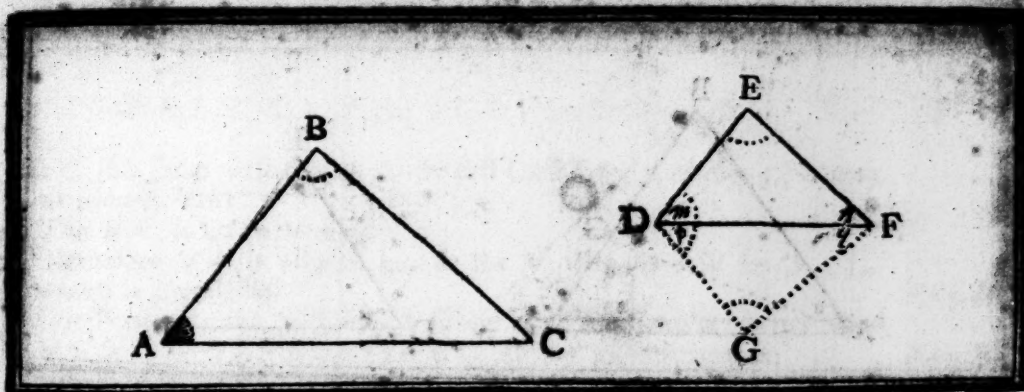
6. From whence it follows that the $\triangle ABC, DEF$ are equiangular. Ax. 1. B. 1.

Which was to be demonstrated. 1.

7. Moreover, the $\angle A, C$ & B opposite to the sides BC, AB, AC , being equal each to each, to the $\angle m, n$ & E opposite to the sides EF, DE, DF ; homologous to the sides BC, AB, AC , each to each, because the one & the other of those $\angle$, are equal each to each to the $\angle p, q, G$ (Prep. 1. P. 32. B. 1. & Arg. 4).

8. It follows, that the $\angle A, m$; also C, n & B, E opposite to the homologous sides are equal. Which was to be demonstrated. 11.

Cor. Therefore these triangles are also similar. (D. 1. B. 6.)



PROPOSITION VI. THEOREM VI.

IF two triangles (ABC, DEF) have one angle (A) of the one equal to one angle (m) of the other, and the sides (BA, AC, & ED, DF), about the equal angles proportionals, the triangles shall be equiangular, and shall have these angles (C & n , also B & E) equal which are opposite to the homologous sides (BA, ED & AC, DF).

Hypothesis.

- I. $\angle A = \angle m$.
- II. $BA : AC = ED : DF$.
- III. BA, ED, AC, DF are homologous.

Thesis.

- I. The $\triangle ABC$, DEF are equiangular.
- II. The $\angle C$ & n , also the $\angle B$ & E are $=$ to one another.

Preparation.

1. At the point D in the straight line DF make $\angle p =$ to $\angle A$, or $=$ to $\angle m$ & at the point F, $\angle q =$ to $\angle C$. P. 23. B.1.
2. Produce the sides DG, FG until they meet in G. Lem. B.1.

DEMONSTRATION.

BECAUSE the $\triangle ABC$, DGF are equiangular (Prep. 1. & P. 32. B. 1), & particularly $\angle C = \angle q$ & $\angle B = \angle G$.

1. $BA : AC = GD : DF$. P. 4. B.5.
- But $BA : AC = ED : DF$. (Hyp. 2).
2. Wherefore, $GD : DF = ED : DF$. P. 11. B.5.
3. Consequently, GD is $=$ to ED. P. 9. B.5.
- Therefore the two $\triangle DEF$, DGF having the two sides ED, DF $=$ to the two sides GD, DF (Arg. 3) & $\angle m =$ to $\angle p$ (Prep. 1).
4. The $\angle n$, q & E, G are $=$, & the $\triangle DEF$, DGF are equiangular. P. 4. B.1.
- But the $\triangle ABC$, DGF being also equiangular (Prep. 1. & P. 32. B.1),
5. It follows, that the $\triangle ABC$, DEF are equiangular. Ax. 1. B.1.

Which was to be demonstrated. 1.

- Moreover, each of the angles C & n being $=$ to $\angle q$ (Prep. 1. & Arg. 4).
6. The $\angle C$ is $=$ to $\angle n$. Ax. 1. B.1.
 7. Consequently, $\angle A$ being $=$ to $\angle m$ (Hyp. 1), $\angle B$ is also $=$ to $\angle E$. P. 32. B.1.
 - And the sides BA, ED & AC, DF opposite to those angles being homologous (Hyp. 3. & D. 12. B. 5.).
 8. It follows, that the $\angle C$ & n , also B & E opposite to those homologous sides are $=$ to one another. Which was to be demonstrated. 11.
- Cor. Therefore those triangles are also similar to each other. (P. 4. Cor. B.6).

CASE II. If the $\angle C$ & F are both obtuse.

By the same reasoning as in the first Case (*Arg. 1. to Arg. 5.*) it may be proved, that

1 The $\angle C$ is $=$ to $\angle m$.

2 Therefore $\angle m$ is also $> \angle$, & the $\angle C + m$ will be $> 2 \angle$, which is impossible.

P.17. B.1.

3 Consequently, the $\angle BAC$ & D are $=$ to one another & the third $\angle C$ is $=$ to $\angle F$, or the $\triangle ABC, DEF$ are equiangular.

P.32. B.1.

Which was to be demonstrated.

PROPOSITION 16. If the $\angle C$ & F are both right angles the $\triangle ABC$ & DEF are equiangular (*Hyp. 1. & P.32. B.1.*)

Cor. Therefore these triangles are similar to one another (P.4. Cor. B.6).

THE $\triangle ABC, DEF$ are similar to one another. Each is also similar to the whole $\triangle ABC$.

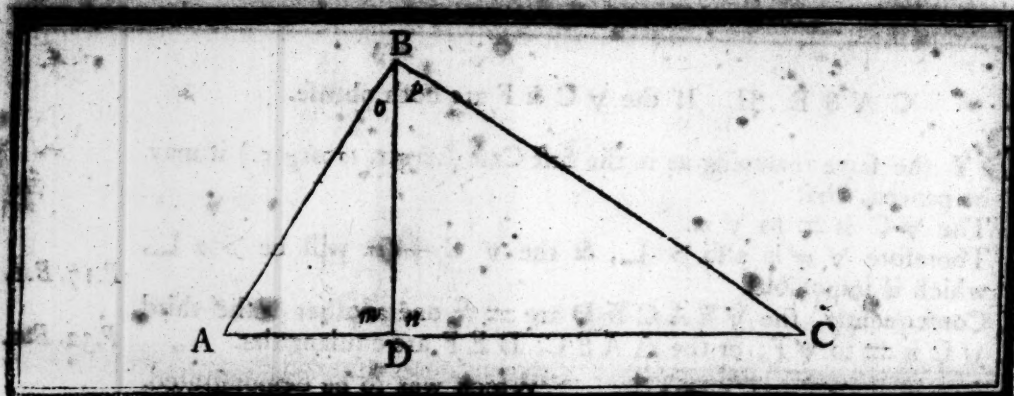
THE $\triangle ABC, DEF$ are similar to one another. Each is also similar to the whole $\triangle ABC$.

DEMONSTRATION.

THE $\triangle ABC$ is in the two right $\triangle ABC, DEF$ the $\angle C$ is $=$ to $\angle F$ (Hyp. 1.) & $\angle A$ is $=$ to $\angle D$ (P.17. B.1.) Consequently, the $\triangle ABC, DEF$ are equiangular (P.32. B.1.)



FROM THE $\triangle ABC$ it is manifest that the $\triangle ABC, DEF$ are similar to one another. Each is also similar to the whole $\triangle ABC$.



PROPOSITION VIII. THEOREM VIII.

IN a right angled triangle ($\triangle ABC$), if a perpendicular (BD) be drawn from the right angle ($\angle ABC$) to the base AC , the triangles ($\triangle ADB$, $\triangle BDC$) on each side of it are similar to the whole triangle ($\triangle ABC$) and to one another.

Hypothesis.

- I. The $\triangle ABC$ is rgle. in B .
- II. BD is $\perp$ upon AC .

Thesis.

The $\triangle ADB$, $\triangle BDC$ are similar to one another, & each is also similar to the whole $\triangle ABC$.

DEMONSTRATION.

BECAUSE in the two rgle. $\triangle ADB$, $\triangle ABC$, the $\angle m$ is $\equiv$ to $\angle ABC$, (*Ax. 10. B. 1.*), & $\angle A$ common to the two $\triangle$.

1. The $\angle o$ is $\equiv$ to $\angle C$ & the two $\triangle ABC$, $\triangle ADB$ are equiangular. *P. 32. B. 1.*
2. Consequently, these two $\triangle$ are also similar. *{ P. 4. B. 6. Cor.*

3. The $\triangle BDC$ is similar to the $\triangle ABC$.

Likewise in the two rgle. $\triangle ADB$, $\triangle BDC$, $\angle n$ being $\equiv$ to $\angle n$ (*Ax. 10. B. 1.*) & $\angle o$ $\equiv$ to $\angle C$ (*Arg. 1.*).

4. The $\angle A$ is $\equiv$ to $\angle p$, & the two $\triangle ADB$, $\triangle BDC$ are equiangular. *P. 32. B. 1.*
5. From whence it follows that these $\triangle$ are similar. *{ P. 4. B. 6. Cor.*

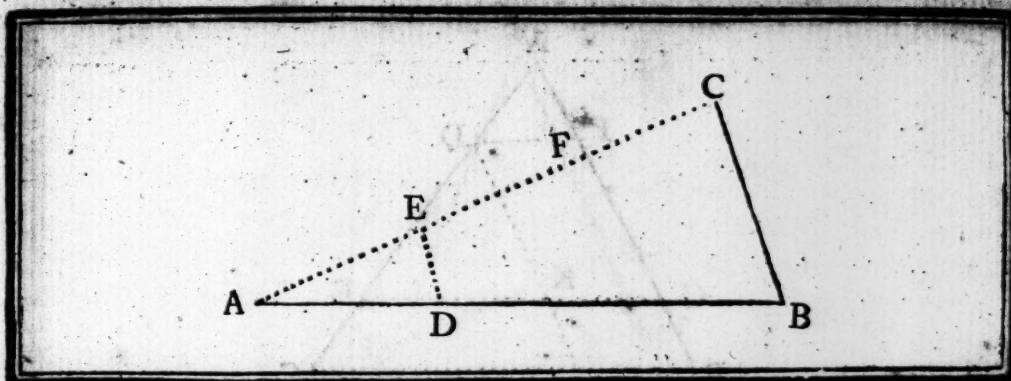
6. Consequently, the $\perp BD$ divides the $\triangle ABC$ into two $\triangle ADB$, $\triangle BDC$ similar to one another (*Arg. 5.*) & similar to the whole $\triangle ABC$ (*Arg. 2. & 3.*).

Which was to be demonstrated.

COROLLARY.

FROM this it is manifest that the perpendicular BD drawn from the Vertex of a right angled triangle to the base, is a mean proportional between the segments AD & DC of the base; for the triangles $\triangle ADB$, $\triangle BDC$ being equiangular, $AD : DB = DB : DC$ (*P. 4. B. 6.*).

Also, each of the sides AB or BC of the triangle $\triangle ABC$ is a mean proportional between the base & the segment AD or DC adjacent to that side. for since each of the triangles $\triangle ADB$, $\triangle BDC$ is equiangular with the whole $\triangle ABC$, $AC : AB = AB : AD$, & $AC : BC = BC : DC$ (*P. 4. B. 6.*).



PROPOSITION IX. PROBLEM I.

FROM a given straight line (A B) to cut off any part required.
(For example the third part).

Given.

The straight line A B.

Sought.

The abscinded straight line' A D,
which may be the third part of A B.

Resolution.

1. From the point A draw an indefinite straight line A C, making with A B any $\angle$ B A C.
2. Take in A C three equal parts A E, E F, F C of any length.
3. Join C B.
4. And thro' E, draw E D p^{lle}. to C B, which will cut the straight line A B so that A D will be the third part.

P^{of}. 1. B. 1.

P. 3. B. 1.

P^{of}. 1. B. 1.

P. 31. B. 1.

DEMONSTRATION.

BECAUSE E D is p^{lle}. to the side C B of the $\triangle$ C A B (P^{re}p. 4).

1. $C E : E A = B D : D A$. P. 2. B. 6.

But C E : is double of E A (Ref. 2);

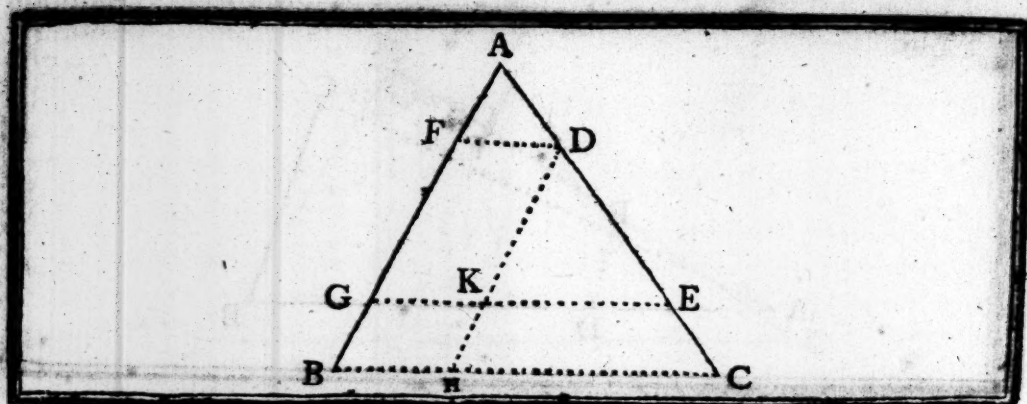
2. Consequently, B D is also double of D A. D. 8. B. 5.

3. Wherefore, A B is triple of A D.

4. And the abscinded straight line A D is the third part of A B.

Which was to be demonstrated.

D d



PROPOSITION X. PROBLEM II.

TO divide a given straight line (A B), similarly to a given straight line (A C) divided in the points (D, E &c)

Given.

- I. The straight line A B.
- II. The straight line A C divided in the points D, E &c.

Sought.

To divide A B similarly to A C in the points F & G, so that $AF : FG = AD : DE$ & that $FG : GB = DE : EC$.

Resolution.

1. Join the given straight lines A B, A C so as to contain any $\angle BAC$. Paf. 1. B. 1.
2. Draw C B, & from the points D & E, the straight lines D F, E G pple. to C B, also D H pple. to A B. P. 31. B. 1.

DEMONSTRATION.

BECAUSE D F is pple. to the side E G of the $\triangle AGE$ (Ref. 2. & P. 30. B. 1), and K E pple. to the side H C of the $\triangle DHC$ (Ref. 2).

1. $AF : FG = AD : DE$

And $DK : KH = DE : EC$

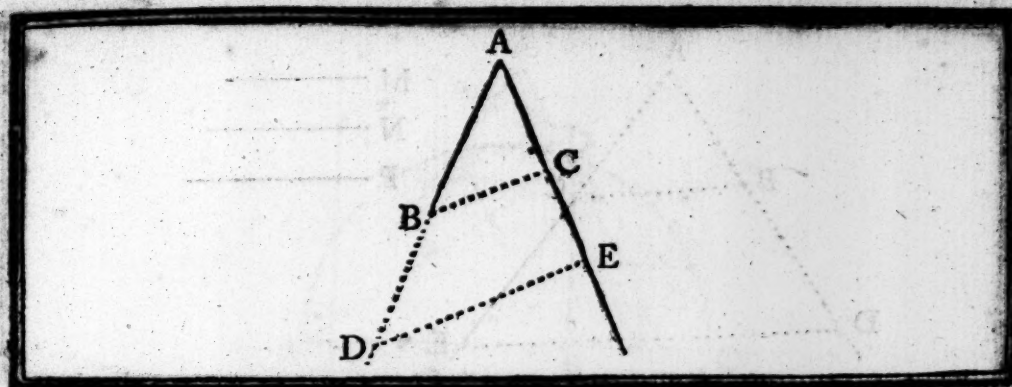
But the figures K F, H G being pperms. (Ref. 2. & D. 35. B. 1.) P. 7. B. 6.

2. It follows, that F G is $=$ to D K & G D $=$ K H. P. 34. B. 1.

3. Therefore, $FG : GB = DE : EC$. P. 7. & 11. B. 5.

4. Consequently, the given straight line A B is divided in the points F & G, so that $AF : FG = AD : DE$ & $FG : GB = DE : EC$.

Which was to be done.



PROPOSITION XI PROBLEM III.

TO find a third proportional (CE) to two given straight lines (AB, AC).

Given.

The two straight lines
AB, AC.

Sought.

The straight line CE, a third proportional
to the two straight lines AB, AC that
is such that $AB : AC = AC : CE$.

Resolution.

1. Join the two straight lines AB, AC so as to contain any $\angle BAC$.
2. Produce them, & make $BD = AC$.
3. Join BC.
4. And from the extremity D of the straight line AD draw DE p^lle. to BC.

P. 3. B.1.

Pof. 1. B.1.

P. 31. B.1.

DEMONSTRATION.

BECAUSE BC is p^lle. to DE (Ref. 4).

$AB : BD = AC : CE$.

P. 2. B.6.

But

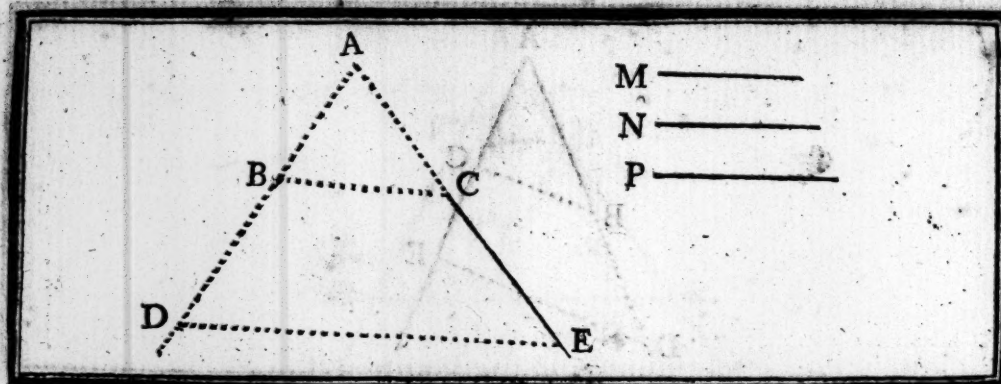
BD is = to AC (Ref. 2);

2. Consequently, $AB : AC = AC : CE$.

P. 7. & 11. B.5.

Which was to be done.





PROPOSITION XII. PROBLEM IV.

TO find a fourth proportional (CE) to three given straight lines (M, N, P).

Given.
The straight lines M, N, P .

Sought.
The straight line CE , a fourth proportional to M, N, P ; that is such, that $M : N = P : CE$.

Resolution.

1. Draw the two straight lines AD, AE , containing any $\angle DAE$. P. 3. B.1.
2. Make $AB = M; BD = N; AC = P$. Pos. 1. B.1.
3. Join BC .
4. From the extremity D of the straight line AD , draw DE , pple to BC . P. 31. B.1.

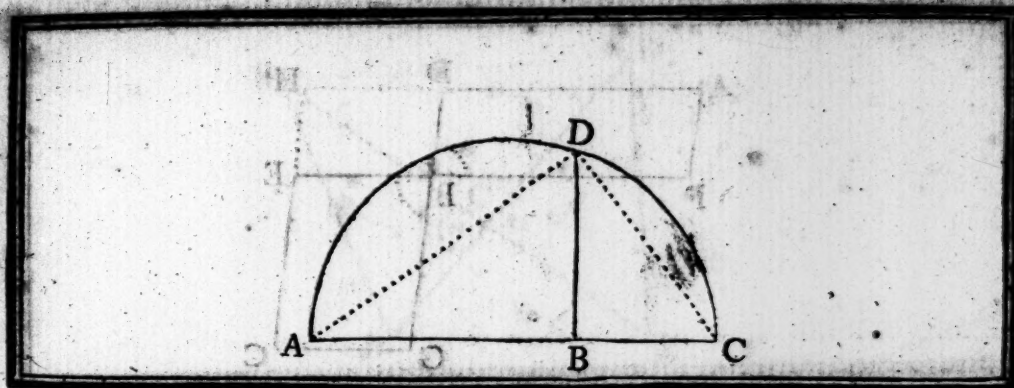
DEMONSTRATION.

BECAUSE BC is pple. to DE (Ref. 4).

1. $AB : BD = AC : CE$. P. 2. B.6.
- But $AB = M, BD = N, \& AC = P$ (Ref. 2);
2. Consequently, $M : N = P : CE$. P. 7. & 11. B.5.

Which was to be done.





PROPOSITION XIII. PROBLEM V.

TO find a mean proportional (BD) ; between two given straight lines (AB, BC).

Given.

The two straight lines AB, BC.

Sought.

The straight line BD, a mean proportional between AB & BC, that is such that $AB : BD = BD : BC$.

Resolution.

1. Place AB, BC in a straight line AC.
2. Describe upon AC the semi $\odot$ ADC.
3. At the point B, in AC, erect the $\perp$ BD meeting the $\odot$ in D.

Pos. 3. B. 1.

P. 11. B. 1.

Preparation.

Join AD, & CD.

Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE the $\angle ADC$ is in a semi $\odot$ (*Ref. 2. & Prep.*).

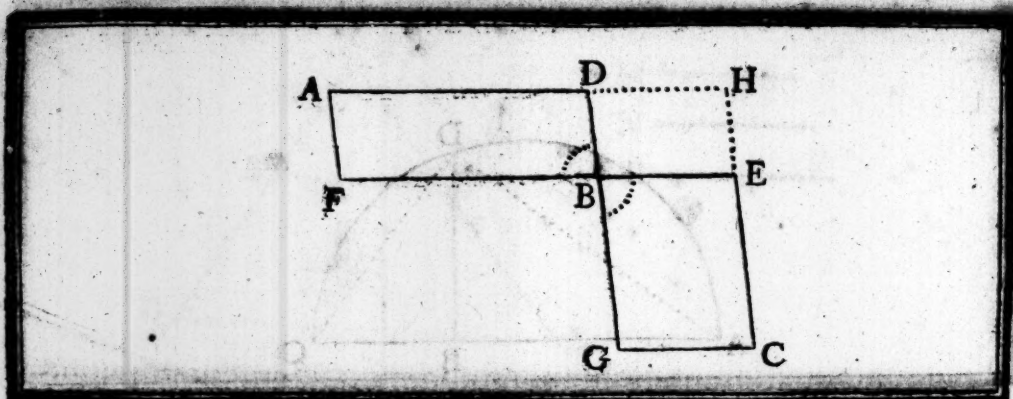
1. It is a right angle.
2. Wherefore, the $\triangle ADC$ is right angled in D, & BD is a $\perp$ let fall from the vertex D of the right angle, on the base AC (*Ref. 3.*).
3. Consequently, $AB : BD = BD : BC$.

P. 31. B. 3.

*{ P. 8. B. 6.
Cor.*

Which was to be done.





PROPOSITION XIV. THEOREM IX.

EQUAL parallelograms (A B, B C), which have one angle of the one (F B D) equal to one angle of the other (G B E), have their sides (F B, B D & G B, B E), about the equal angles reciprocally proportional, (that is, $FB : BE = GB : BD$). And parallelograms that have one angle of the one (F B D) equal to one angle of the other (G B E) and the sides (F B, B D & G B, B E), about the equal angles reciprocally proportional, are equal.

Hypothesis.

Thesis.

I. The pgr. $A \rightarrow B$ is $\equiv$ to the pgr. $B \rightarrow C$.

$$FB : BE = GB : BD.$$

II. $\forall F B D$ is $\equiv$ to $\forall G B E$.

Preparation.

1. Place the two pgs. A B, B C so as the sides F B, B E may be in a straight line F E.

2. Complete the pgr. D E.

Pos. 2. B. 1.

I. DEMONSTRATION.

BECAUSE the $\angle$ FBD, GBE are equal (*Hyp. 2*); & FB, BE are in a straight line FE (*Prep. 1*).

1. Therefore, GB, BD are in a straight line GD .

P. 14. B. 1.

But the pgr. A B being $\equiv$ to the pgr. B C (*Hyp.* 1).

2. The pgr. $\overline{AB} : \text{pgr. } \overline{DE} = \text{pgr. } \overline{BC} : \text{pgr. } \overline{DE}$.

P. 7. B. 5.

But the pgrs. AB, DE also BC, DE have the same altitude ($D. 4. B. 6$).

3. Hence pgr. $AB : \text{pgr. } DE = FB : BE$ & pgr. $BC : \text{pgr. } DE = GB : BD$. P. 1. B-6.

4. Consequently, $\overline{FB} : BE = GB : BD$ (Arg. 2).

P.11. B.5.

Which was to be demonstrated.

Hypothesis.

Thesis.

$$I. FB : BE = GB : BD.$$

The pgr. A B is = to the pgr. B C.

II. $\forall F B D$ is $\equiv$ to $\forall G B E$.

II. DEMONSTRATION.

1. It may be demonstrated as before, that GB, BD are in the line GD.

But the pgrs. AB, DE, & BC, DE, have the same altitude (D.4 B.6).

2. Hence, $\text{pgr. } AB : \text{pgr. } DE = FB : BE$, & $\text{pgr. } BC : \text{pgr. } DE = GB : BD$. *P. 1. B.6.*

But $FB : BE = GB : BD$ (Hyp.).

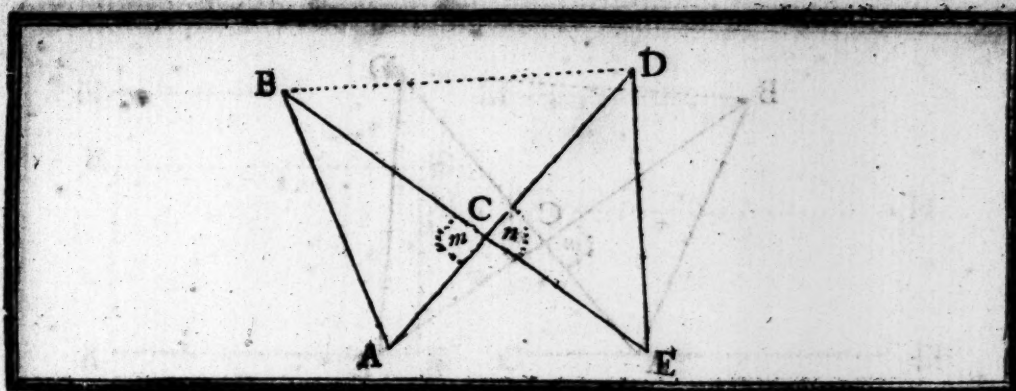
3. Wherefore, the pgr. $A B : \text{pgr. } \bar{D} \bar{E} = \text{pgr. } B C : \text{pgr. } D E$.

P. 11. B. 5.

4. Consequently, the pgr. A B is $\equiv$ to the pgr. B C.

P. g. B. 5.

Which was to be demonstrated.



PROPOSITION XV. THEOREM X.

EQUAL triangles ($\triangle ACB$, $\triangle ECD$) which have one angle of the one (m) equal to one angle of the other (n): have their sides (AC , CB , & EC , CD), about the equal angles, reciprocally proportional; & the triangles ($\triangle ACB$, $\triangle ECD$) which have one angle in the one (m) equal to one angle in the other (n), and their sides (AC , CB , & EC , CD), about the equal angles reciprocally proportional, are equal to one another.

CASE I.

Hypothesis.

- I. The $\triangle ACB$ is $\equiv$ to $\triangle ECD$.
 II. $\angle m$ is $\equiv$ to $\angle n$.

Thesis.

The sides AC , CB & EC , CD , are reciprocally proportional, or $AC : CD \equiv EC : CB$.

Preparation.

1. Place the $\triangle ACB$, $\triangle ECD$ so that the sides AC , CD may be in the same straight line AD .
2. Draw the straight line BD .

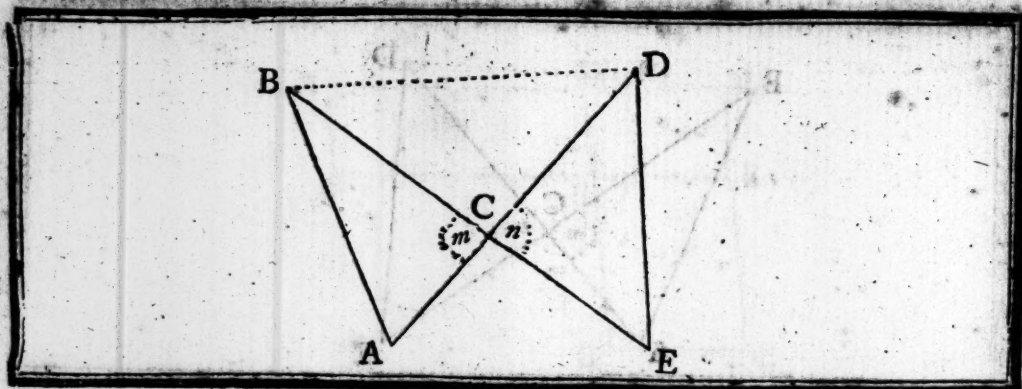
Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE $\angle m \equiv \angle n$ (Hyp. 2.), & the straight lines AC , CD are in the same straight line AD (Prep. 1).

1. The lines EC , CB are also in a straight line EB . P. 14. B. 1.
 But the $\triangle ACB$ being $\equiv$ to the $\triangle ECD$ (Hyp. 1).
2. The $\triangle ACB : \triangle CBD \equiv \triangle ECD : \triangle CBD$. P. 7. B. 5.
 But the $\triangle ACB$, $\triangle CBD$ also $\triangle ECD$, $\triangle CBD$ have the same altitude (Prep. 2. Arg. 1. & D. 4. Rem. B. 6).
3. Wherefore the $\triangle ACB : \triangle CBD \equiv AC : CD$. } P. 1. B. 6.
 & the $\triangle ECD : \triangle CBD \equiv EC : CB$.
4. Consequently, $AC : CD \equiv EC : CB$ (Arg. 2. & P. 11. B. 5).

Which was to be demonstrated.



CASE II.

Hypothesis.

I. $AC : CD = EC : CB$.II. $\angle m = \angle n$.

Thesis.

The $\triangle ACB$, is = to the $\triangle ECD$.

Preparation.

1. Place the two $\triangle ACB$, $\triangle ECD$ so that the sides AC , CD , may be in the same straight line AD .
2. Draw the straight line BD .

DEMONSTRATION.

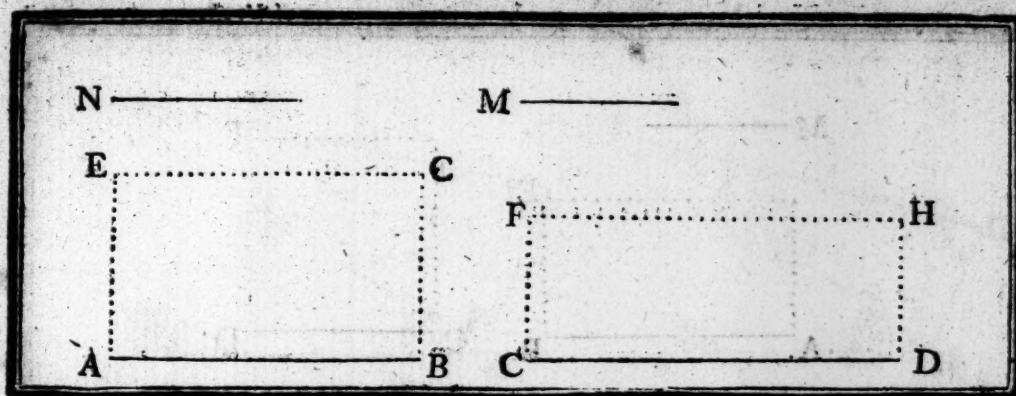
1. **I**T may be demonstrated, as in the first Case, that EC , CB are in the same straight line EB .
And because the $\triangle ACB$, $\triangle CBD$, also the $\triangle ECD$, $\triangle CBD$ have the same altitude (*Prep. 2. Arg. 1. & D. 4. Rem. B. 6*).
2. The $\triangle ACB : \triangle CBD = AC : CD$.
Likewise $\triangle ECD : \triangle CBD = EC : CB$.
But $AC : CD = EC : CB$. (*Hyp. 1*).
3. Wherefore $\triangle ACB : \triangle CBD = \triangle ECD : \triangle CBD$.
4. Consequently, the $\triangle ACB$ is = to the $\triangle ECD$.

} P. 1. B. 6.

P. 11. B. 5.
P. 9. B. 5.

Which was to be demonstrated.





PROPOSITION XVI. THEOREM XI.
IF four straight lines (A B, C D, M, N) be proportionals, the rectangle contained by the extremes (A B. N) is equal to that of the means (C D. M). And if the rectangle contained by the extremes (A B. N) be equal to the rectangle contained by the means (C D. M), the four straight lines (A B, C D, M, N) are proportionals.

Hypothesis.

$$A B : C D = M : N.$$

Thesis.

$$Rgle. A B. N = Rgle. C D. M.$$

Preparation.

1. At the extremities A & C, of A B, C D, erect the $\perp$ A E, C F. *P. 11. B. 5.*
2. Make A E = N, & C F = M. *P. 3. B. 1.*
3. Complete the rgles. E B, F D. *P. 31. B. 1.*

I. DEMONSTRATION.

BECAUSE $A B : C D = M : N$ (*Hyp.*) : & $M = C F$ & $N = A E$ (*Prep. 2.*).

1. $A B : C D = C F : A E$. *P. 7. & 11. B. 5.*
2. Therefore the sides of the rgles E B, F D about the equal $\angle$ A & C, (*Prep. 1. & Ax. 10. B. 1.*) are reciprocal. *D. 2. B. 6.*
3. Consequently, the rgle. E B = rgle. F D, or the rgle under A B. A E { *P. 14. B. 6.*
 = the rgle. under C D. C F. *{ D. 1. B. 1.*
- Consequently, A E being = N & C F = M (*Prep. 2.*).
4. The rgle. under A B. N is also = to the rgle. under C D. M. *Ax. 2. B. 2.*

Which was to be demonstrated.

Hypothesis.

The rgle. A B. N is = to the rgle. C D. M.

Thesis.

$$A B : C D = M : N.$$

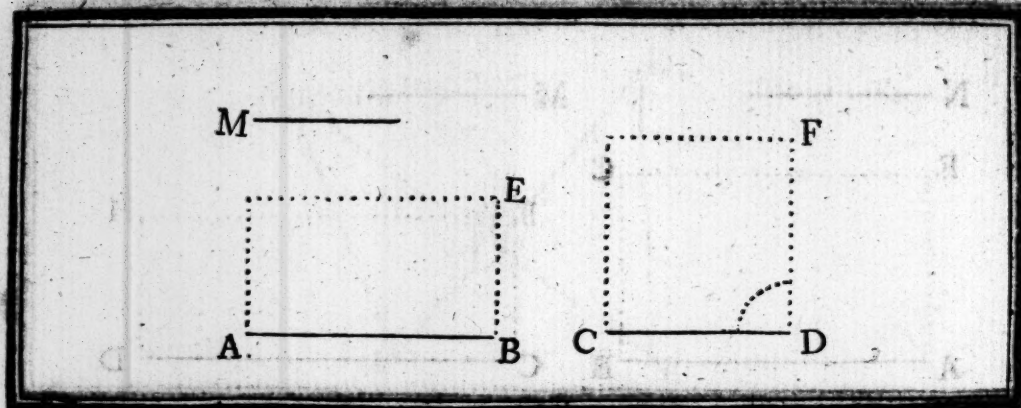
II. DEMONSTRATION.

BECAUSE the rgle. A B. N is = to the rgle C D. M (*Hyp.*) : & A E = N, & C F = M (*Prep. 2.*).

1. The rgle. under A B. A E is = to the rgle under C D. C F. *Ax. 2. B. 1.*
 But these sides being about the equal $\angle$ E A B, F C D (*Prep. 1. & Ax. 10. B. 1.*).
2. $A B : C D = C F : A E$. *P. 14. B. 6.*
- And C F being = M & A E = N (*Prep. 2.*).
3. $A B : C D = M : N$. *P. 7. & 11. B. 5.*

Which was to be demonstrated.

E e



PROPOSITION XVII. THEOREM XII.

IF three straight lines (AB, CD, M) be proportionals, the rectangle (AB.M) contained by the extremes is equal to the square of the mean (CD): And if the rectangle contained by the extremes (AB.M) be equal to the square of the mean (CD), the three straight lines (AB, CD, M) are proportionals.

Hypothesis.
 $AB : CD :: CD : M.$

Thesis.
 The rgle. AB.M is $\equiv$ to the $\square$ of CD.

Preparation.

1. At the extremities B & D of AB, CD erect the $\perp$ BE, DF. *P. 11. B. 1.*
2. Make BE = M & DF = DC. *P. 3. B. 1.*
3. Complete the rgles. EA, FC. *P. 31. B. 1.*

I. DEMONSTRATION.

BECAUSE $AB : CD :: M$ (*Hyp.*) & $CD :: DF$ & $M :: BE$ (*Prep. 2*).

1. $AB : CD :: DF : BE.$ *P. 7. & 11. B. 5.*
 Therefore the sides of the rgles. EA, FC about the equal $\angle$ B & D (*Prep. 1. & Ax. 10. B. 1*) are reciprocal. *D. 2. B. 6.*
2. Consequently, the rgle. EA is $\equiv$ to the rgle. FC, or the rgle. under AB. BE $\equiv$ the rgle CD. DF. *{ P. 14. B. 1. D. 1. B. 6. Ax. 2. B. 2. }*
3. Wherefore, BE being $\equiv$ M & DF $\equiv$ CD (*Prep. 2*), the rgle. AB. M is also $\equiv$ to the $\square$ of CD.

Hypothesis.
 The rgle. AB. M is $\equiv$ to the $\square$ of CD.

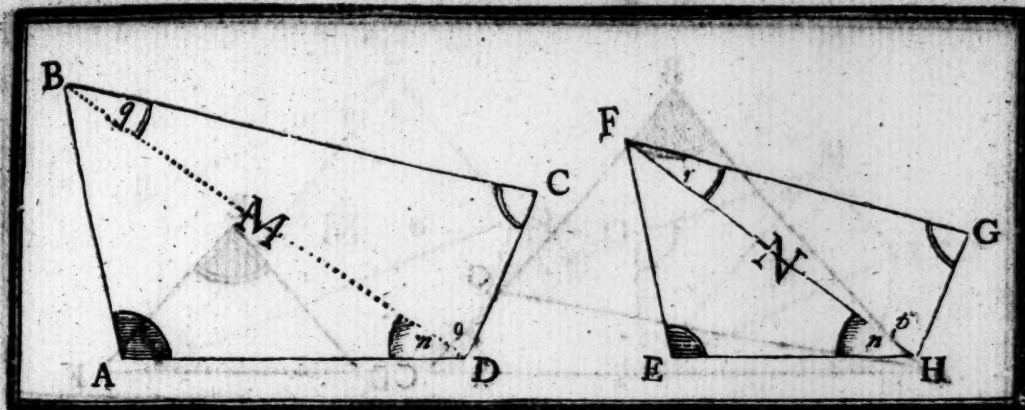
Thesis.
 $AB : CD :: CD : M$

II. DEMONSTRATION.

BECAUSE the rgle. under AB.M is $\equiv$ to the $\square$ of CD (*Hyp.*), & that BE is $\equiv$ M & DF $\equiv$ CD (*Prep. 2*).

1. The rgle. under AB. BE is $\equiv$ to the rgle. under CD. DF. *Ax. 2. B. 2.*
 But these sides are about the equal $\angle$ EBA, FDC (*Ax. 10. B. 1. & Prep. 1*).
2. Therefore, $AB : CD :: DF : BE.$ *P. 14. B. 6.*
 And since DF = CD & BE = M (*Prep. 2*).
3. $AB : CD :: CD : M.$ *P. 7. & 11. B. 5.*

Which was to be demonstrated.



PROPOSITION XVIII. PROBLEM VI.

UPON a given straight line (AD) to describe a rectilineal figure (M) similar, and similarly situated to a given rectilineal figure (N).

Given.

- I. The straight line AD.
- II. The rectilineal figure N,

Sought.

The rectilineal figure M similar to a rectilineal figure N & similarly situated.

Resolution.

1. Join HF.
2. At the points A & D in AD, make $\angle A = \angle E$ & $\angle m = \angle n$, wherefore the remaining $\angle ABD$ will be $=$ to the remaining $\angle EFH$.
3. At the points D & B in DB make $\angle o = \angle p$ & $\angle q = \angle r$, consequently the remaining $\angle C$ will be $=$ to the remaining $\angle G$.

Pos. 1. B. 1.

P. 23. 32.

&

Lem. B. 1.

DEMONSTRATION.

BECAUSE the $\triangle ABD$ is equiangular to the $\triangle EFH$, & the $\triangle DBC$ equiangular to the $\triangle HFG$ (Ref. 2. & 3).

1. $BD : FH = BA : FE = AD : EH$.
- & $BD : FH = DC : HG = CB : GF$.

P. 4. B. 6.

2. Consequently, $BA : FE = AD : EH = DC : HG = CB : GF$.

P. 11. B. 5.

But $\angle m$ being $= \angle n$ (Ref. 2), & $\angle o = \angle p$ (Ref. 3).

3. The whole $\angle m + \angle o$ is $=$ to the whole $\angle n + \angle p$.

As. 2. B. 1.

4. Likewise $\angle ABC = \angle EFG$.

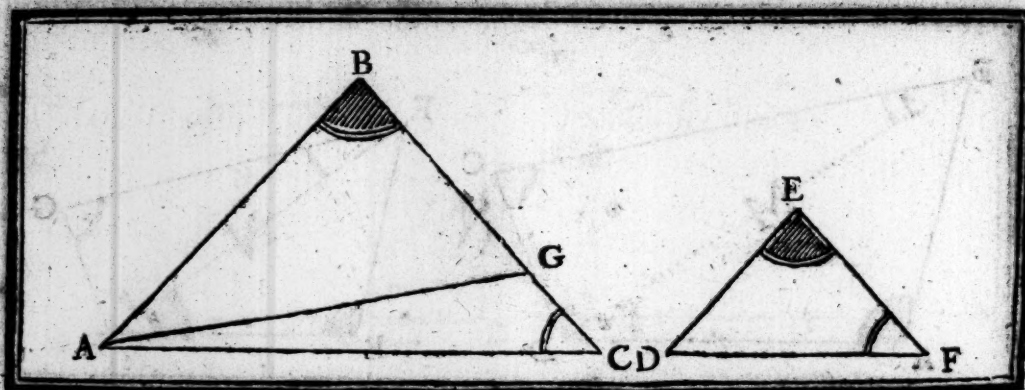
Moreover, $\angle A = \angle E$ (Ref. 2), & $\angle C = \angle G$ (Ref. 3).

4. Wherefore, the rectilineal figure M is equiangular to the rectilineal figure N, & their sides about the equal $\angle$ are proportionals.

6. Therefore, the rectilineal figure M described upon the given line AD is similar to the rectilineal figure EG, & is similarly situated.

D. 1. B. 6.

Which was to be done.



PROPOSITION XIX. THEOREM XIII.

SIMILAR triangles (ABC , DEF) are to one another in the duplicate ratio of their homologous sides (CB , FE or AC , DF , &c).

Hypothesis.

The triangles ABC , DEF are similar,
So that $\angle C = \angle F$, & the sides
 AC , DF & CB , FE are homologous.

Thesis.

The $\triangle ABC$ is to the $\triangle DEF$ in
the duplicate ratio of CB to FE
that is as $CB^2 : FE^2$.*

Preparation.

Take CG a third proportional to CB , FE , & draw AG .

{ P. 11. B. 6.
Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE $AC : CB = DF : FE$ (Hyp. & D. 1. B. 6).

1. Alternando $AC : DF = CB : FE$. P. 16. B. 5.

But $CB : FE = FE : CG$ (Prep.).

2. Consequently, $AC : DF = FE : CG$. P. 11. B. 5.

3. Therefore, the sides of the $\triangle AGC$, DEF about the equal $\angle C$ & F (Hyp.) are reciprocal (P. 2. B. 6).

4. Hence the $\triangle AGC$ is $=$ to the $\triangle DEF$. P. 15. B. 6.

But the $\triangle ABC$, AGC having the same altitude.

5. The $\triangle ABC : \triangle AGC = CB : CG$. P. 1. B. 6.

6. Consequently, the $\triangle ABC : \triangle DEF = CB : CG$. P. 7. B. 5.

But since $CB : FE = FE : CG$ (Prep.).

7. $CB : CG$ in the duplicate ratio of CB to FE , or as $CB^2 : FE^2$ * D. 10. B. 5.

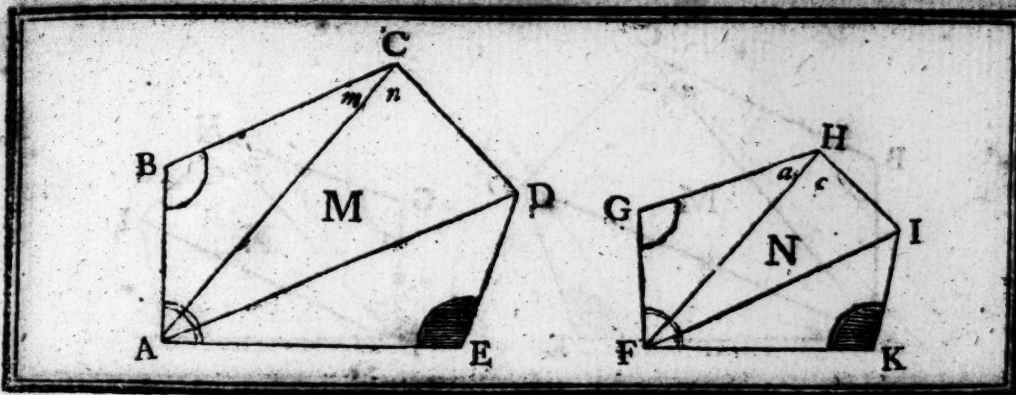
8. Wherefore, the $\triangle ABC : \triangle DEF$ in the duplicate ratio of CB to FE , or as $CB^2 : FE^2$.* P. 11. B. 5.

Which was to be demonstrated.

COROLLARY.

FROM this it is manifest, that if three lines (CB , FE , CG) be proportionals, as the first is to the third, so is any $\triangle$ upon the first to a similar, & similarly described $\triangle$ upon the second.

* See Cor. 2. of the following proposition.



PROPOSITION XX. THEOREM XIV.
SIMILAR polygons (M & N) may be divided by the diagonals (AC, AD; FH, FI) into the same number of similar triangles (ABC, ACD, ADE, & FGH, FHI, FIK) having the same ratio to one another, that the polygons (M & N) have; and the polygons (M & N) have to one another the duplicate ratio of that which their homologous sides (AB, FG; or BC, GH &c.) have.

Hypothesis.

The polyg. M is similar to the polyg. N;
 so that the $\angle A, B, C, \&c.$ are $=$ to the
 $\angle F, G, H, \&c.$ each to each & the
 sides AB, FG; or BC, GH, &c.
 are homologous.

Thesis.

- I. Those polygons may be divided into the same number of similar Δ .
- II. Whereof, each to each has the same ratio which the polygons have.
- III. And the polyg. M : polyg. N in the duplicate ratio of the homologous sides AB, FG; or as $AB^2 : FG^2$.*

Preparation.

Draw AC, FH, likewise AD, FI.

Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE $\angle B = \angle G$ & $AB : BC = FG : GH$ (Hyp. & D. 1. B. 6).

1. The ΔABD is equiangular to the ΔFGH .

2. Wherefore those Δ are similar, & $\angle m = \angle a$.

But the whole $\angle m + n$ is $=$ to the whole $\angle a + c$ (Hyp).

3. Consequently, $\angle n$ is $=$ to $\angle c$.

Since then by the simil. of the ΔABC & ΔFGH (Arg. 2),

$$AC : BC = FH : GH,$$

& by the simil. of the polyg. M & N, $BC : CD = GH : HI$.

4. It follows, Ex Aequo, that

$$AC : CD = FH : HI.$$

That is, the sides about the equal $\angle n$ & c are proportionals.

5. Therefore the ΔACD is equiangular to the ΔFHI .

And consequently is similar to it.

6. For the same reason, all the other $\Delta ADE, FIK, \&c.$ are similar.

7. Therefore, similar polygons may be divided into the same number of similar Δ .

Which was to be demonstrated. 1.

* See Cor. 2. of this proposition.

P. 6. B. 6.

P. 4. B. 6.

Cor.

Ax. 3. B. 1.

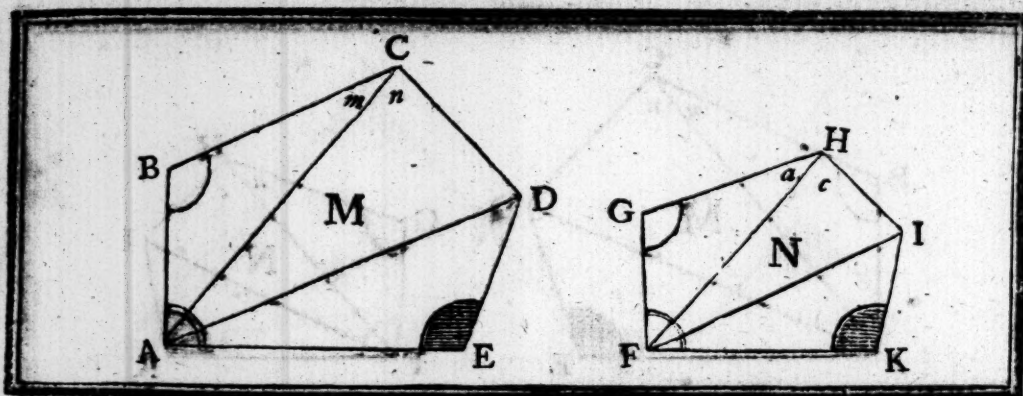
D. 1. B. 6.

P. 22. B. 5.

P. 6. B. 6.

P. 4. B. 6.

Cor.



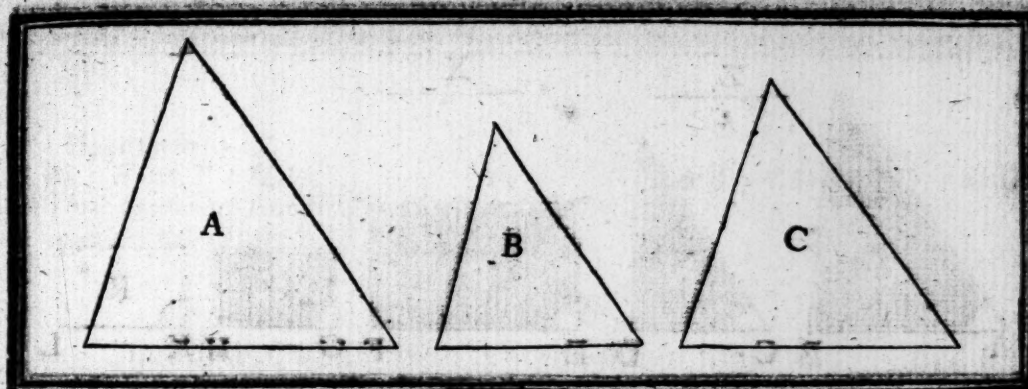
- Likewise, because the $\triangle ABC, FGH$ are similar (*Arg. 2*).
6. The $\triangle ABC : \triangle FGH = AC^2 : FH^2$.* } P. 19. B. 6.
 And the $\triangle ACD : \triangle FHI = AC^2 : FH^2$.* }
 7. Therefore, the $\triangle ABC : \triangle FGH = \triangle ACD : \triangle FHI$. P. 11. B. 5.
 It may be demonstrated after the same manner, that
 8. The $\triangle ADE : \triangle FIK = \triangle ACD : \triangle FHI$.
 9. Wherefore, $\triangle ABC : \triangle FGH = \triangle ACD : \triangle FHI = \triangle ADE : \triangle FIK$. P. 11. B. 5.
 10. Therefore, comparing the sum of the anteced. to that of the conseq.
 $\triangle ABC + \triangle ACD, \&c. : \triangle FGH + \triangle FHI, \&c. = \triangle ABC : \triangle FGH, \&c.$ P. 12. B. 5.
 That is, the polyg. M : polyg. N = $\triangle ABC : \triangle FGH =$
 $\triangle ACD : \triangle FHI, \&c.$ P. 7. B. 5.
 Which was to be demonstrated. 11.
 Since then the $\triangle ABC : \triangle FGH = AB^2 : FG^2$ * (P. 19. B. 6).
 11. The polyg. M : polyg. N = $AB^2 : FG^2$ *. P. 11. B. 5.
 Which was to be demonstrated. 111.

COROLLARY I.

AS this Demonstration may be applied to quadrilateral figures, & the same truth has already been proved in triangles (P. 19), it is evident universally, that similar rectilineal figures are to one another in the duplicate ratio of their homologous sides. Wherefore, if to AB, FG two of the homologous sides a third proportional X be taken; because AB is to X in the duplicate ratio of AB : FG; & that a rectilineal figure M is to another similar rectilineal figure N, in the duplicate ratio of the same sides AB : FG; it follows, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure described upon the first to a similar & similarly described rectilineal figure upon the second. (P. 11. B. 5).

* COROLLARY II.

ALL squares being similar figures (D. 30. B. 1. & D. 1. B. 6), similar rectilineal figures M & N, are to one another as the squares of their homologous sides AB, CD (expressed thus $AB^2 : CD^2$.) for those figures are in the duplicate ratio of these same sides.



PROPOSITION XXI. THEOREM XV.

RECTILINEAL figures (A, C) which are similar to the same rectilinear figure (B), are also similar to one another.

Hypothesis.

The rectilinear figures A & C are similar to the figure B.

Thesis.

The rectilinear figure A is similar to the rectilinear figure C.

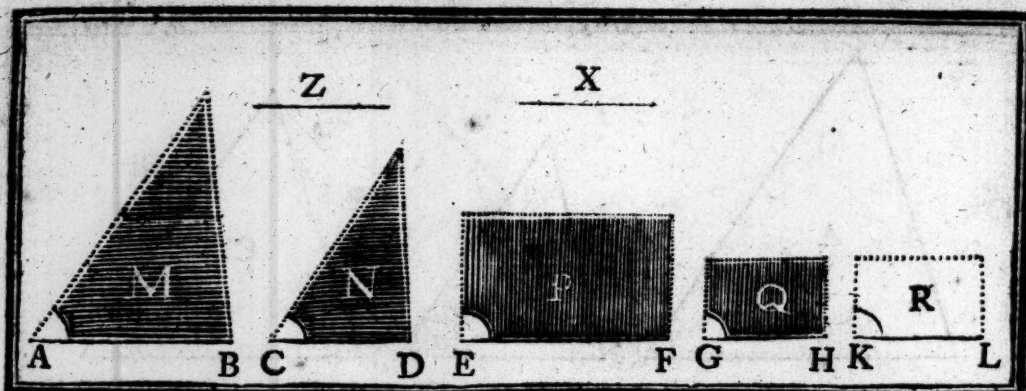
DEMONSTRATION.

BECAUSE each of the figures A & C is similar to the figure B (Hyp.).

1. Each of those figures will be also equiangular to the figure B, & will have the sides about the equal $\angle$, proportional to the sides of the figure B. D. 1. B.6.
2. Consequently, those figures A & C will be also equiangular to one another, and their sides about the equal $\angle$, will be proportional. { Ax. 1. B.1.
P. 11. B.5.
3. Consequently, the figures A & C are similar. D. 1. B.6.

Which was to be demonstrated.





PROPOSITION XXII. THEOREM XVI.

IF four straight lines (AB, CD, EF, GH) be proportionals, the similar rectilineal figures & similarly described upon them (M, N, & P, Q) shall also be proportionals. And if the similar rectilineal figures (M, N, & P, Q) similarly described upon four straight lines be proportionals, those straight lines shall be proportional.

I.

Hypothesis.

- I. $AB : CD = EF : GH$.
- II. The figures M & N described upon AB, CD. also the figures P & Q described upon EF, GH. are similar, & similarly situated.

Thesis.

$$M : N = P : Q$$

Preparation.

To the lines AB, CD take a III proportional Z.
To the lines EF, GH take a III proportional X.

} P.11. B.6.

DEMONSTRATION.

- B**ECAUSE $AB : CD = EF : GH$. { (Hyp. 1).
& $CD : Z = GH : X$. { (Hyp. 1. Prep. & P.11. B.5).
f. $AB : Z = EF : X$. P.22. B.5.
But the figures M, N, & P, Q being similar & similarly described upon the straight lines AB, CD, & EF, GH (Hyp. 2).
2. $AB : Z = M : N$ { P.20. B.6.
& $EF : X = P : Q$ { Cor. 1.
3. Wherefore, $M : N = P : Q$. (Arg. 1). P.11. B.5.

II.

Hypothesis.

- I. $M : N = P : Q$
 II. *Those figures are similar & similarly described upon the straight lines AB, CD & EF, GH.*

Thesis.

$$AB : CD = EF : GH.$$

Preparation.

1. To AB, CD, EF take a IVth proportional KL. P.12. B.6.
 2. Upon KL describe the rectil. figure R, similar to the rectil. figures P or Q, & similarly situated. P.18. B.6.

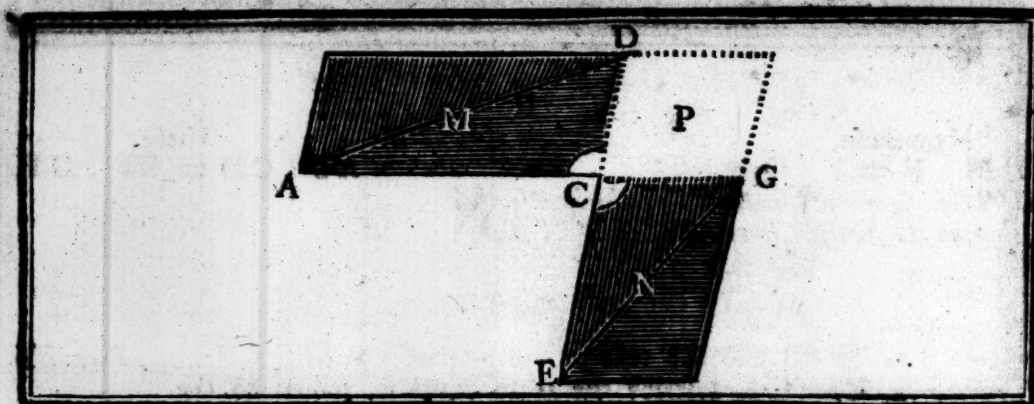
DEMONSTRATION.

BECAUSE $AB : CD = EF : KL$ (*Prep. 1*), & upon those straight lines have been similarly described the figures M, N, & P, R, similar each to each (*Hyp. 2. & Prep. 2*).

1. $M : N = P : R$ (*Ist. part of this proposition.*)
 But $M : N = P : Q$ (*Hyp. 1*).
 2. Consequently, $P : R = P : Q$ P.11. B.5.
 3. Wherefore, $R = Q$ P. 9. B.5.
 Moreover, those figures being similar & similarly described upon the straight lines GH, KL (*Prep. 2*).
 4. $Q : R = \square \text{ of GH} : \square \text{ of KL}$. { P.20. B.6.
 And $Q \text{ being} = R$ (*Arg. 3*). { Cor. 2.
 4. The $\square$ of GH is = to the $\square$ of KL. { P.16. B.5.
 5. Consequently, $GH = KL$. { Cor.
 Since then $AB : CD = EF : KL$ (*Prep. 1*), & $GH = KL$ (*Arg. 5*). { P.46. B.1.
 6. $AB : CD = EF : GH$. { Cor.

Which was to be demonstrated.





PROPOSITION XXIII. THEOREM XVII.

EQUIANGULAR parallelograms (M & N) have to one another the ratio which is compounded of the ratios of their sides (AC, CD & EC, CG) about the equal angles.

Hypothesis.

The pgrs. M & N are equiangular,
so that $\angle ACD = \angle ECG$.

Thesis.

Pgr. M : Pgr. N $\equiv$ AC . CD : EC . CG.

Preparation.

1. Place AC & CG in the same straight line AG ;
therefore EC & CD are also in a straight line ED.
2. Complete the pgr. P.

P. 14. B. 1.
Pof. 1. B. 1.

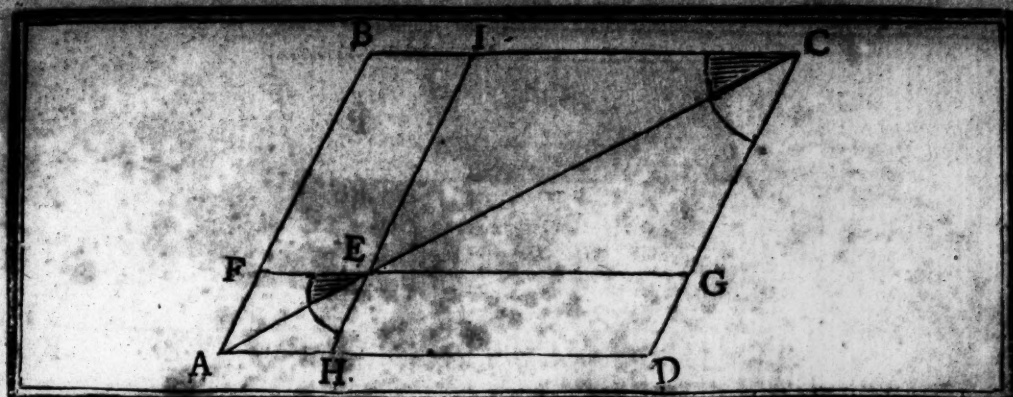
DEMONSTRATION.

BECAUSE the pgrs. M, P, N form a series of three magnitudes

1. $M : MP = N : NP.$ D. 5. B. 5.
2. And alternando $M : N = MP : NP.$ P. 16. B. 5.
3. Consequently the ratio of the first M to the last N, is compounded of the ratios M : P & P : N. D. 5. B. 6.
- But since $AC : CG = M : P$
& $DC : CE = P : N.$ } P. 1. B. 6.
4. The ratio of the sides AC : CG is the same as that of the pgrs. M : P ; & the ratio of the sides DC : CE, the same as that of the pgrs. P : N.
Since then the ratio of M : N is compounded of the ratios M : P, & P : N (Arg. 1).
5. This same ratio is compounded of their equals ; the ratios AC : CG & CD : EC, of the sides about the equal $\angle ACD, ECG.$
6. Consequently, $M : N = AC . CD : EC . CG.$ D. 5. B. 6.

Which was to be demonstrated.

Cor. The same truth is applicable to the triangles (ACD, ECG) having an angle (ACD) equal to an angle (ECG) : for the diagonals (AD, EG) divide the pgrs. into two equal parts (P. 34. B. 1).



PROPOSITION XXIV. THEOREM XVIII.

THE parallelograms (FH, IG) about the diagonal (AC) of any parallelogram (BD), are similar to the whole, and to one another.

Hypothesis.

- I. BD is a pgr.
- II. FH, IG are pgrs. about the diagonal AC.

Thesis.

- I. The pgrs. AFEH, EICG are similar to the pgr. ABCD.
- II. And similar to one another.

DEMONSTRATION.

BECAUSE FE is pple. to BC (*Hyp.* 1. & 2. & *P.* 30. *B.* 1).

1. The $\triangle AFE$ is equiang. to the $\triangle ABC$ in the order of the letters. In like manner, because HE is pple. to DC.

2. The $\triangle AHE$ is equiang. to the $\triangle ADC$, in the order of the letters.
3. Therefore the pgr. AFEH is also equiangular to the pgr. ABCD, in the order of the letters.

And because in the $\triangle AHE, ADC$, the $\angle AHE$ & $\angle D$ are equal (*Arg.* 2), as also in the $\triangle AFE, ABC$, the $\angle AFE$ & $\angle B$ (*Arg.* 1).

4. $AH : HE = AD : DC$ & $AF : EF = AB : CB$.

Moreover, because the $\angle AEH, ACD$, also $\angle FEA, BCA$ are equal (*Arg.* 1. & 2).

5. $HE : AE = DC : AC$ & $AE : EF = AC : CB$.

6. Therefore, *ex æquo*, $HE : EF = DC : CB$.

And because the $\angle EAH, EFA$ are common to the two $\triangle AHE, ADC$ & $\triangle AFE, ABC$.

7. $HA : EA = DA : CA$ & $EA : AF = CA : AB$.

8. Therefore, *ex æquo*, $HA : AF = DA : AB$.

9. Wherefore the pgrs. AFEH, ABCD have their angles equal, each to each in the order of the letters (*Arg.* 3); & the sides about the equal angles, proportionals (*Arg.* 4. 6. 8.).

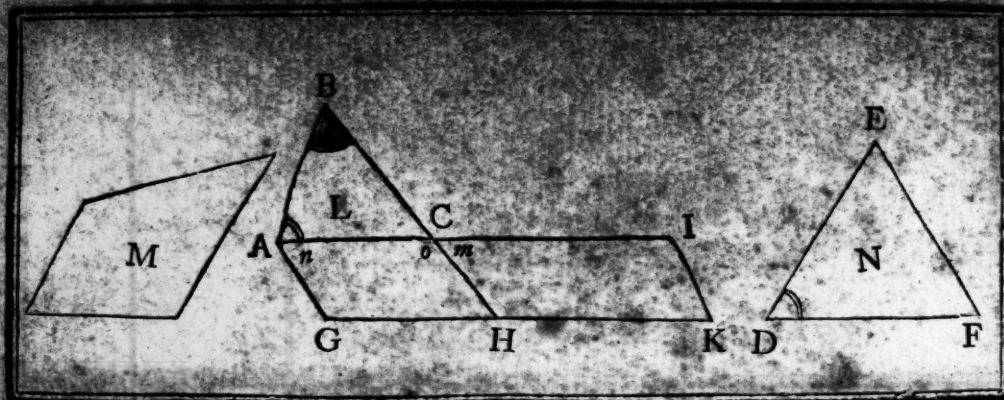
10. Consequently, those pgrs. are similar.

11. It may be demonstrated after the same manner that the pgrs. EICG, ABCD are similar.

Which was to be demonstrated. 1.

12. Consequently, the pgrs. AFEH, EICG are also similar to one another. *P.* 21. *B.* 1.

Which was to be demonstrated. 11.



PROPOSITION XXV. PROBLEM VII.

TO describe a rectilinear figure (N), which shall be similar to a given rectilinear figure (L), and equal to another (M).

Given.

- I. The rectilinear figure L.
- II. The rectilinear figure M.

Sought.

The rectil. figure N, similar to the rectil. figure L, & = to the rectil. figure M.

Resolution.

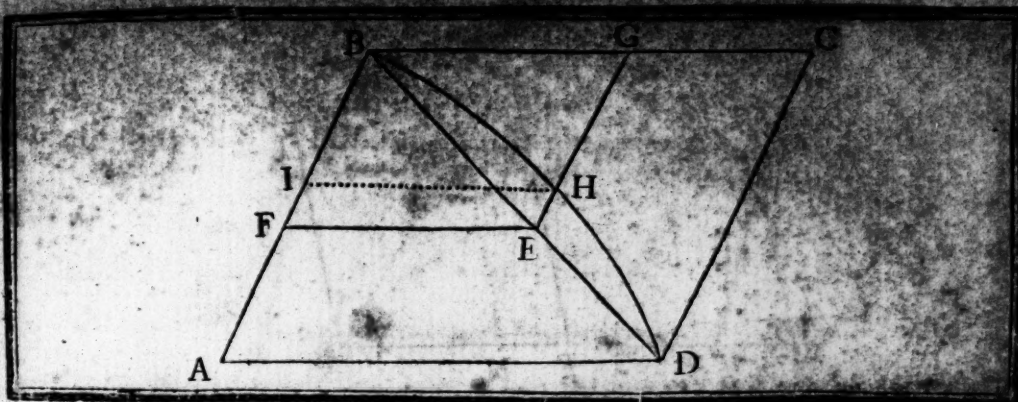
1. Upon the straight line AC, describe the pgr. AH = to the given rectilinear figure L. P.44. B.1.
2. And on the straight line CH a pgr. CK = to the given rectilinear figure M, having an $\angle m =$ to the $\angle n$. P.45. B.1.
3. Consequently, the sides AC, CI, & GH, HK will be in a straight line. P.14. 29. $\S$ 34. B.1.
4. Between AC, & CI find a mean proportional DF. P.13. B.6.
5. Upon this straight line DF, describe the rectil. figure N, similarly & similar to the rectilinear figure L. P.18. B.6.

DEMONSTRATION.

BECAUSE the pgrs. AH, CK have the same altitude (Ref. 2. $\S$ 3).

1. pgr. AH : pgr. CK = AC : CI. P. 1. B.6.
But the pgr. AH = rectil. L, & the pgr. CK = rectil. M (Ref. 1. $\S$ 2).
2. Consequently, L : M = AC : CI. P.11. B.5.
But AC : DF = DF : CI (Ref. 4.), & upon the straight lines AC, DF have been similarly described the similar figures L & N, (Ref. 5).
3. Consequently, L : N = AC : CI. { P.20. B.6.
4. Hence, L : N = L : M (Arg. 2). { Cor.
5. Wherefore, N = M. { P.11. B.5.
6. Therefore, there has been described a rectilinear figure N, similar to the rectilinear figure L (Ref. 5), & equal to the rectilinear figure M (Arg. 5). { P.14. B.5.

Which was to be done.



PROPOSITION XXVI. THEOREM XIX

IF two similar parallelograms (A C, F G) have a common angle (F B G), and be similarly situated, they are about the same diagonal (B D).

Hypothesis.

- I. A C is a pgr. & B D its diagonal.
- II. F G is a pgr. similar to A C & having an $\angle$ F B G common with A C.

Thesis.

The pgr. F G is placed about the diagonal B D of the pgr. A C.

DEMONSTRATION.

If not, let another line B H D different from B E D be the diagonal of the pgr. A C, cutting the side G E in the point H.

Preparation.

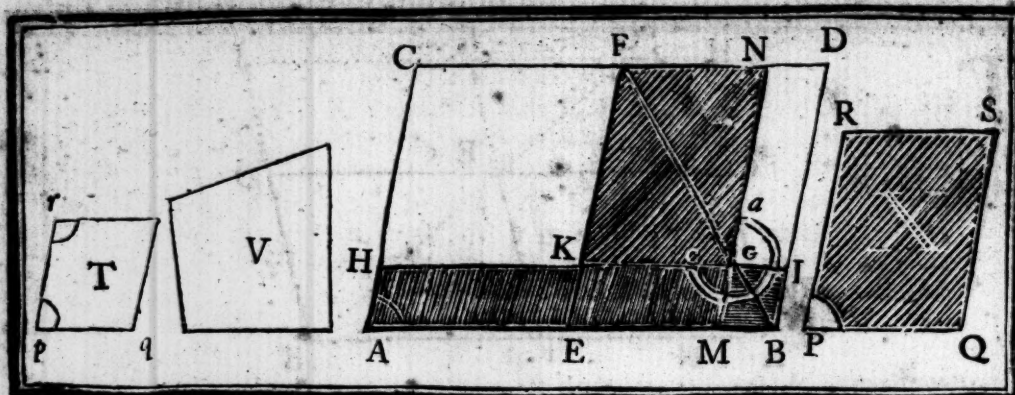
Thro' the point H draw H I pple. to C B or D A.

P. 31. B. 1.

THE pgrs. A C, I G being about the same diagonal B H D, & $\angle$ F B G being common to the two pgrs. (*Sup. & Prep.*).

1. The pgr. A C is similar to the pgr. I G. P. 24. B. 6.
2. Consequently, $CB : BA = GB : BI$. D. 1. B. 6.
But the pgrs. A C & F G being also similar, & $\angle$ B common to the two pgrs. (*Hyp. 2*).
3. It follows, that $CB : BA = GB : BF$. D. 1. B. 6.
4. Consequently, $GB : BI = GB : BF$. P. 11. B. 5.
5. Therefore, $BI = BF$. P. 14. B. 5.
6. Which is impossible. Ax. 8. B. 1.
7. Hence, a line B H D, different from the line B E D is not the diagonal of the pgr. A C.
8. Consequently, the line B E D is the diagonal, & the pgr. F G is placed about it.

Which was to be demonstrated.



PROPOSITION XXVIII. PROBLEM VIII.

TO a given straight line (A B) to apply a parallelogram (A G) equal to a given rectilineal figure (V), and deficient by a parallelogram (M I), similar to a given parallelogram (T); but the given rectilineal figure (V) must not be greater than the parallelogram (A F) applied to half of the given line, having its defect (E D) similar to the given parallelogram (T).

Given.

- I. The straight line A B, & the pgr. T.
- II. The rectilineal figure V, not $>$ pgr. E D, similar to T, applied to A E, half of A B.

Sought.

The construction of a pgr. A G, applied to A B, which may be $=$ to V, & deficient by a pgr. M I similar to T.

Resolution.

1. Divide A B into two equal parts in E. P.10. B.1.
2. Upon E B describe a pgr. E D, similar to the pgr. T, & similarly situated. P.18. B.6.
3. Complete the pgr. A D. P.21. B.1.

The pgr. A F will be either $=$ or $>$ V; since it cannot be $<$ V, by the determination.

C A S E I. If A F be $=$ V.

There has been applied to A B, a pgr. A F $=$ to the rectilineal V, & deficient by a pgr. E D similar to the pgr. T.

C A S E II. If A F be $>$ V, & consequently E D $>$ V, A F being $=$ E D. P.36. B.1.

4. Describe a pgr. X similar to the pgr. T (or to the pgr. E D) (Ref. 2), & similarly situated, & equal to the excess of E D, or its equal A F, above V (i. e. make $X = E D - V$), & let R S, F D & R P, F E be the homologous sides. P.45. B.1.

And because X is simil. to E D & $<$ E D; (E D being $= V + X$).

The sides R S, R P are $<$ their homologous sides F D, F E.

5. Make then F N = R S, & F K = R P. P. 3. B.1.
6. And complete the pgr. N K. P.31. B.1.

DEMONSTRATION.

THE pgr. KN, being equal & similar to the pgr. X (*Ref.* 4, 5, & 6); which is itself similar to the pgr. ED (*Ref.* 4).

1. The pgr. KN is similar to the pgr. ED. P. 21. B. 6.
2. Wherefore those two pgrs. KN, ED, are about the same diagonal. P. 26. B. 6.
Draw this diagonal FGB, & complete the description of the figure.
Since then the pgr. MI, is also about the same diagonal FB.
3. It is similar to the pgr. ED. P. 24. B. 6.
4. Consequently similar to the pgr. T (*Ref.* 2). P. 21. B. 6.
But the pgr. DG being $\equiv$ to the pgr. EG (*P.* 43. B. 1), if the common pgr. MI be added on both sides.
5. The pgr. MD will be $\equiv$ to the pgr. EI. Ax. 2. B. 1.
But the pgr. AK being also $\equiv$ to the pgr. EI (*P.* 36. B. 1).
6. The pgr. MD is $\equiv$ to the pgr. AK. Ax. 1. B. 1.
And adding to both sides the common pgr. EG.
7. The gnomon *abc* will be $\equiv$ to the pgr. AG. Ax. 2. B. 1.
But the pgr. ED being $\equiv$ to the figures V & X taken together (*Ref.* 4), or to V & KN, since X is $\equiv$ KN (*Ref.* 5, & 6); if KN be taken away from both sides.
8. The remaining gnomon *abc* $\equiv$ V. Ax. 3. B. 1.
9. Consequently, the pgr. AG is $\equiv$ to V (*Arg.* 7).
But pgr. AG has for defect pgr. MI, similar to pgr. T (*Arg.* 4).
10. Therefore, there has been applied to AB a pgr. AG $\equiv$ V, deficient by a pgr. MI, similar to the pgr. T. D. 8. B. 6.

Which was to be done.

R E M A R K.

SEVERAL Editors of New Elements of Euclid have left out this proposition & the following, as useless; because they were ignorant of their use. They are notwithstanding absolutely necessary for the analysis of the ancients, corresponding to the analytic resolution of equations of the second degree.

This XXVIIIth proposition corresponds to the case, where the last term of the equation is positive.

For reducing the given space V to an equiangular pgr. T; let $V = nl$; the ratio of the sides QP, PR of the pgr. X (or T), $m : n$; $AB = a$, $AM = x$ & $MB = a - x$. Consequently, since the defect MI, should be similar to the pgr. T or to the pgr. X.

$$QP : PR = BM : MG \quad (D. 1. B. 6).$$

$$m : n = a - x : \frac{n}{m} (a - x).$$

And because the pgr. GA ($\equiv$ MA. MG) should be equal to the given space V ($= nl$), there results the following equation (*P.* 23. B. 6).

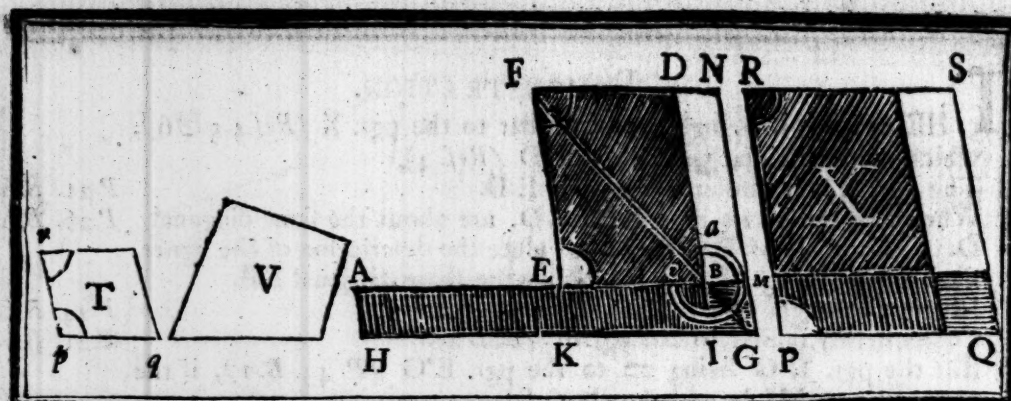
$$\frac{n}{m} (a - x) x = V \text{ or } nl.$$

Which is reduced to $\frac{n}{m} xx - \frac{n}{m} ax + V = 0$.

Or substituting for V its value, & multiplying by m & dividing by n.

$$xx - ax + ml = 0.$$

G g



PROPOSITION XXIX. PROBLEM IX.

TO a given straight line (A B), to apply a parallelogram (A G), equal to a given rectilinear figure (V), exceeding by a parallelogram (M I), similar to another given (T).

Given.

- I. The straight line A B, & the pgr. T.
- II. The rectilinear figure V.

Sought.

The construction of a pgr. A G, applied to A B, equal to the rectilinear figure V, & having for excess a pgr. M I, similar to T.

Resolution.

1. Divide A B into two equal parts in E. P.10. B.1.
2. Upon E B, describe a pgr. E D, similar to the pgr. T, & similarly situated.
3. Describe a pgr. X (or P S) = V + E D, similar & similarly situated to the pgr. T; & consequently similar to the pgr. E D (Ref. 2. P. 21. B. 6); & let the sides RS, FD, RP, FE be homologous. P.18. B.1.
4. Since X, (as = V + E D), is > E D; the side R S is > F D, & the side R P > F E; wherefore having produced F D & F E, make F N = R S & F K = R P; & complete the pgr. F K G N, which will be equal & similar to the pgr. X. P.31. B.1.

DEMONSTRATION.

THE pgr. K N being equal and similar to the pgr. X, which is itself similar to the pgr. E D (Ref. 3).

1. The pgr. K N is similar to the pgr. E D. P.21. B.6.
2. Wherefore those two pgrs K N, E D are about the same diagonal. P.26. B.6.
Draw this diagonal F B G, & complete the description of the figure. Since X is = to V + E D; & X = pgr. K N (Ref. 3. & 4).
3. The pgr. K N = V + E D. Ax.1. B.1.
Therefore taking away from both sides the common pgr. E D.
4. The remaining gnomon a b c is = to the rectilinear figure V. Ax.3. B.1.
But because A E = E B (Ref. 1).
5. The pgr. A K = the pgr. E I. P.36. B.1.
6. Consequently, this pgr. A K is = to the pgr. N B. P.43. B.1.

- Therefore adding to both sides the common pgr. M K.
7. There will result the pgr. A G $\equiv$ to the gnomon $a b c$. Ax.2. B.1.
- But the gnomon $a b c$ is $\equiv$ to the rectilineal figure V (Arg. 4).
8. Consequently, the pgr. A G is $\equiv$ to the rectilineal figure V. Ax.1. B.1.
- Since then this pgr. A G has for excess the pgr. M I, similar to the pgr. E D (P. 24. B. 6.); & consequently similar to the pgr. T (Ref. 2. P. 21. B. 6).
9. There has been applied to A B, a pgr. A G $\equiv$ to the rectilineal figure V, having for excess a pgr. M I, similar to the pgr. T.

Which was to be done.

R E M A R K.

IF as in the foregoing case A B be made $\equiv a$, the given square V (reduced to a pgr. equiangular to the pgr. T) $\equiv n l$; the ratio of the sides Q P, P R of the pgr. X (which is the same as that of the sides of the pgr. T) $m : n$; & A M $\equiv x$, consequently, M B $\equiv x - a$. there will result an equation of the same kind.

For since the defect M I should be similar to the pgr. T or X, we will have as before the following proportion.

$$Q P : P R = M B : M G \text{ (D. 1. B. 6.)}$$

$$m : n = x - a : \frac{n}{m} (x - a).$$

And because the pgr. A G ($\equiv$ A M. M G) should be equal to the given space V ($\equiv n l$), there results the following equation,

$$\frac{n}{m} (x - a) x = V \text{ (P. 23. B. 6.)}$$

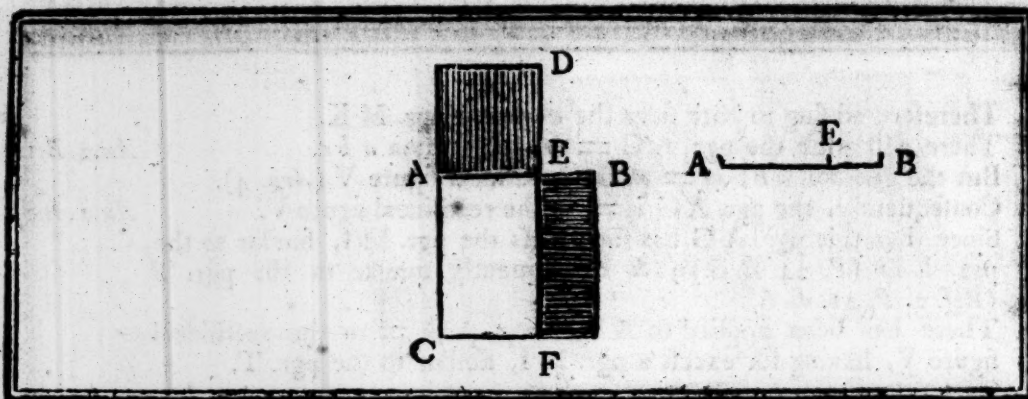
which is reduced to $\frac{n}{m} x x - \frac{n}{m} a x - V = 0$.

And substituting for V its value $n l$, then multiplying by m & dividing by n .

$$x x - a x - m l = 0.$$

From whence it appears that the XXIXth Prop. corresponds to the Case, in which the last term of the equation is negative.





PROPOSITION XXX. PROBLEM X.

To cut a given straight line (A B) in extreme and mean ratio (in E).

Given.

The straight line A B.

Sought.

The point E, such that
 $BA : AE = AE : BE$.

Resolution.

1. Upon the straight line A B describe a square B C. P. 46. B. 1.
2. Apply to the side C A, a pgr. C D = to the square B C, P. 29. B. 6.
 whose excess A D is similar to B C, which will consequently be a square.

DEMONSTRATION.

BECAUSE $BC = CD$ (Ref. 2); by taking away the common rgle. C E from each.

1. The remainder $BF = AD$. Ax. 3. B. 1.
 But BF is also equiangular with A D (P. 15. B. 1).
2. Therefore their sides F E, E B, E D, A E about the equal angles, are reciprocally proportional, that is $FE : ED = AE : EB$. P. 14. B. 6.
 But F E is $= CA$ (P. 34. B. 1), or $= BA$, & $ED = AE$. D. 30. B. 1.
3. Wherefore, $BA : AE = AE : EB$. P. 7. & 11. B. 5.
 But because $BA > AE$ (Ax. 8. B. 1).
4. The straight line A E is $> EB$. P. 14. B. 5.
5. Consequently, the straight line A B is cut in extreme & mean ratio in E.

Which was to be done.

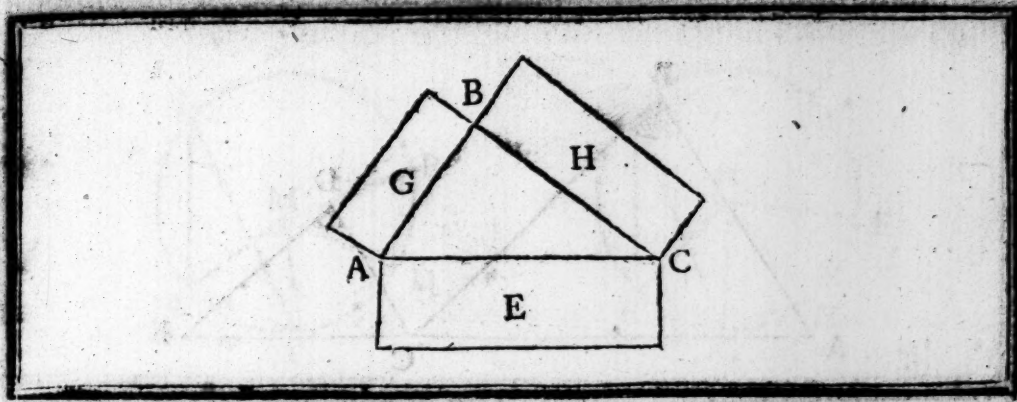
Otherwise.

Divide B A in E, so that the rect. A B. B E be $=$ to the $\square$ of A E. P. 11. B. 2.

DEMONSTRATION.

- B**ECAUSE $BA \cdot BE$ is $=$ to the $\square$ of A E (Ref.).
1. $BA : AE = AE : BE$. P. 17. B. 6.
 And because $BA > AE$ (Ax. 8. B. 1).
 2. The straight line A E is $> BE$. P. 14. B. 5.
 3. Consequently, the straight line A B is cut in extreme & mean ratio in E. D. 3. B. 6.

Which was to be done.



PROPOSITION XXXI. THEOREM XXI.

IN every right angled triangle (ABC), the rectilineal figure (E) described upon the hypotenuse (AC) is equal to the sum of the similar and similarly described figures (G & H), upon the sides (AB , BC) containing the right angle.

Hypothesis.

- I. ABC is a rgle. Δ in B .
- II. The fig. E is described upon the hypoth. AC of this Δ .
- III. And the figures G & H are similar to E , & similarly described upon the two other sides AB , BC .

Thesis.

$$\text{fig. } E = \text{fig. } G + H.$$

DEMONSTRATION.

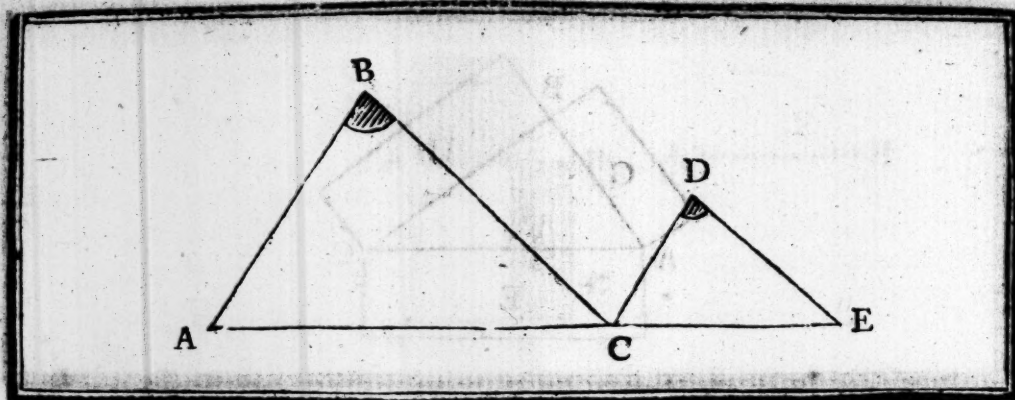
BECAUSE the figures E , G , H are similar, & similarly described upon the homologous sides AC , AB , BC (*Hyp.* 3).

1. $G : E = \square \text{ of } AB : \square \text{ of } AC.$ { *P. 20. B. 6.*
- And $H : E = \square \text{ of } BC : \square \text{ of } AC.$ { *Cor. 2.*
2. Consequently, $G + H : E = \square \text{ of } AB + \square \text{ of } BC : \square \text{ of } AC.$ { *P. 24. B. 5.*
- But because the ΔABC is rgle. in B (*Hyp.* 1).
3. The $\square \text{ of } AB + \square \text{ of } BC$ is = to the $\square \text{ of } AC.$ { *P. 47. B. 1.*
4. Therefore, the figure E is = to the figures $G + H.$ { *P. 16. B. 5.*

Cor.

Which was to be demonstrated.





PROPOSITION XXXII. THEOREM XXII.

IF two triangles (ABC , CDE), which have two sides (AB , BC) of the one, proportional to two sides (CD , DE) of the other, be joined at one angle (C), so as to have their homologous sides (AB , CD , BC , DE) parallel to one another, the remaining sides (AC , CE) shall be in a straight line.

Hypothesis.

- I. $AB : BC = CD : DE$.
- II. The $\triangle ABC$, CDE , are joined in C .
- III. So that AB is *plle.* to CD , & BC *plle.* to DE .

Thesis.

The remaining sides AC , CE of those $\triangle$ are in a straight line AE .

DEMONSTRATION.

BECAUSE the *plles.* AB , CD are cut by the straight line BC , & the *plles.* BC , DE by the straight line DC (*Hyp.* 2).

1. The $\angle B$ is $=$ to $\angle BCD$ & $\angle D$ is $=$ to $\angle BCD$.

P. 29. B. 1.

2. Consequently, $\angle B$ is $=$ to $\angle D$.

Ax. 1. B. 1.

And besides $AB : BC = CD : DE$ (*Hyp.* 1).

3. The $\triangle ABC$, CDE are equiangular.

P. 6. B. 6.

4. Therefore, $\angle A$ is $=$ to $\angle DCE$, being opposite to the homologous sides BC , DE .

Adding then to both sides $\angle B$, or its $=$ $\angle BCD$ (*Arg.* 1), together with the common $\angle BCA$.

5. The $\angle A + B + BCA$ will be $=$ to the $\angle DCE + BCD + BCA$. Ax. 2. B. 1.

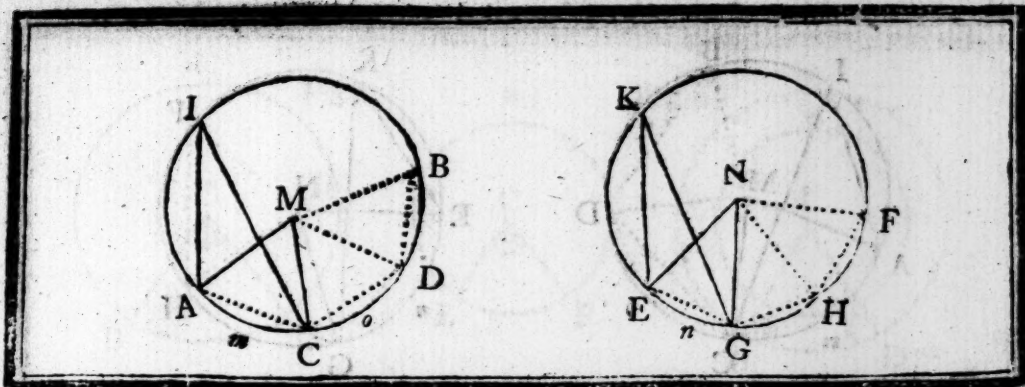
But the $\angle A + B + BCA$ are $=$ to $2\angle$ (P. 32. B. 1).

6. Consequently the $\angle DCE + BCD + BCA$ are also $=$ to $2\angle$. Ax. 1. B. 1.

7. Wherefore the straight lines AC , CE are in the same straight line AE .

P. 14. B. 1.

Which was to be demonstrated.



PROPOSITION XXXIII. THEOREM XXIII.

IN equal circles (AIBC, EKFG), angles, whether at the centres or circumferences (AMC, ENG or AIC, EKG), as also the sectors (AMCm, ENGn) have the same ratio with the arches (AmC, EnG) on which they stand, have to one another.

Hypothesis.

Thesis.

- I. The $\odot$ AIBC, EKFG are $\equiv$ to one another. I. $\angle AMC : \angle ENG = AmC : EnG$.
- II. The $\angle$ s at the centers AMC, ENG & the $\angle$ s at the $\odot$ AIC, EKG stand upon the arches AmC, EnG. II. $\angle AIC : \angle EKG = AmC : EnG$.
- III. Sect. AMCm : Sect. ENGn $\equiv$ AmC : EnG.

Preparation.

1. Join the chords AC, EG. Pos. 1. B. 1.
2. In the $\odot$ AIBC, draw the chords CD, DB &c, each $\equiv$ to AC, & in the $\odot$ EKFG a pareil number of cords GH, HF &c, each $\equiv$ to EG. P. 1. B. 4.
3. Draw MD, MB &c, also NH, NF &c. Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE on one side the cords AC, CD, DB, & on the other the cords EG, GH, HF are $\equiv$ to one another (Prep. 2).

1. The arches AmC, CD, DB are all equal on the one side, as the arches EnG, GH, HF are on the other. P. 28. B. 3.
2. Consequently, the $\angle$ AMC, CMD, DMB &c, & ENG, GNH, HNF &c, are also $\equiv$ to one another, on one side & the other.
3. Wherefore, the $\angle$ AMB & the arch ACDB, are equimult. of the $\angle$ AMC & of the arch AmC.
4. Likewise, $\angle$ ENF & the arch EGHF are equimult. of $\angle$ ENG, & of the arch EnG.

But because the $\odot$ AIBC, EKFG are equal (Hyp. 1).

According as the arch ACDB is $>$, $=$ or $<$ the arch EGHF; $\angle$ AMB is also $>$, $=$ or $<$ $\angle$ ENF.

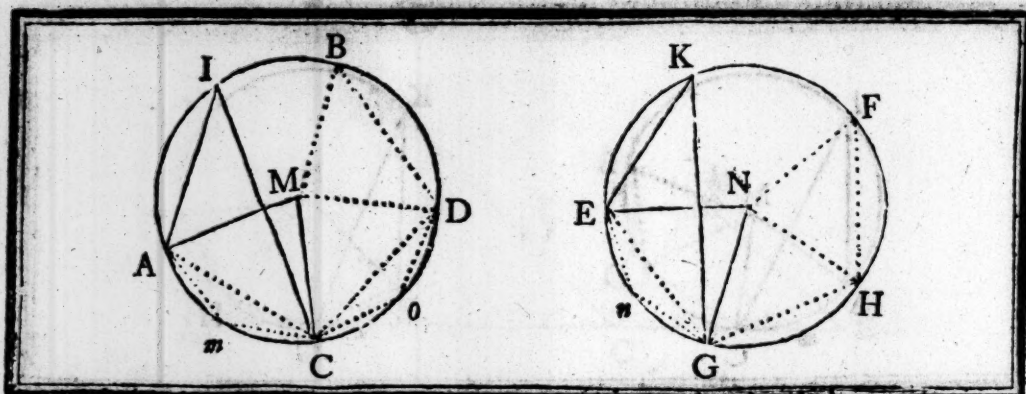
5. Wherefore, $\angle AMC : \angle ENG = AmC : EnG$. P. 27. B. 3.

Which was to be demonstrated. 1.

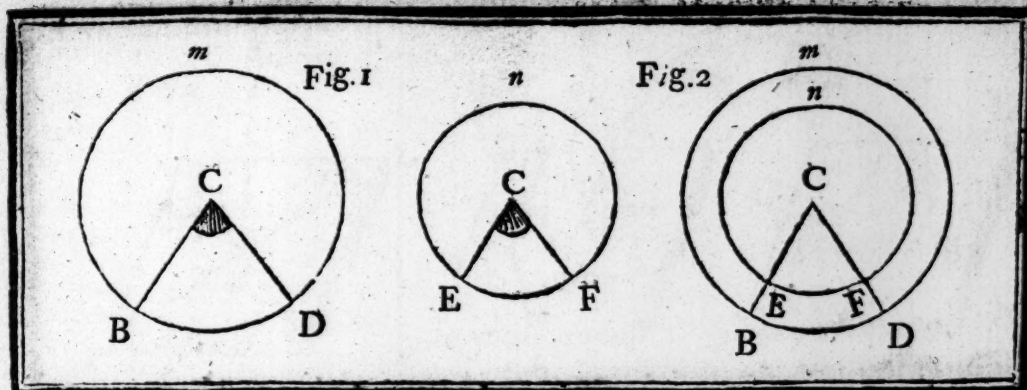
Moreover, $\angle$ AMC being double of $\angle$ AIC, & $\angle$ ENG double of $\angle$ EKG (P. 20. B. 3).

6. It follows that $\angle AMC : \angle ENG = \angle AIC : \angle EKG$. P. 15. B. 5.
7. Consequently, $\angle AIC : \angle EKG = AmC : EnG$. P. 11. B. 5.

Which was to be demonstrated. 11.



- PREP. 4.** In the arches AC , CD , take the points m & o , & join $A m$, $C m$; $C o$, $D o$ & c. Pos. 1. B. 1.
- Since then the two sides AM , MC are $=$ to the two sides CM , MD (*D. 15. B. 1.*), & the $\angle AMC$, CMD are equal (*Arg. 2.*).
8. The base AC is $=$ to the base CD , & the $\triangle AMC$ $=$ to the $\triangle CMD$. P. 4. B. 1.
- Moreover, because the arch $A m C$ is $=$ to the arch $C o D$ (*Arg. 1.*).
9. The complement $AIBDC$ of the first is $=$ to the complement $CAIBD$ of the second. Ax. 3. B. 1.
10. Wherefore $\angle A m C$ is $=$ to $\angle C o D$. P. 27. B. 3.
11. Therefore the segment $A m C$ is similar to the segment $C o D$. Ax. 2. B. 3.
- Besides they are subtended by equal cords (*Arg. 8.*).
12. Consequently, the segment $A m C$ is $=$ to the segment $C o D$. P. 24. B. 3.
- But since the $\triangle AMC$ is also $=$ to the $\triangle CMD$ (*Arg. 8.*).
13. The sector $AMC m$ is $=$ to the sector $CMD o$. Ax. 2. B. 1.
- Likewise, the sector DMB is equal to each of the two foregoing $AMC m$, $CMD o$.
14. Therefore the sectors AMC , CMD , DMB are $=$ to one another.
15. It is demonstrated after the same manner, that the sectors ENG , GNH , GNF are $=$ to one another.
16. Wherefore, the sect. $AMBDC$, & the arch $ACDB$ are equimult. of the sect. $AMC m$, & of the arch $A m C$, the sect. $ENFHG$, & the arch $EGHF$ are equimult. of the sect. $ENG n$, & of the arch EnG . But because the $\odot AIBC$, EKG are equal (*Hyp. 1.*).
- If the arch $ACDB$ be $=$ to the arch $EGHF$, the sect. $AMBDC$ is also $=$ to the sect. $ENFHG$, as is proved by the reasoning employed in this third part of the demonstration to arg. 12 inclusively.
- And, if the arch $ACDB$ be $>$ the arch $EGHF$, the sect. $AMBDC$ is also $>$ the sect. $ENFHG$, & if less, less.
- Since then there are four magnitudes, the two arches $A m C$, EnG , & the two sect. $AMC m$, $ENG n$. And of the arch $A m C$, & sect. $AMC m$, the arch $ACDB$ & sect. $AMBDC$ are any equimult. whatever; & of the arch EnG , & sector $ENG n$, the arch $EGHF$, & the sect. $ENFHG$ are any equimult. whatever.
- And it has been proved that, if arch $ACDB$ be $>$, $=$ or $<$ the arch $EGHF$, sect. $AMBDC$ is also $>$, $=$ or $<$ the sect. $ENFHG$.
17. It follows, that sect. AMC : sect. $ENG = AMC : EnG$. D. 5. B. 5.
- Which was to be demonstrated. III.



COROLLARY I.

THE angle at the center, is to four right angles, as the arch upon which it stands, is to the circumference.

For (Fig. 1), $\angle BCD : 4\text{R} = BD : \text{circumference}$ to a quadrant of the $\odot$.

Wherefore, quadrupling the consequents.

$$\angle BCD : 4\text{R} = BD : \odot.$$

P.15. B.5.

COROLLARY II.

THE arches EF, BD of unequal circles, are similar, if they subtend equal angles C & C, either at their centers, or at their $\odot$ (Fig. 2).

For $EF : \odot E n F = \angle ECF : 4\text{R}$.

But $\angle BCD$ or $\angle ECF : 4\text{R} = BD : \odot B m D$. } (Cor. 1.)

Consequently, $EF : \odot E n F = BD : \odot B m D$.

P.11. B.5.

Therefore, the arches EF, BD are similar.

COROLLARY III.

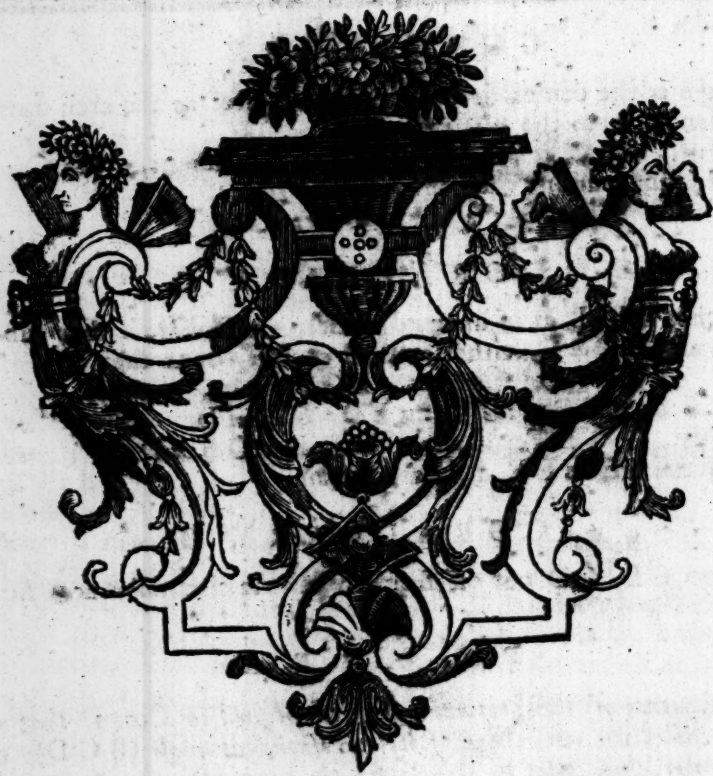
TWO rays CB, CD cut off from concentric circumferences similar arches EF, BD (Fig. 2).

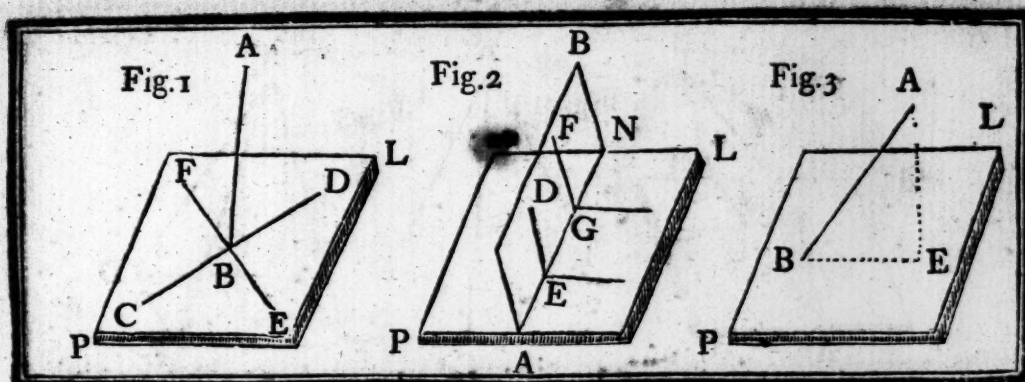
REMARK.

IT is in consequence of the proportionality established in Cor. 1. that an arch of a circle (BD) is called the MEASURE of its correspondent angle (BCD); that is of the angle at the center, subtended by this arch; the circumference of a circle being the only curve, whose arches, increase or diminish in the ratio of the correspondent angles, about the same point.

The whole circle is conceived to be divided into 360 equal parts, which are called DEGREES; and each of these degrees into 60 equal parts, called MINUTES; and each minute into 60 equal parts, called SECONDS &c. in consequence of this hypothesis, & the correspondence established between the arches, & the angles, we are obliged to conceive all the angles about a point in the same plane (that is the sum of 4R), as divided into 360 equal parts, in such a manner, that the angle of a degree is no more than the 360th part of 4R , or the 90th of a R , & consequently, of an amplitude to be subtended by the 360th part of the circumference.

H h





DEFINITIONS.

I.

A SOLID is that which hath length, breadth and thickness.

II.

That which bounds a Solid is a superficies.

III.

A straight line (A B) is perpendicular to a plane (P L) (Fig. 1), if it be perpendicular to all the lines (C D & F E), meeting it in this plane; that is, The line (A B) will be perpendicular to the plane (P L), if it be perpendicular to the lines (C D & F E) which being drawn in the plane (P L) pass through the point (B), so that the angles (A B C, A B D, A B E & A B F) are right angles.

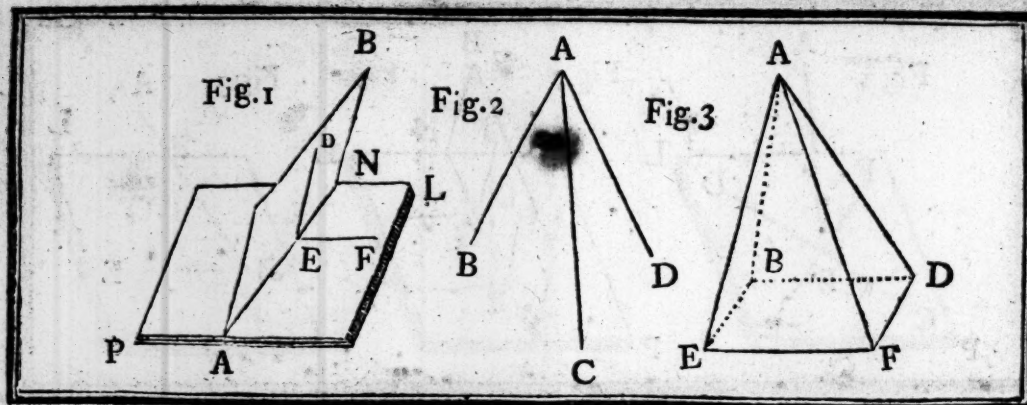
IV.

A plane (A B) (Fig. 2.) is perpendicular to a plane (P L), if the lines (D E & F G) drawn in one of the planes (as in A B) perpendicularly to the common section (A N) of the planes, are also perpendicular to the other plane (P L).

The common section of two planes is the line which is in the two planes: as the line (A N), which is not only in the plane (A B), but also in the plane (P L); therefore if the lines D E & F G drawn perpendicular to A N in the plane A B are also perpendicular to the plane P L; the plane A B will be perpendicular to the plane P L.

V.

The inclination of a straight line (A B) to a plane, (Fig. 3.) is the acute angle (A B E), contained by the straight line (A B), and another (B E) drawn from the point (B), in which A B meets the plane (P L), to the point (E) in which a perpendicular (A E) to the plane (P L) drawn from any point (A) of the line (A B) above the plane, meets the same plane.



DEFINITIONS.

VI.

THE inclination of a plane (A B) (Fig. 1) to a plane (P L); is the acute angle (D E F) contained by two straight lines (E D & E F) drawn in each of the planes, (that is D E in the plane A B & E F in the plane P L) from a same point (E), perpendicular to their common section (A N).

VII.

Two planes are said to have the same or a like inclination to one another, which two other planes have, when their angles of inclination are equal.

VIII.

Parallel planes are such which do not meet one another tho' produced.

IX.

Similar solid figures are those which are contained by the same number of surfaces, similar and homologous.

X.

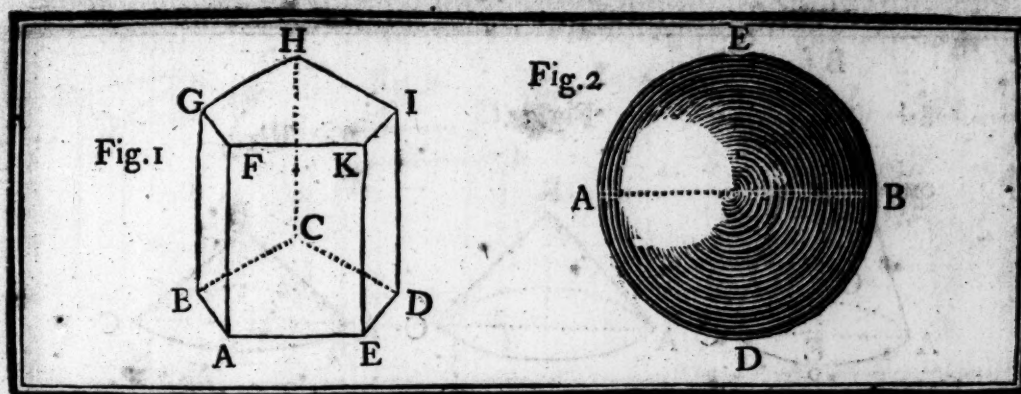
Equal & similar Solids are those which are contained by the same number of equal, similar and homologous surfaces.

XI.

A solid Angle (A) is that which is made by the meeting of more than two plane angles (B A C, C A D & B A D), which are not in the same plane, in one point (A).

XII.

A Pyramid (E B A D F) (Fig. 3.) is a solid contained by more than two triangular planes (B A D, B A E &c.) having the same vertex (A), and whose bases (viz. the lines E B, B D &c.) are in the same plane (E B D F).



DEFINITIONS,

XIII.

A Prism is a solid figure (AHE) (Fig. 1.) contained by plane figures, of which two that are opposite (viz. GHIKF & BCDA) are equal similar, and parallel to one another; and the other sides (as GA, AK, KD, &c.) are parallelograms.

If the opposite parallel planes be triangles, the prism is called a triangular one, (and it is only of those prisms that Euclid treats in the XIth and XIIth Book), if the opposite planes are polygons, they are called polygon prisms.

XIV.

A Sphere is a solid figure (AEBD) (Fig. 2.) whose surface is described by the revolution of a semicircle (AEB) about its diameter, which remains unmoved.

XV.

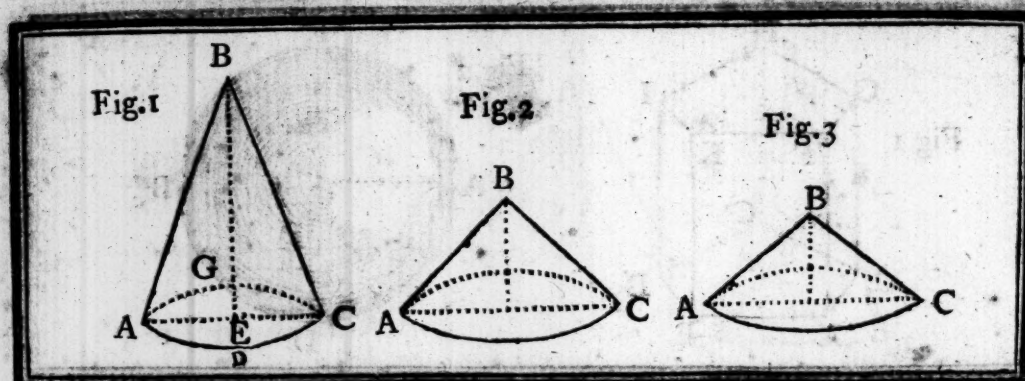
The Axis of a Sphere is the fixed diameter (AB) about which the semicircle revolves whilst it describes the superficies of the sphere.

XVI.

The Center of a Sphere is the same with that of the semicircle which described its superficies.

XVII.

The Diameter of a Sphere is any straight line which passes thro' the center, and is terminated both ways by the superficies of the sphere.



DEFINITIONS.

XVIII.

A *Cone* is a solid figure (ABCD) (Fig. 1, 2, & 3.) described by the revolution of a right angled triangle (ABE), about one of the sides (BE) containing the right angle, which side remains fixed.
 If the fixed side (BE) of the triangle (ABE) (Fig. 2.) be equal to the other side (AE) containing the right angle, the cone is called a right angled cone; if (BE) be less than (AE) (Fig. 3.) an obtuse angled, and if (BE) be greater than (AE) (Fig. 1.) an acute angled cone.

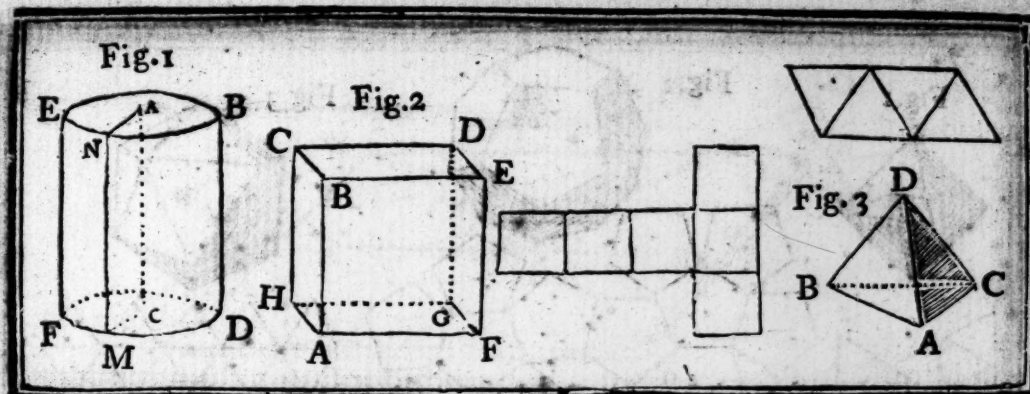
XIX.

The Axis of a Cone is the fixed straight line (BE) about which the triangle (ABE) revolved whilst it described the superficies of the cone.

XX.

The Base of a Cone is the circle (AGCD) (Fig. 1.) described by that side (BE) containing the right angle, which revolves.





DEFINITIONS.

XXI.

A *Cylinder* is a solid figure (EBDF) (Fig. 1.) described by the revolution of a right angled parallelogram (ANMC) about one of its sides (AC) which remains fixed.

XXII.

The Axis of a Cylinder is the fixed straight line (AC) about which the parallelogram revolved, whilst it described the superficies of the cylinder.

XXIII.

The Bases of a Cylinder (viz. ENB, & FMD) are the circles described by the two opposite sides (NA, MC) of the parallelogram, revolving about the points A & C.

XXIV.

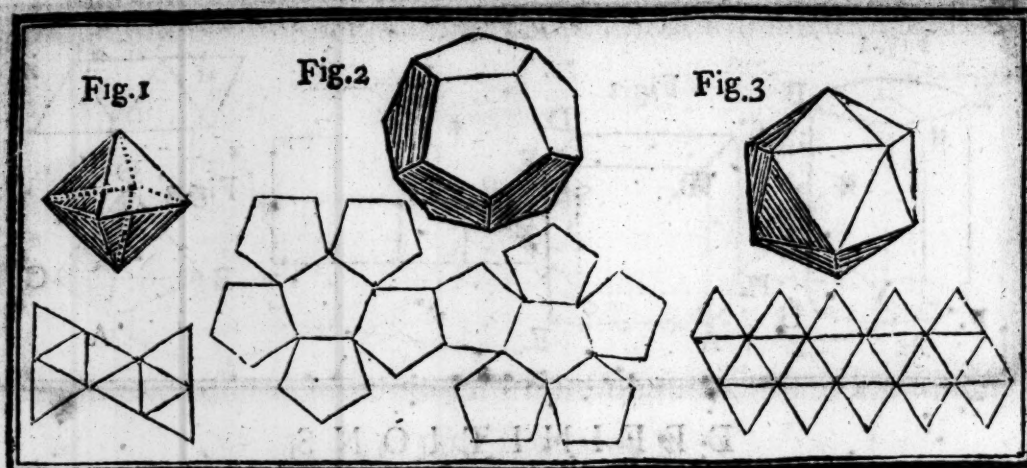
Similar Cones and Cylinders are those which have their axes and the diameters of their Bases proportionals.

XXV.

A Cube or Exahedron (Fig. 2.) is a solid figure contained by six equal squares.

XXVI.

A Tetrahedron is a pyramid (BDCA) (Fig. 3.) contained by four equal and equilateral triangles (viz. $\triangle BDC$, $\triangle BAD$, $\triangle ADC$ & $\triangle BAC$).



DEFINITIONS.

XXVII.

A *Octahedron* (Fig. 1.) is a solid figure contained by eight equal and equilateral triangles.

XXVIII.

A *Dodecahedron* (Fig. 2.) is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

A *Icosahedron* (Fig. 3.) is a solid figure contained by twenty equal and equilateral triangles.

XXX.

A *Parallelepiped* is a solid figure contained by six quadrilateral figures whereof every opposite two are parallel.

XXXI.

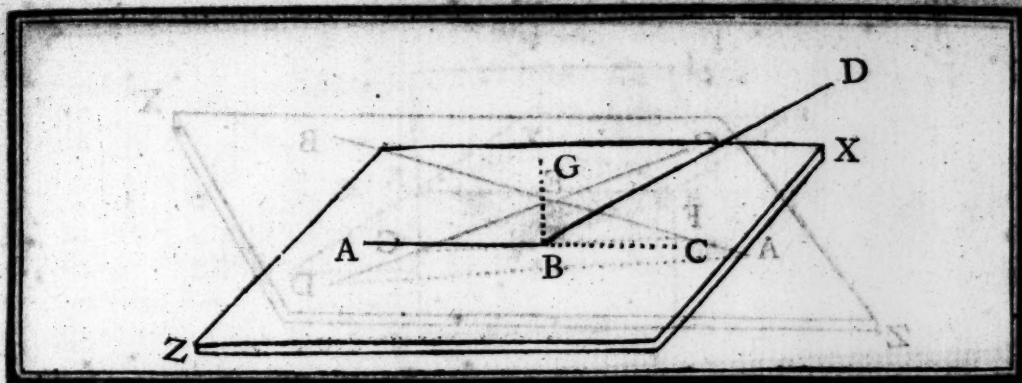
A *Solid is said to be inscribed in a Solid*, when all the angles of the inscribed solid touch the angles, the sides, or the planes of the solid in which it is inscribed.

XXXII.

A *Solid is said to be circumscribed about a Solid*, when the angles, the sides, or the planes of the circumscribed solid touch all the angles of the inscribed solid.

EXPLICATION of the SIGNS.

∞ Similar.
 ▭ Parallelepiped.



PROPOSITION 1. THEOREM 1.

ONE part (A B) of a straight line cannot be in a plane (Z X); and another part above it.

Hypothesis.

A B is a part of a straight line situated in the plane Z X.

Thesis.

Another part of this straight line (as B C) will be in the same plane Z X.

DEMONSTRATION.

If not

It will be above the plane as B D is.

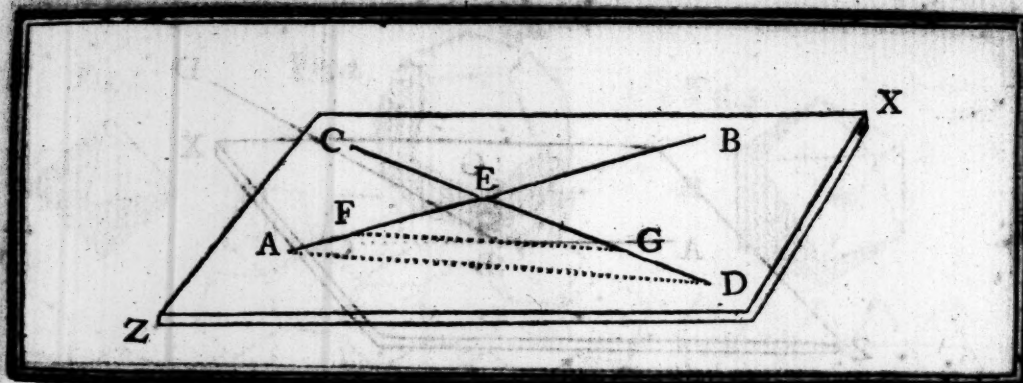
Preparation.

1. At the point B in A B erect in the plane Z X the $\perp$ G B. } P.11. B.1.
2. At the point B in B G erect in the plane Z X the $\perp$ B C. }

BECAUSE $\angle$ A B G is a $\perp$, likewise $\angle$ G B C, & they meet in the same point B.

1. The lines A B & B C are in the same straight line A C. } P.14. B.1.
But the line B D is a part of the straight line above the plane (Sup.).
2. Therefore the lines B D & B C have a common segment A B.
3. Consequently, $\angle$ D B G = $\angle$ G B A = G B C, that is, the part = to the whole. } Ax.10.B.1.
4. Which is impossible. } Ax. 8.B.1.
5. Therefore, B D cannot be a part of the straight line A B (Arg. 1).
And as the same demonstration may be applied to any other part of B C.
6. It follows, that all the parts of a straight line are in the same plane.

Which was to be demonstrated.



PROPOSITION II. THEOREM II.

TWO straight lines which cut one another in (E); are in one plane (Z X) and three straight lines which constitute a triangle (EAD) are in the same plane (Z X).

Hypothesis.

- I. AB & CD cut one another in E.
II. EAD is a Δ .

Thesis.

- I. AB & CD, are in the same plane.
II. The whole Δ EAD is in the plane Z X.

DEMONSTRATION.

If not,

The lines AB & CD are not in the same plane,
Likewise a part of the Δ EAD, as AFGD.

Preparation.

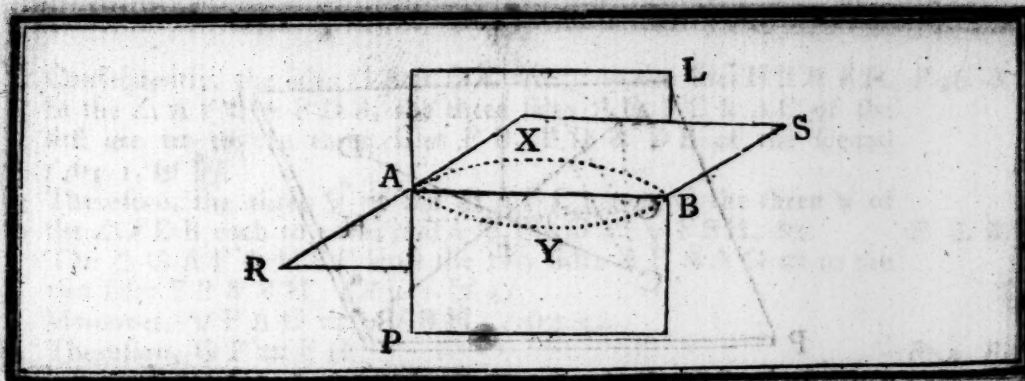
Draw GF.

BECAUSE the part AFGD of the Δ EAD is not in one plane (Z X) with EFG (*Sup.*).

1. It follows, that the parts GD, CG of the line CD are in different planes, & the parts AF, FB of the straight line AB, are in different planes, as also AFGD & FEG.
2. Which is impossible.
3. Since then the parts of the two lines & of the Δ can not be in different planes.
4. They must consequently be in the same plane.

P. I. B. 11.

Which was to be demonstrated. I. & II.



PROPOSITION III. THEOREM III.

IF two planes (RS & PL) cut one another, their common section is a straight line (AB).

Hypothesis.

RS & PL are two planes
which cut one another.

Thesis.

Their common section AB,
is a straight line.

DEMONSTRATION.

If it be not,

The section will be two straight lines.

As AXB for the plane RS; & A Y B for the plane PL.

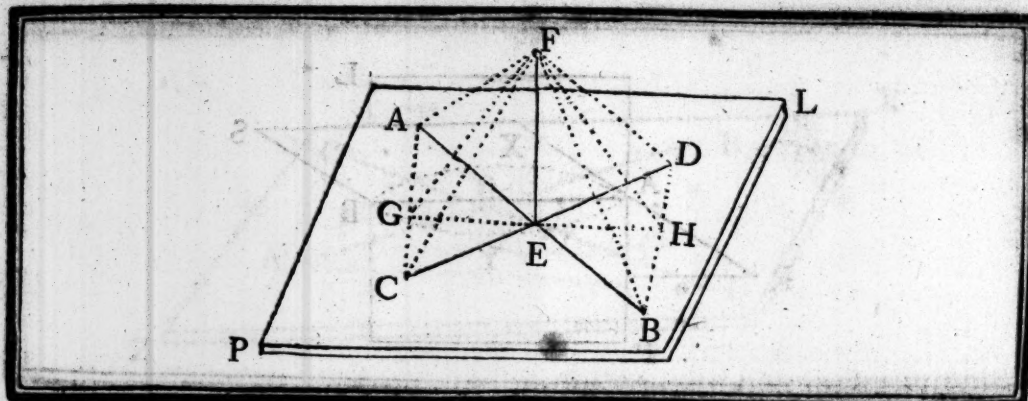
BECAUSE the straight lines AXB & A Y B have the same extremities A & B.

1. Those two straight lines AXB & A Y B include a space AXY.
2. Which is impossible.
3. Consequently, the section of the planes PL & RS can not be two straight lines AXB & A Y B.
4. Therefore their common section, is a straight line AB.

Ax. 12. B. 1.

Which was to be demonstrated.





PROPOSITION IV. THEOREM IV.

IF two straight lines (AB & CD) intersect each other, and at the point (E) of their intersection a perpendicular (EF) be erected upon those lines (AB & CD): it will be also perpendicular to the plane (PL) which passes through those lines (AB & CD).

Hypothesis.

Thesis.

- I. AB & CD are straight-lines situated in the plane PL .
 II. They intersect each other in E .
 III. EF is $\perp$ to those lines at the point E .
 EF is $\perp$ to the plane PL .

Preparation.

1. Take EC at will, & make EB , ED & AE each equal to EC .
2. Join the points A & C , also B & D .
3. Thro' the point E in the same plane PL , draw the straight line GH , terminated by the straight lines AC & BD , at the points G & H .
4. Draw AF , GF , CF , DF , HF & BF .

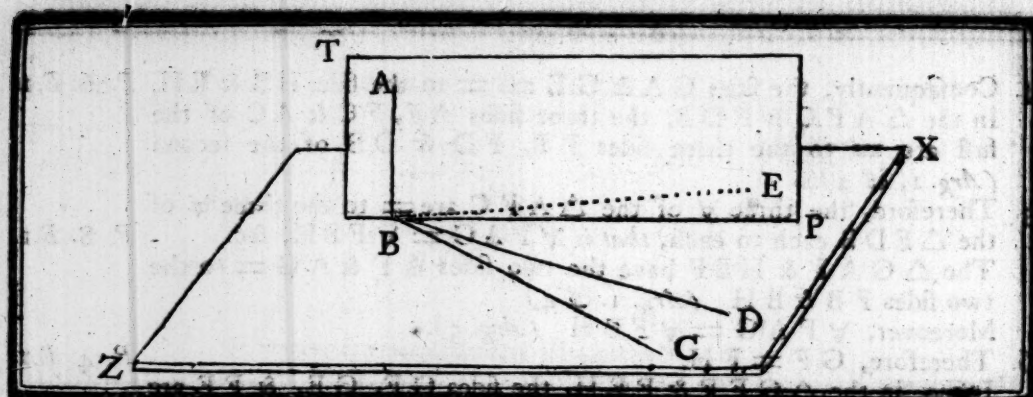
DEMONSTRATION.

- T**HE $\triangle AEF$, CEF , BEF , & DEF have the side EF common.
 The sides AE , CE , BE , & DE equal (*Prep. 1*) & the adjacent $\angle AEF$, CEF , BEF , & DEF equal (*Hyp. 3*).
 1. Consequently the bases AF , CF , BF , & DF are equal. P. 4. B.1.
 In the $\triangle AEC$ & DEB , the sides AE , CE , ED & EB are = (*Prep. 1.*) & the $\angle AEC$ & DEB also equal. P.15. B.1.
 2. Therefore, $AC = BD$. } P. 4. B.1.
 3. And $\angle EAC = \angle EBD$. } P.15. B.1.
 The $\triangle GAE$ & EBH have $\angle AEG = \angle HEB$.
 $\angle EAG = \angle EBH$ (*Arg. 3.*) & $AE = EB$ (*Prep. 1*).

4. Consequently, the sides GA & GE are $\equiv$ to the sides HB & EH . *P. 26. B. 1.*
 In the $\triangle AFC$ & FDB , the three sides AF , FC & AC of the first are $\equiv$ to the three sides FB , FD & DB of the second (*Arg. 1, § 2*).
5. Therefore, the three $\sphericalangle$ of the $\triangle AFC$ are $\equiv$ to the three $\sphericalangle$ of the $\triangle FDB$ each to each, *that is* $\sphericalangle FAG = \sphericalangle FBH$, &c. *P. 8. B. 1.*
 The $\triangle GAF$ & HBF have the two sides AF & $AG =$ to the two sides FB & BH (*Arg. 1, § 4*).
 Moreover, $\sphericalangle FAG = \sphericalangle FBH$ (*Arg. 5*).
6. Therefore, $GF = FH$. *P. 4. B. 1.*
 In fine, in the $\triangle GFF$ & FEH , the sides GF , GE , & FE are $\equiv$ to the sides FH , EH , & EF (*Arg. 4, § 6*).
7. Consequently, the three $\sphericalangle$ of the $\triangle GFE$ are $\equiv$ to the three $\sphericalangle$ of the $\triangle FEH$, each to each, *that is* $\sphericalangle FEG = \sphericalangle FEH$, &c. *P. 8. B. 1.*
 But those $\sphericalangle FEG$ & FEH are formed by the straight line EF falling upon GH (because GE & EH are in the same straight line) (*Prep. 3*).
8. Therefore, those $\sphericalangle FEG$ & FEH are $\perp$, & $FE \perp$ upon GH . $\left\{ \begin{array}{l} P. 13. B. 1. \\ D. 10. B. 1. \end{array} \right.$
 But HG is in the same plane, with the lines AB & CD (*Prep. 3*).
 And EF is $\perp$ upon those lines (*Hyp. 3*).
9. Consequently, EF is $\perp$ upon the same plane PL . *D. 3. B. 11.*

Which was to be demonstrated.





PROPOSITION V. THEOREM V.

IF three straight lines (BC , BD , & BE) meet all in one point (B), And a straight line (AB) is perpendicular to each of them in that point; these three straight lines (BC , BD , & BE) are in one and the same plane (ZX).

Hypothesis.

- I. BC , BD , & BE meet in B .
- II. AB is $\perp$ to those lines.

Thesis.

BC , BD , & BE are in the same plane ZX .

DEMONSTRATION.

If not,

One of those three as BE is in a different plane.

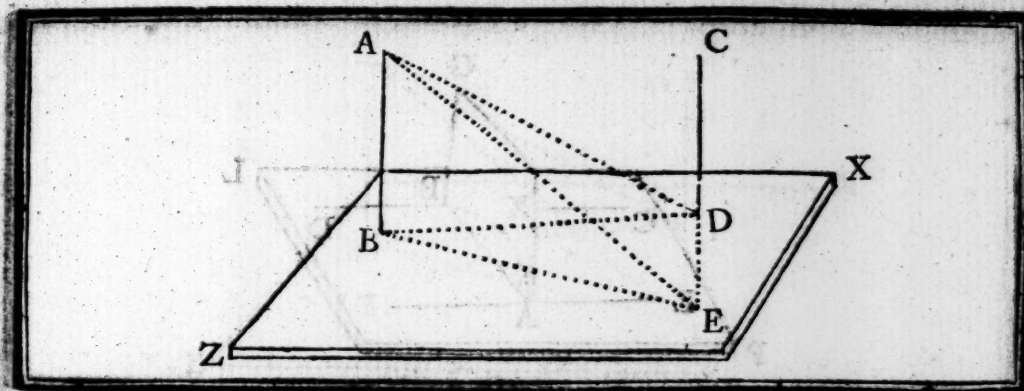
Preparation.

Let a plane TP pass thro' the $\perp AB$ & the line BE .

BECAUSE TP & ZX are different planes which meet in B .

1. They will cut each other when produced, & their common section will be a straight line BP , common to the two planes. P. 3. B.11.
But AB is $\perp$ to BD & BC (*Hyp.* 11).
2. Consequently, AB will be also $\perp$ to the plane ZX , in which those lines are. P. 4. B.11.
3. Therefore, AB is $\perp$ to BP & $\forall ABP$ a $\perp$ (*Arg.* 1).
But $\forall ABE$ is a $\perp$ (*Hyp.* 11).
And BE is in the same plane with AB & BP (*Prep.* & *Arg.* 1).
4. Consequently, $\forall ABE = \forall ABP$, that is, the part = the whole.
5. Which is impossible. Ax.8. B. 1.
6. Therefore, BE is not in a different plane from that in which BD & BC are.
7. Consequently, those three lines are in the same plane ZX .

Which was to be demonstrated.



PROPOSITION VI. THEOREM VI.

IF two straight lines ($A B$ & $C D$) be perpendicular to a plane ($Z X$), they shall be parallel to one another.

Hypothesis.

$A B$ & $C D$ are $\perp$ to the plane $Z X$.

Thesis.

$A B$ & $C D$ are parallel.

Preparation.

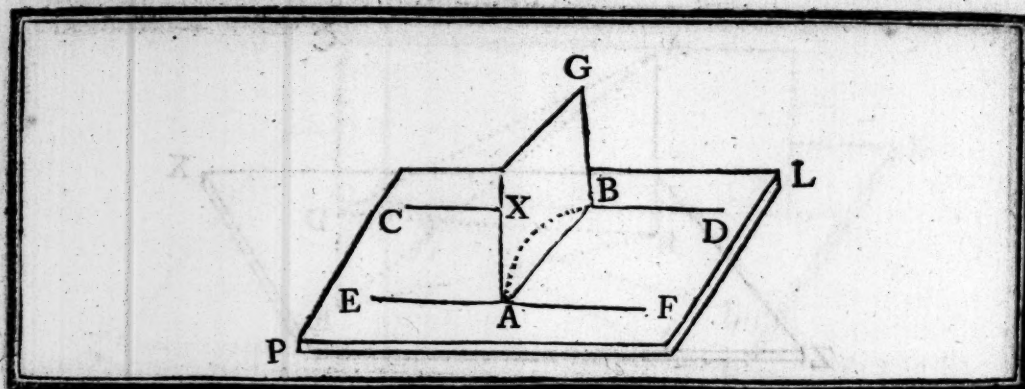
1. Join the points B & D in the plane $Z X$.
2. At the point D in $B D$ in this same plane, erect the $\perp D E$. *P. 11. B. 1.*
3. Make $D E = A B$. *P. 3. B. 1.*
4. Draw $A D$, $A E$, & $B E$.

DEMONSTRATION.

BECAUSE in the $\triangle A B D$ & $B D E$, the side $D E$ is $= A B$ (*Prep. 3.*), $B D$ is common to the two $\triangle$, & the $\angle A B D$ & $B D E$ are $\perp$ (*Hyp. prep. 2. & D. 3. 11.*)

1. The side $A D$ is $= B E$. *P. 4. B. 1.*
In the $\triangle A B E$ & $A D E$, the side $A E$ is common, $A B$ is $= D E$, & $B E = A D$ (*Prep. 3. & Arg. 1.*)
2. Consequently, $\angle A B E$ is $= \angle A D E$. *P. 8. B. 1.*
But $\angle A B E$ is a $\perp$. *D. 3. B. 11.*
3. Therefore, $\angle A D E$ is also a $\perp$. *Ax. 1. B. 1.*
But $\angle C D E$ is a $\perp$. *D. 3. B. 11.*
4. Consequently, $D E$ is $\perp$ to $C D$, $D A$ & $D B$ (*Hyp. prep. 2. & Arg. 3.*).
5. Therefore, those lines $C D$, $D A$ & $D B$ are in the same plane, that is $C D$ is in the plane which passes thro' $D A$ & $D B$. *P. 5. B. 11.*
6. Likewise $A B$ is also in the same plane which passes thro' $D A$ & $D B$. *P. 2. B. 11.*
7. Therefore, $A B$ & $C D$ are in the same plane.
But the interior $\angle A B D$ & $B D C$ are $\perp$ (*Hyp.*)
8. Consequently, $A B$ is parallel to $C D$. *P. 28. B. 1.*

Which was to be demonstrated.



PROPOSITION VII. THEOREM VII.

IF two points (A & B) in two parallels (DC & FE) be joined by a straight line (AB); it will be in the same plane (PL) with the parallels.

Hypothesis.

- I. A & B are two points taken at will in the parallels EF & CD.
- II. AB is a straight line which joins those points.

Thesis.

AB is in the same plane PL, with the plles. CD & EF.

DEMONSTRATION.

If not,

It will be in a different plane AG, as the line AXB is.

BECAUSE AXB is in the plane AG, different from the plane PL, & its extremities A & B are in the lines CD & EF, situated in the plane PL.

1. The line AXB will be common to the two planes, *that is*, AXB is the common section of the two planes AG & PL.

P. 3. B.11.

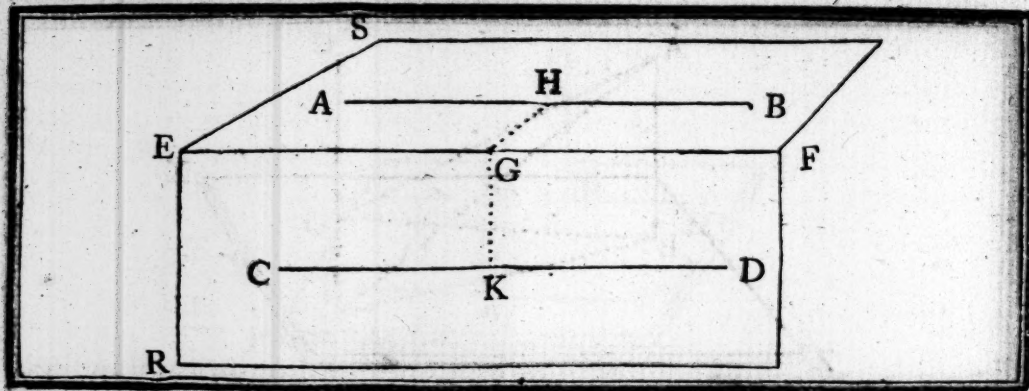
But AB is also a straight line having the same extremities A & B (Hyp. 11).

3. Which is impossible.
4. Wherefore, the straight line (AB) which joins the points A & B, is not in a plane AG different from that in which the parallels CD & EF are.
5. Therefore, AB is in the same plane PL with the plles. CD & EF.

Ax.12. B.4.

Which was to be demonstrated.





PROPOSITION IX. THEOREM IX.

THE lines (A B & C D) which are each of them parallel to the same straight line (E F) though situated in different planes (S F & R F) are parallel to one another.

Hypothesis.

- I. A B is in the plane S F, & C D in the plane R F.
- II. A B & C D are each p^{lle.} to E F.

Thefis.

A B is p^{lle.} to C D.

Preparation.

1. From the point H of the line A B in the plane F S let fall a $\perp$ H G upon E F.
2. From the point G in the plane R F let fall the $\perp$ G K upon C D. P. 11. B. 1.

DEMONSTRATION.

BECAUSE E G or E F is $\perp$ to G H & G K (*Prep. 1. & 2.*).

1. E G will be $\perp$ to the plane which passes thro' those lines. P. 4. B. 11.
But A B is p^{lle.} to E F (*Hyp. 2.*).

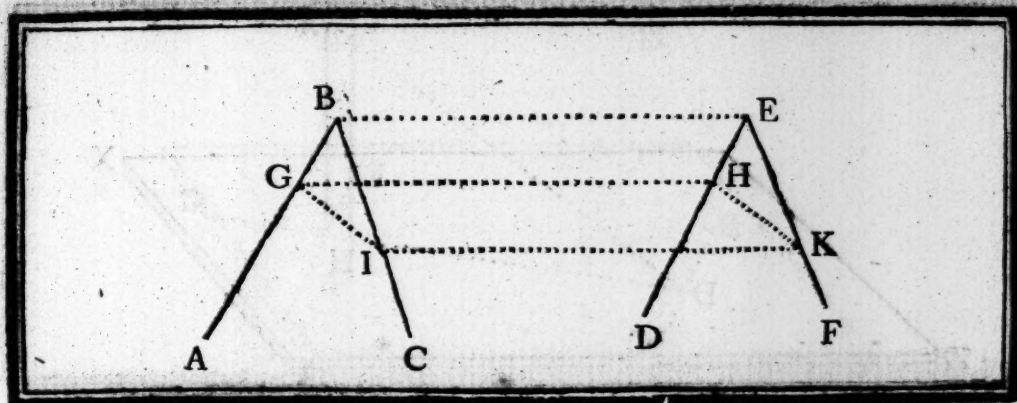
2. Therefore, A B is $\perp$ to the plane which passes thro' those lines H G & G K. P. 8. B. 11.

3. In like manner, C D is also $\perp$ to this same plane. Therefore, the lines A B & C D being $\perp$ to the same plane (*Arg. 2. & 3.*).

4. They are p^{lle.} to one another. P. 6. B. 11.

Which was to be demonstrated.

27



PROPOSITION X. THEOREM X.

IF two straight lines (AB & BC) which meet one another (in B) be parallel to two others (DE & EF) which meet one another in (E); and are not in the same plane with the first two; the first two and the other two shall contain equal angles (ABC & DEF).

Hypothesis.

AB & BC meet one another in B, in a different plane from that in which DE & EF are, which also meet one another in E.

Thesis.

$\angle ABC$ is $= \angle DEF$.

Preparation.

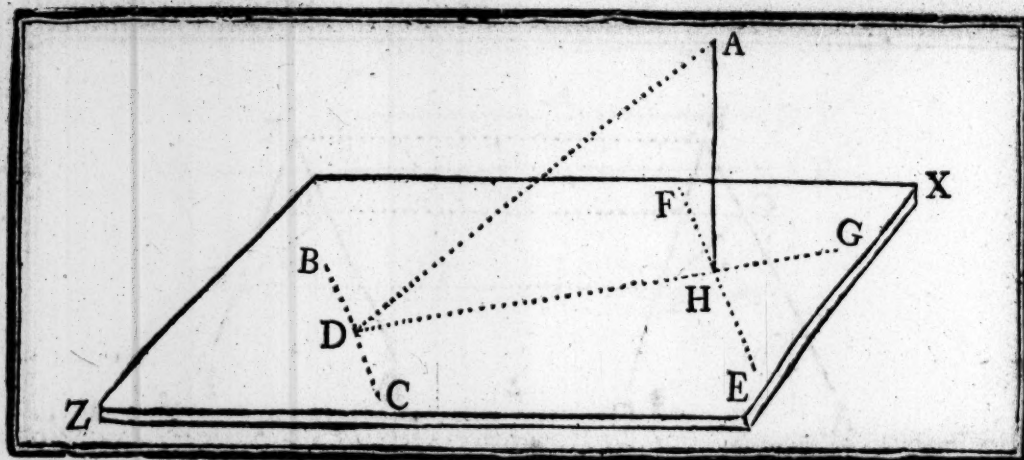
1. Cut off at will from the straight lines AB & BC the parts BG & BI. P. 3. B. 1.
2. Make HE = BG, & EK = BI.
3. Join the points BE, GH, GI, HK & IK. Pof. 1. B. 1.

DEMONSTRATION.

THE line BG being = & p^lle. to HE, (*Prep. 2. & Hyp.*).

1. GH will be = & p^lle. to BE. P. 33. B. 1.
2. In like manner, IK is = & p^lle. to BE. { P. 9. B. 11.
3. Consequently, GH is = & p^lle. to IK. { Ax. 1. B. 1.
4. Therefore, GI is = & p^lle. to KH. P. 33. B. 1.
 And because in the $\triangle GBI$ & $\triangle HEK$ the three sides BG, BI, & GI of the first, are = to the three sides HE, EK, & HK of the last, each to each, (*Prep. 2. & Arg. 4.*).
5. $\angle GBI$ or $\angle ABC$ is = to $\angle HEK$ or $\angle DEF$. P. 8. B. 1.

Which was to be demonstrated.



PROPOSITION XI. PROBLEM I.

TO draw a straight line (A H) perpendicular to a plane (Z X) from a given point (A) above it.

Given.

- I. The plane Z X.
- II. A point A above it.

Sought.

The straight line A H let fall from the point A, $\perp$ to the plane Z X.

Resolution.

1. In the plane Z X draw at will the straight line B C: P.12. B. 1.
2. From the point A let fall upon B C the $\perp$ A D.
3. At the point D in the plane Z X erect upon B C the $\perp$ D G. P.11. B. 1.
4. From the point A let fall upon D G the $\perp$ A H. P.12. B. 1.

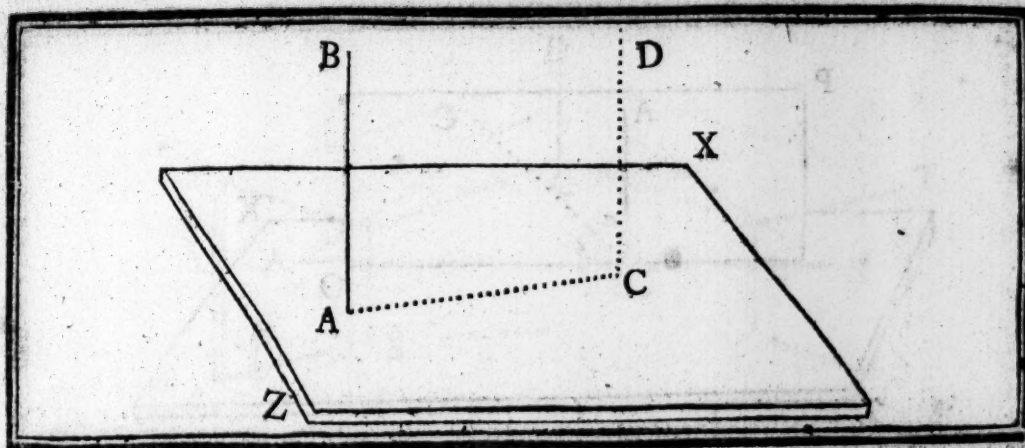
Preparation.

Thro' the point H draw the straight line F E p^lle. to B C. P.31. B. 1.

DEMONSTRATION.

- B**ECAUSE the straight line B C is $\perp$ to D A & D G (Ref. 2. & 3).
1. It will be $\perp$ to the plane which passes thro' those lines. P. 4. B.11.
But F E is p^lle. to B C (Prep).
 2. Therefore, F E is also $\perp$ to this same plane which passes thro' D G & D A. P 8. B.11.
But A H is in the same plane with D A & D G (P. 2. B. 11) & meets F E in H (Ref. 4. & Prep).
 3. Therefore, $\angle F H A$ is a $\perp$. D. 3. B.11.
And because $\angle A H D$ is a $\perp$ (Ref. 4).
 4. A H is $\perp$ to the two lines F E & D G situated in the plane Z X which intersect each other in H.
 5. Therefore, A H is $\perp$ to the plane Z X. P. 4. B.11.

Which was to be done.



PROPOSITION XII. PROBLEM II.

FROM a given point (A) in a plane (XZ) to erect a perpendicular (BA).

Given.

A point A in the plane XZ.

Sought.

A straight line BA drawn from the point A $\perp$ to the plane XZ.

Resolution.

1. Take at will a point D above the plane XZ.
2. From this point D; let fall upon this plane the $\perp$ DC, *P. 11. B. 11.*
3. Join the points A & C. *Pos. 1. B. 1.*
4. From the point A draw AB p^{lle}. to DC, *P. 31. B. 1.*

DEMONSTRATION.

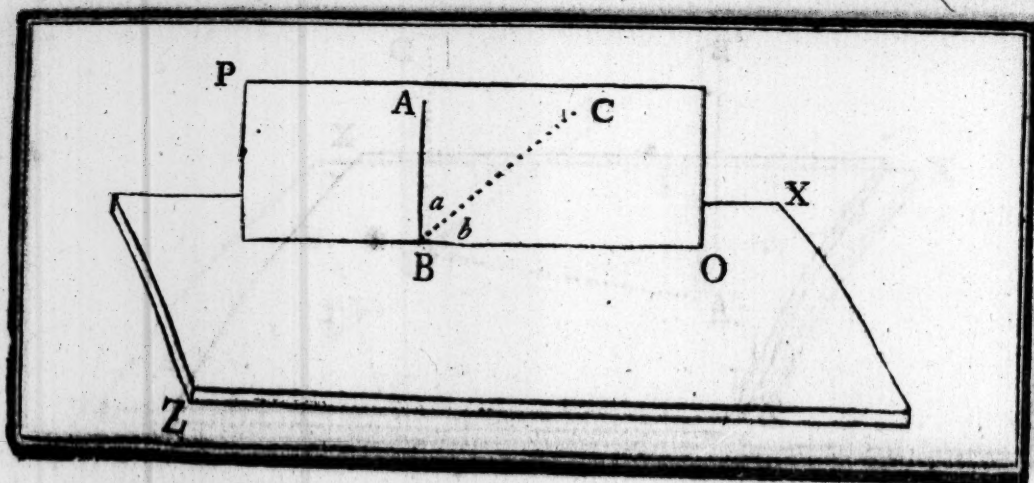
BECAUSE the line AB is p^{lle}. to DC (*Ref. 4*),
And that DC is $\perp$ to the plane XZ (*Ref. 2*).

1. AB will be also $\perp$ to the same plane XZ.

P. 8. B. 11.

Which was to be done.





PROPOSITION XIII. THEOREM XI.

FROM the same point (B) in a given plane (Z X) there cannot be drawn on the same side of it more than one perpendicular (A B).

Hypothesis.

A B is $\perp$ at the point B, to the plane X Z.

Thesis.

It is impossible to draw from the point B another $\perp$ to the plane X Z on the same side that A B is.

DEMONSTRATION.

If not,

There may be drawn from the point B another $\perp$.

Preparation.

From the point B erect a $\perp$ B C different from A B.

BECAUSE the lines A B & B C meet at the point B.

1. They are in the same plane P O.

But they are each $\perp$ to the plane X Z (*Sup.*).

2. Consequently, the $\angle a + b$, & b are each $\perp$.

3. Therefore, $\angle a + b = \angle b$, that is, the whole = to the part.

4. Which is impossible.

But A B is $\perp$ to the plane X Z (*Hyp.*).

5. Therefore, B C is not $\perp$ to X Z.

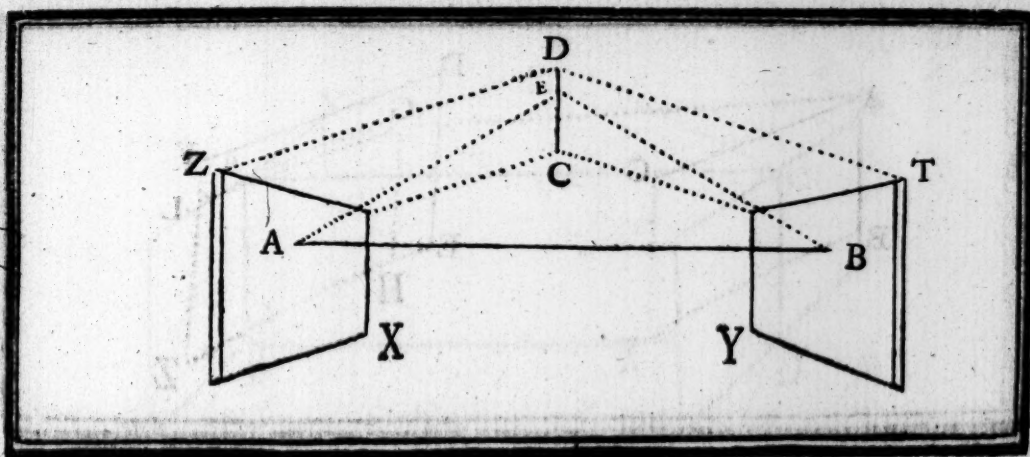
6. Consequently, it is impossible to draw from a point B any other line on the same side as A B, that will be $\perp$ to the plane X Z.

Which was to be demonstrated.

P.2. B.11.

D.10. B. 1.

Ax.8. B. 1.



PROPOSITION XIV. THEOREM XII.

PLANES (ZX & TY) to which the same straight line (AB) is perpendicular; are parallel to one another.

Hypothesis.

AB is $\perp$ to the planes XZ & TY.

Thesis.

The plane XZ is p^{lle}. to the plane TY.

DEMONSTRATION.

If not,

The planes XZ & TY produced will meet one another. D. 8. B.11.

Preparation.

1. Produce the planes XZ & TY until they meet in DC.
2. Take a point E in the section DC.
3. Draw EA & EB.

BECAUSE AB is $\perp$ to the plane TY (*Hyp.*) & EB is in this plane (*Prep. 3*).

D. 3. B.11.

1. $\angle ABE$ is a $\perp$.

2. Likewise $\angle BAE$ is a $\perp$.

3. Consequently, the $\triangle BAE$ has two $\perp$.

4. Which impossible.

P. 17. B. 1.

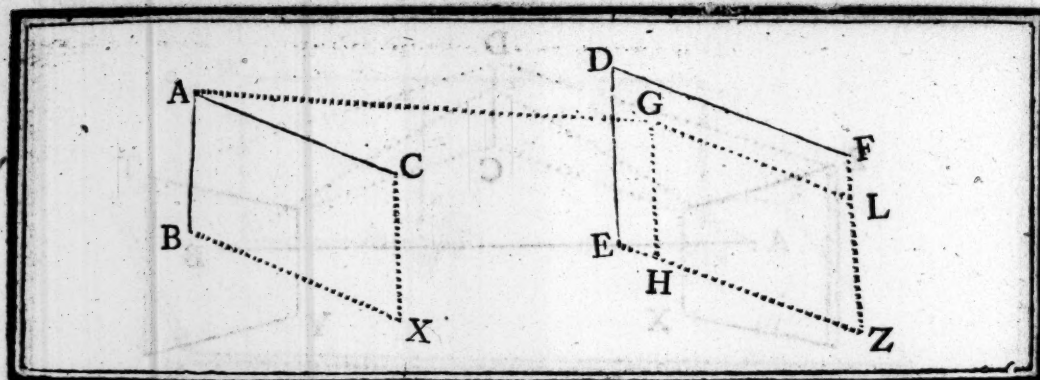
5. From whence it follows that the lines AE & EB do not meet one another, no more than the planes TY & XZ.

P. 1. B.11.

6. Therefore, those planes are p^{lle}.

D. 8. B.11.

Which was to be demonstrated.



PROPOSITION XV. THEOREM XIII.

IF two straight lines (AB & AC) situated in the same plane (AX), and meeting one another (in A), be parallel, to two straight lines (DE & DF) meeting one another, and situated in another plane (DZ); those planes (AX & DZ) will be parallel.

Hypothesis.

AB & AC situated in the plane AX & meeting each other in A , are pple. to DE & DF meeting each other in D , & situated in the plane DZ .

Thesis.

The plane AX in which are the lines AB & AC is pple. to the plane DZ in which are the lines DE & DF .

Preparation.

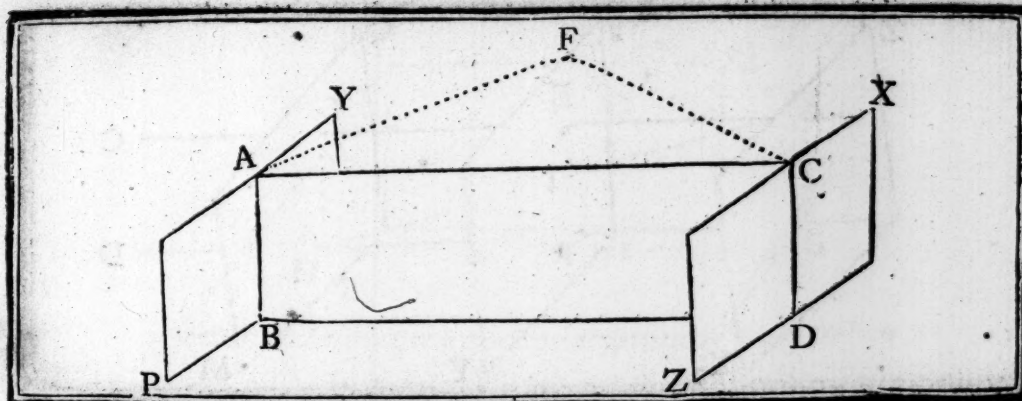
1. From the point A let fall upon the plane DZ the $\perp$ AG . P. 11. B. 11.
2. Draw GH pple. to DE , & GL pple. to DF . P. 31. B. 1.

DEMONSTRATION.

BECAUSE the lines GH & GL are pple. to DE & DF (Prep. 2).

1. They will be also pple. to AB & AC . P. 9. B. 11.
And GL being pple. to AC .
2. The $\angle CAG + \angle AGL$ are $= 2 \angle$. P. 29. B. 1.
But $\angle AGL$ is a $\angle$ (Prep. 1).
3. Consequently, $\angle CAG$ is also a $\angle$.
4. It may be demonstrated after the same manner that $\angle BAG$ is a $\angle$.
5. Therefore, GA is $\perp$ to the plane AX . P. 4. B. 11.
But GA is also $\perp$ to the plane DZ (Prep. 1).
6. Wherefore, the plane AX is pple. to the plane DZ . P. 14. B. 11.

Which was to be demonstrated.



PROPOSITION XVI. THEOREM XIV.

IF two parallel planes ($Z X$ & $Y P$) be cut by another plane ($A B D C$), the common sections with it ($C D$ & $A B$) are parallels.

Hypothesis.

- I. The planes $Z X$ & $Y P$ are *plle.*
- II. They are cut by the plane $A B C D$.

Thesis.

The common sections $C D$ & $A B$ are *plle.*

DEMONSTRATION.

If not,

The lines $A B$ & $C D$ being produced will meet somewhere.

Preparation.

Produce them until they meet in F .

Pos. 2. B. 1.

BECAUSE the straight lines $B A F$ & $D C F$ meet in F .

1. The planes $P Y$ & $Z X$ in which those lines are, will also meet one another: ($B A F$ being entirely in the plane $P Y$, & $D C F$ entirely in the plane $Z X$).

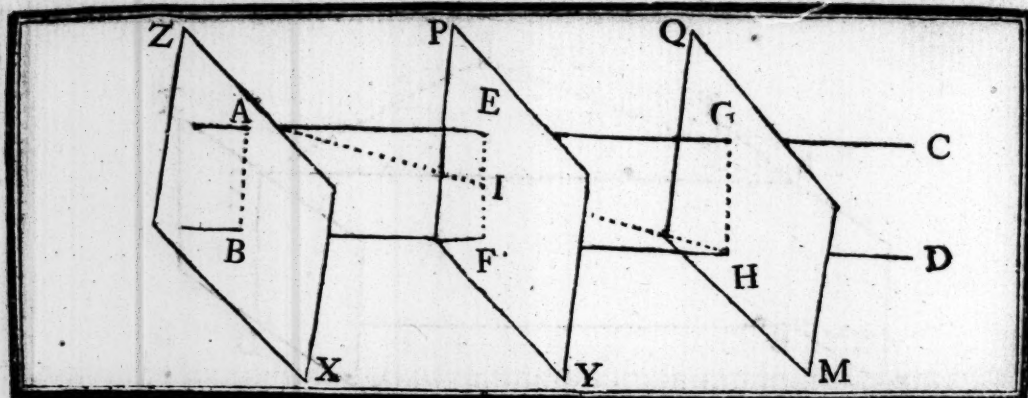
P. 1. B. 11.

2. Which is impossible (*Hyp. 1*).
3. Wherefore, $A B$ & $C D$ do not meet one another.
4. Therefore, $A B$ & $C D$ are *plle.*

D. 35. B. 1.

Which was to be demonstrated.





PROPOSITION XVII. THEOREM XV.

IF two straight lines (AC & BD) be cut by parallel planes (XZ, PY & QM): they shall be cut in the same ratio, (*that is*, $AE : EG = BF : FH$ &c).

Hypothesis.

- I. AC & BD are two straight lines.
- II. Cut by the p^{lle}. planes XZ, PY & QM.

Thesis.

$$AE : EG = BF : FH.$$

Preparation

1. Join the points A & B, also G & H.
2. Draw AH which will pass thro' the plane PY in the point I.
3. Draw EI & IF.

} P^of. I. B. 1.

DEMONSTRATION.

BECAUSE the p^{lle}. planes ZX & PY are cut by the plane of the $\triangle ABH$.

1. AB is p^{lle}. to IF.
2. Likewise, EI is p^{lle}. to GH.
3. Consequently, $AI : IH = BF : FH$.
4. And, $AI : IH = AE : EG$.
5. Therefore, $AE : EG = BF : FH$.

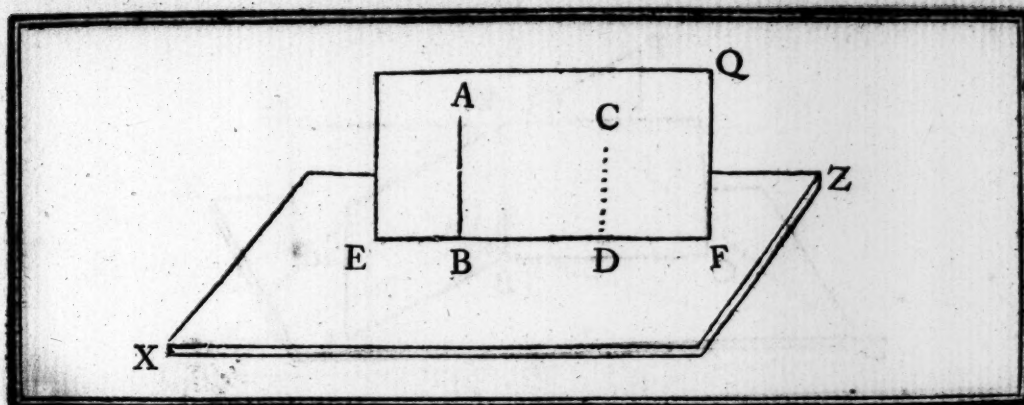
P. 16. B. 11.

} P. 2. B. 6.

P. 11. B. 5.

Which was to be demonstrated.





PROPOSITION XVIII. THEOREM XVI.

IF a straight line (AB) is perpendicular to a plane (ZX): every plane (as QE) which passes thro' this line (AB) shall be perpendicular to this plane (ZX).

Hypothesis.

AB is $\perp$ to the plane ZX .

Thefis.

Every plane (as QE) which passes thro' the $\perp AB$ is $\perp$ to the plane ZX .

Preparation.

1. Let a plane QE pass thro' AB , which will cut the plane ZX in EF . P. 3. B. 1.
2. Take in this straight line EF , a point D at will.
3. From this point D , draw in the plane QE , the line DC p^{er}ple. to AB . P. 31. B. 1.

DEMONSTRATION.

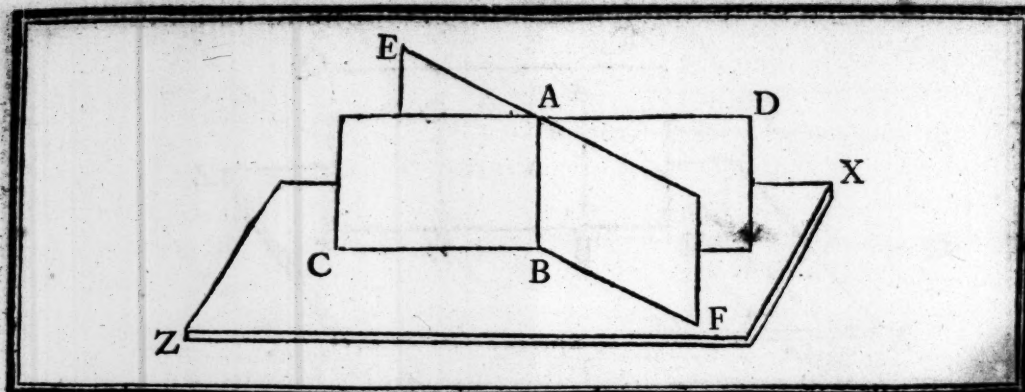
BECAUSE the straight line AB is $\perp$ to the plane ZX , & DC is p^{er}ple. to AB (*Hyp. & Prep. 3*).

1. The line DC is $\perp$ to the plane ZX . P. 8. B. 11.
2. Consequently, CD is also $\perp$ to the common section EF . D. 3. B. 11.
3. Therefore, the plane EQ in which the lines AB & CD are, is $\perp$ to the plane ZX . D. 4. B. 11.

And as the same demonstration may be applied to any other plane which passes thro' the $\perp AB$, we may conclude,

4. That every plane which passes thro' this line is $\perp$ to the plane ZX .

Which was to be demonstrated.



PROPOSITION XIX. THEOREM XVII.

IF two planes (CD & EF) cutting one another be each of them perpendicular to a third plane (ZX); their common section (AB) shall be perpendicular to the same plane (ZX).

Hypothesis.

- I. The planes CD & EF are $\perp$ to the plane ZX.
- II. They cut one another in AB.

Thesis.

The common section AB is $\perp$ to the plane ZX.

DEMONSTRATION.

BECAUSE CB, the common section of the plane CD with the plane XZ is also in the plane XZ.

1. There may be erected at the point B in CB a $\perp$ (P. 11. B. 11.)

P. 3. B. 11.

which will be in the plane CD (Hyp. 1.)

P. 18. B. 11.

And because the line FB the common section of the planes FE & XZ is also in the plane XZ.

P. 3. B. 11.

2. There may be erected at the same point B & at the same side with the foregoing another $\perp$ which will fall in the plane FE, But from the point B only one $\perp$ can be raised.

P. 18. B. 11.

P. 13. B. 11.

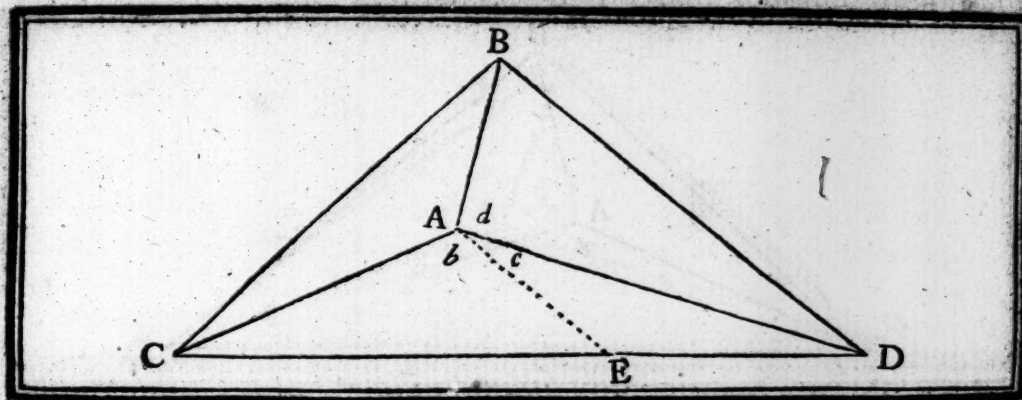
3. Consequently, those $\perp$ must coincide, *that is*, those two lines must form but one which is common to the two planes.

But those planes have only the line AB in common (Hyp. 2.)

4. Therefore AB is $\perp$ to the plane XZ.

Which was to be demonstrated.





PROPOSITION XX. THEOREM XVIII.

IF three plane angles (CAB, BAD & DAC) form a solid angle A: any two of those angles (as BAD & CAB) are greater than the third (CAD).

Hypothesis.

The three plane $\angle CAB$, d & $c + b$
form a solid $\angle A$.

Thesis.

$\angle CAB + d > \angle b + c$.

DEMONSTRATION.

CASE I.

When the three angles CAB, d , & $c + b$ are equal.

BECAUSE the $\angle CAB$, d & $c + b$ are equal.
1. It follows that $\angle CAB + d$ will be $> \angle c + b$.

Ax. 4. B. 1.

CASE II.

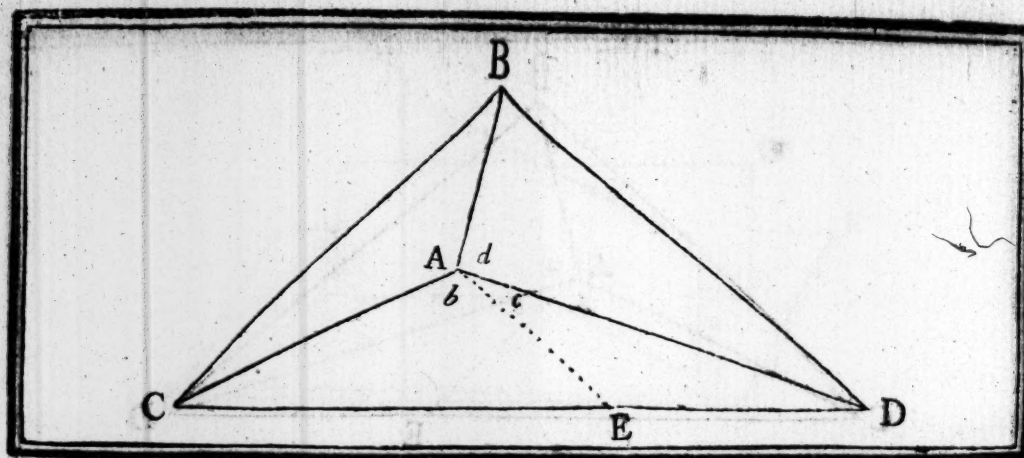
When of the three angles CAB, d & $c + b$ two as CAB & d are equal, & the third $c + b$ is less than either of them.

BECAUSE $\angle CAB$ is $> \angle c + b$.
1. $\angle CAB + \angle d$ will be much $> \angle c + b$.

Ax. 4. B. 1.

Which was to be demonstrated.





CASE III.

When the three angles are unequal, & $b + c$ is $> CAB$ or d .

Preparation.

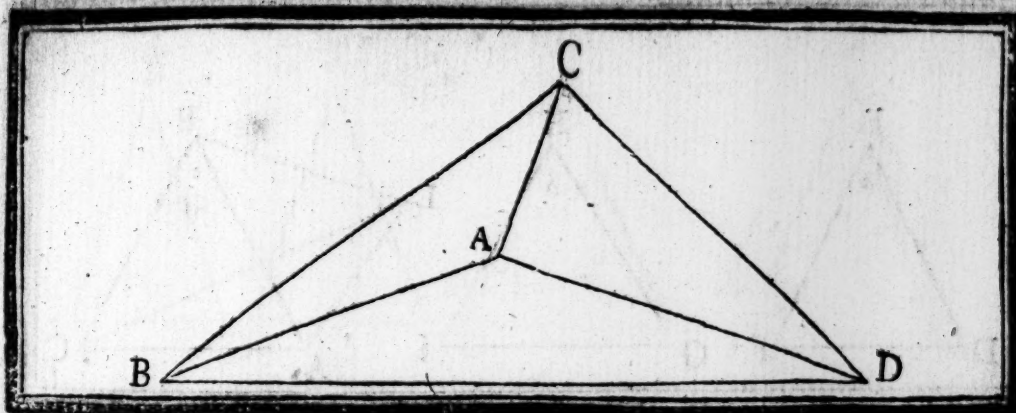
1. At the point A in AC make $\angle b = \angle CAB$ in the plane CAD. P. 23. B. 1.
2. Make $AE = AB$. P. 3. B. 1.
3. From the point C draw thro' E the straight line CED. } P. 1. B. 1.
4. From the points C & D draw CB & BD.

THE $\triangle BCA$ & $\triangle CAE$ have the sides AB & AE equal (*Prep. 2*).
The side CA common & $\angle b = \angle CAB$ (*Prep. 1*).

1. Consequently, the side BC is = to the side CE. P. 4. B. 1.
But in the $\triangle CBD$ the sides CB + BD are $> CD$. P. 20. B. 1.
Therefore, if from CB + BD be taken away the part CB, & from CD a part = to CE.
2. The remainder BD will be $> ED$. Ax. 5. B. 1.
In the $\triangle BAD$ & $\triangle EAD$, the sides AB & AE are = (*Prep. 2*).
& AD common.
But the base BD is $>$ the base ED (*Arg. 2*).
3. Therefore, $\angle d$ is $> \angle c$. P. 25. B. 1.
If therefore, $\angle CAB$ be added one side, & its equal $\angle b$ on the other.
4. $\angle CAB + d$ will be $> \angle b + c$ or CAD. Ax. 4. B. 1.

Which was to be demonstrated.





PROPOSITION XXI. THEOREM XIX.

ALL the plane angles (BAC , CAD & DAB) which form a solid angle (A); are less than four right angles.

Hypothesis.

The $\angle BAC$, CAD & DAB
form a solid $\angle A$

Thesis.

The plane $\angle BAC + CAD + DAB$
are $< 4 \angle$.

Preparation.

1. In the sides BA , AC , & AD take the three points B , C , D .

2. Draw BC , BD & CD .

Pos. 1. B. 1.

3. Let a plane BCD pass thro' those lines, which will form with the planes BAC , CAD & BAD , three solid $\angle$; viz. the solid $\angle B$, formed by the plane $\angle CBA$, ABD & CBD ; the solid $\angle C$, formed by the plane $\angle BCA$, ACD & BCD , & infine, the solid $\angle D$, formed by the plane $\angle CDA$, ADB & BDC .

D. 11. B. 11.

DEMONSTRATION.

BECAUSE the solid $\angle D$, is formed by the plane $\angle CDA$, ADB & BDC .

1. The $\angle CDA + ADB$ are $> \angle BDC$.

2. Likewise, $\angle ABD + ABC$ are $> \angle DBC$.

3. And $\angle ACB + ACD$ are $> \angle BCD$.

P. 20. B. 11.

4. Hence, the six plane $\angle CDA + ADB + ABD + ABC + ACB + ACD$ are $>$ the three plane $\angle BDC + DBC + BCD$.

But those three plane $\angle BDC + DBC + BCD$ are $= 2 \angle$. *P. 32. B. 1.*

5. Therefore, the six plane $\angle CDA + ADB + ABD + \&c.$ are $> 2 \angle$ (*Arg. 4.*)

But the nine $\angle$ of the $\triangle BCA$, CAD & DAB viz. the six already mentioned (*Arg. 5.*) & the three remaining $\angle BAC$, CAD & DAB are together $=$ to $6 \angle$.

P. 32. B. 1.

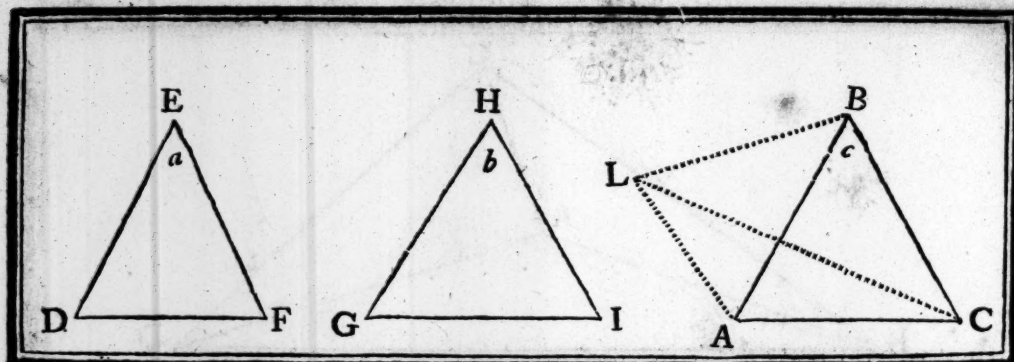
If therefore the six $\angle CDA + ADB + ABD + ABC + ACB + ACD$ which are together $> 2 \angle$ be taken away.

6. The remaining plane $\angle BAC + CAD + DAB$ will be $< 4 \angle$.

But those plane $\angle BAC$, CAD & DAB form a solid $\angle A$.

7. Consequently, the plane $\angle$ which form a solid $\angle A$ are $< 4 \angle$.

Which was to be demonstrated.



PROPOSITION XXII. THEOREM XX.

IF every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made of the straight lines (DF, GI & AC) which subtend those angles.

Hypothesis.

- I. Any two of the three given $\forall a, b, c$, are $>$ the third, as $b + a > c$, or $a + c > b$, or $b + c > a$.
- II. The sides HG, HI, DE, EF, AB & BC which contain those $\forall$, are equal.

Thesis.

A Δ may be made of the straight lines GI, DF & AC, which subtend those $\forall$.

DEMONSTRATION.

The three given $\forall a, b, \& c$ are either equal, or unequal.

CASE I. If the $\forall a, b, \& c$ be equal.

BECAUSE the sides which contain the $\forall$, are equal (Hyp. 2.)

1. The Δ DEF, GHI & ABC are equal. P. 4. B. 1.
2. Therefore $DF = GI = AC$.
3. Consequently, $DF + AC > GI$. Ax. 4. B. 1.
4. Wherefore a Δ may be made of those straight lines DF, AC & GI. P. 22. B. 1.

CASE II. If the given $\forall a, b, \& c$ be unequal

Preparation.

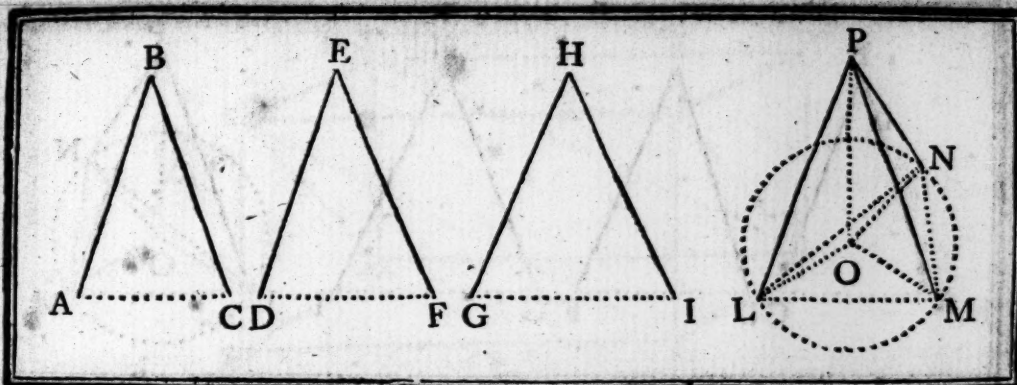
1. At the vertex of one of the $\forall$ as B, make $\forall ABL = \forall a$. P. 23. B. 1.
2. Make $BL = DE$. P. 3. B. 1.
3. Draw LC & LA. Pos. 1. B. 1.

DEMONSTRATION.

BECAUSE the two $\forall a + c$ are $> \forall b$ (Hyp. 1.) & $LB = HG = BC = HI$ (Prep. 2. & Hyp. 2.)

1. The base LC will be $> GI$. P. 24. B. 1.
But $LC < LA + AC$. P. 20. B. 1.
2. Much more then GI is $< LA + AC$.
But $LA = DF$ (Prep. 1. & P. 4. B. 1.).
3. Therefore GI is $< DF + AC$. Ax. 1. B. 1.
4. Consequently, a Δ may be made of the straight lines DF, AC & GI.

Which was to be demonstrated.



PROPOSITION XXIII. PROBLEM III.

TO make a solid angle (P), which shall be contained by three given plane angles (ABC, DEF & GHI), any two of them being greater than the third, and all three together ($\angle ABC + \angle DEF + \angle GHI$) less than four right angles.

Given.

- I. Three $\angle ABC$, DEF & GHI, any two of which are greater than the third, as $\angle B + \angle E > \angle H$, $\angle B + \angle H > \angle E$, & $\angle E + \angle H > \angle B$.
- II. $\angle B + \angle E + \angle H < 4 \angle L$.

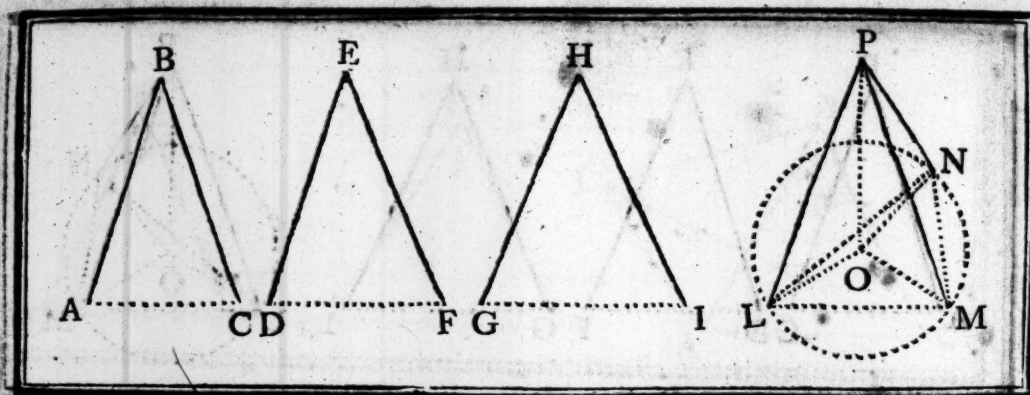
Sought.

A solid $\angle P$, contained by the three plane $\angle B$, E & H.

Resolution.

1. Take AB at will, & make the sides BC, DE, EF, GH & HI equal to one another & to AB. P. 3. B. 1.
2. Draw the bases AC, DF, & GI. Pos. 1. B. 1.
3. With those three bases AC, DF & GI make a $\triangle LMN$ so that NM be = GI, NL = AC, & LM = DF. P. 27. B. 1.
4. Inscribe the $\triangle LMN$ in a $\odot LMN$. P. 22. B. 11.
5. From the center O, to the $\angle L$, M & N, draw the straight lines LO, ON & OM. P. 5. B. 4.
6. At the point O, erect the $\perp OP$ to the plane of the $\odot LMN$. P. 12. B. 11.
7. Cut OP so that the $\square$ of LO + the $\square$ of PO be = to the $\square$ of AB.
8. Draw the straight lines LP, PN & PM.

M m



DEMONSTRATION.

BECAUSE PO is $\perp$ to the plane of the $\odot LMN$ (Ref. 6.)

1. The $\triangle POL$ will be right angled in O (Ref. 5. & 8.)

2. Consequently, the $\square$ of PO + the $\square$ of OL is = to the $\square$ of LP . P. 47. B. 1.

But the $\square$ of PO + the $\square$ of OL = $\square AB$, (Ref. 7.) { Ax. 1.

3. Therefore the $\square$ of AB is = to the $\square$ of LP , & $AB = LP$. { P. 40. B. 1.

4. Likewise PN & PM are each = to AB . { Cor. 3.

But NM is = to GI , $NL = AC$, & $LM = DF$, (Ref. 3).

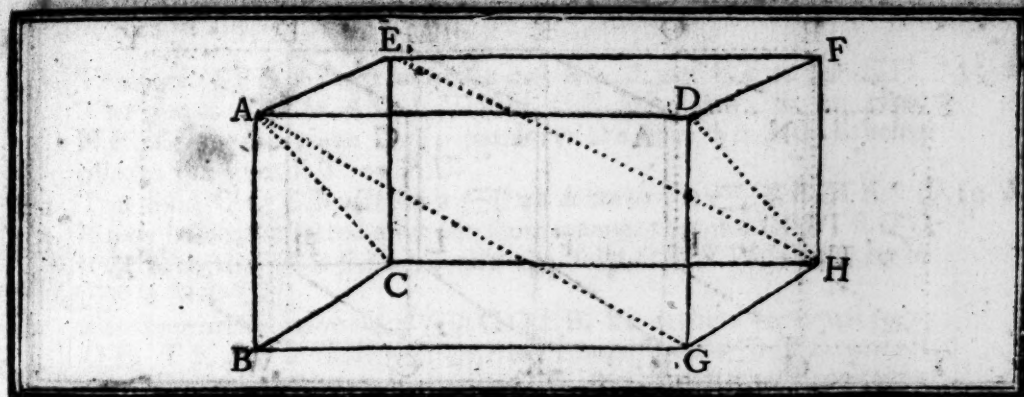
5. Consequently, $\triangle NMP$ is = to the $\triangle GHI$, $\triangle NPL = \triangle ABC$, $\triangle LPM = \triangle DEF$, $\angle NPM = \angle H$, $\angle LPN = \angle B$, & $\angle LPM = \angle E$. { P. 8. B. 1.

But those three $\angle NPM$, $\angle LPN$ & $\angle LPM$ form a solid $\angle P$.

6. Therefore a solid $\angle P$ has been made, contained by the three given plane $\angle B$, E & H .

Which was to be done.





PROPOSITION XXIV. THEOREM XXI.

IN every parallelepiped (A H); the opposite planes (B D & C F; B E & F G; A F & B H) are similar & equal parallelograms.

Hypothesis.

In the given $\square$ B F, the plane B D is opposite to C F, B E to F G & A F to B H.

Thesis.

The opposite planes B D, C F; B E & F G; A F & B H are $=$ & $\propto$ pgrs. each to each.

Preparation.

Draw the opposite diagonals E H & A G, also A C & D H.

DEMONSTRATION.

BECAUSE the pple. planes B D & C F are cut by the plane A B C E.

1. The line B A is pple. to E C.

P. 16. B. 11.

2. Likewise C H is pple. to G B.

And the same pple. planes B D & C F being also cut by the plane D G H F.

3. The line D G will be pple. to F H.

4. Likewise A E is pple. to B C & D F pple. to G H.

And because those pple. planes (Arg. 1. 2. & 4.) are the opposite sides of the quadrilateral figures A E C B & D F H G.

5. Those quadrilateral figures A E C B & D F H G, are pgrs.

D. 35. B. 1.

6. Likewise the other opposite planes B D & C F; A F & B H are pgrs.

And since A B & B G are pple. to E C & C H, each to each (Arg. 1. & 2).

7. $\angle$ A B G is $=$ to $\angle$ E C H.

P. 10. B. 11.

But A B is $=$ to E C & B G $=$ C H.

P. 34. B. 1.

8. Therefore the $\triangle$ A B G is $=$ & $\propto$ to the $\triangle$ E C H.

But the pgr. B D is double of the $\triangle$ A B G. } (P. 41. B. 1.)

And the pgr. C F is double of the $\triangle$ E C H. }

{ P. 4. B. 1.
{ P. 4. B. 6.

But those pgrs. have each an $\angle$ common with the equiangular $\triangle$.

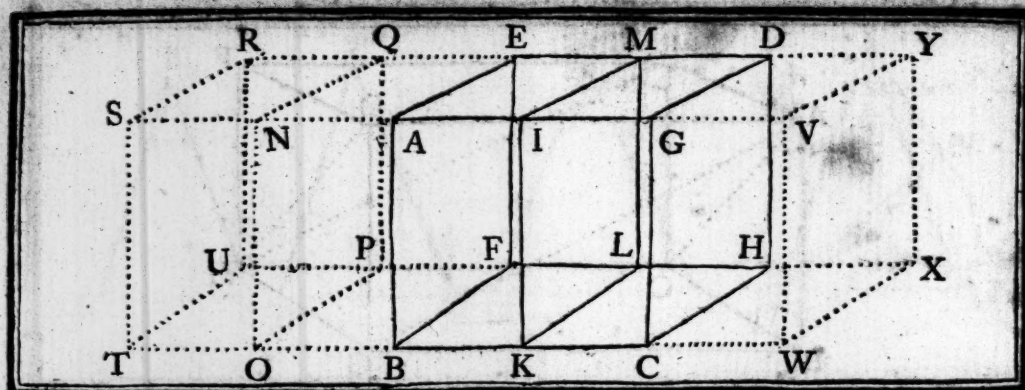
9. Consequently, the pgrs. B D & C F are $=$ & $\propto$.

D. 1. B. 6.

10. It may be demonstrated after the same manner that the pgr. B D is $=$ & $\propto$ to the pgr. C F, & pgr. A F is $=$ & $\propto$ to the pgr. B H.

11. Therefore the opposite planes of a $\square$ are $=$ & $\propto$ pgrs.

Which was to be demonstrated.



PROPOSITION XXV. THEOREM XXII.

IF a parallelepiped (BEDC) be cut by a plane (KIML) parallel to the opposite planes (AEFB & CGDH); it divides the whole into two parallelepipeds (*viz.* the $\square$ BEMK & KMDC), which shall be to one another as their bases (BFLK & KLHC).

Hypothesis.

The $\square$ BEDC is divided into two $\square$ BM & MC, by a plane KM, p^{lle.} to the opposite planes BE & CD.

Thesis.

The $\square$ BM: $\square$ MC = base BL: base LC.

Preparation.

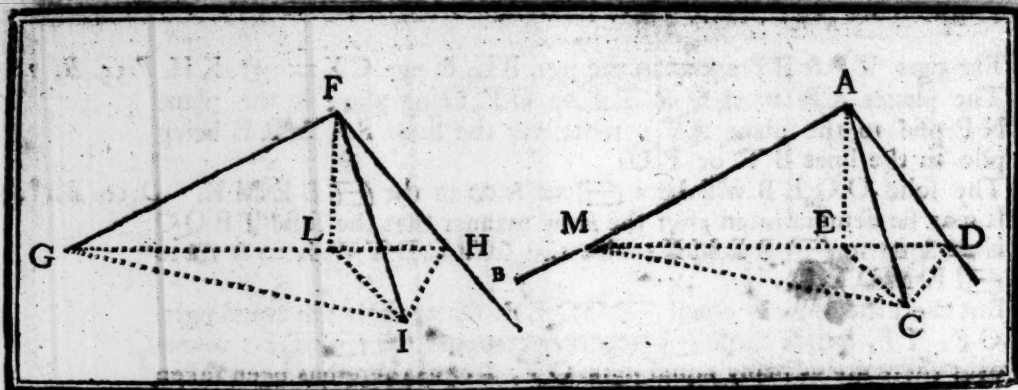
1. Produce BC both ways, as also FH. Pof. 2. B. 1.
2. In BC produced take any number of lines = to BK & CK: as BO & TO each = to BK & CW = KC. P. 3. B. 1.
3. Thro' those points T, O & W, draw the straight lines TU, OP & WX p^{lle.} to BF or CH, until they meet the other p^{lle.} produced in the points U, P & X. P. 3. B. 1.
4. Thro' the lines TU, OP & WX let the planes TR, OQ & WY pass, p^{lle.} to the planes BE & CD, which will meet the plane AEDG in SR, NQ & VY.

DEMONSTRATION.

BECAUSE the lines BO & TO, are each = to BK & CW = KC (*Prep. 2.*) & the lines OP, TU & WX p^{lle.} to BF or CH, meet FH produced, in the points, P, U & X (*Prep. 3.*).

1. The pgrs. TP & BP are $\equiv$ to the pgr. BL; & pgr. CX $\equiv$ pgr. KH. *P. 35. B. 1.*
The planes AR or AQ & TF or OF being pple; & the plane NP pple. to the plane AF; moreover the lines SA & RE being pple. to the lines BT or FU.
2. The solid OQEB will be a $\square \equiv$ & $\propto$ to the $\square$ BEMK. *D. 10. B. 11.*
3. It may be demonstrated after the same manner that the solid TRQQ is $\equiv$ & $\propto$ to $\square$ BEMK; also the solid CDYW is $\equiv$ & $\propto$ to $\square$ KMDC.
But there are as many equal $\square$ OQEB, &c. as there are equal pgrs. OF, TP, &c. & those $\square$ together compose the $\square$ TE: moreover there are as many equal pgrs. OF, &c. as there has been taken straight lines, each $\equiv$ to BK, which together are $\equiv$ to TB.
4. Consequently, the $\square$ TE is the same multiple of the $\square$ BEMK that the parts (TO, OB) of the line TB taken together, are multiples of the line BK.
5. Likewise the $\square$ CDYW is the same multiple of the $\square$ KMDC that the line WC is of the line KC.
6. Therefore according as the $\square$ TREB is $>$, $=$ or $<$ the $\square$ BEMK, the line TB will be $>$, $=$ or $<$ the line BK.
And according as the $\square$ CDYW is $>$, $=$ or $<$ $\square$ KMDC, the line CW will be $>$, $=$ or $<$ the line KC.
7. Consequently, the $\square$ BEMK : $\square$ KMDC = BK : KC. *D. 5. B. 5.*
But BK : KC = base BL : base KH. *P. 1. B. 6.*
8. Therefore $\square$ BEMK : $\square$ KMDC = base BL : base KH. *P. 11. B. 5.*
Which was to be demonstrated,





PROPOSITION XXVI. PROBLEM IV.

AT a given point (A) in a given straight line (AB), to make a solid angle equal to a given solid angle (F).

Given.

- I. A point A in a straight line AB.
- II. A solid angle F.

Sought.

At the point A, a solid angle = to the solid angle F.

Resolution.

1. From any point I in one of the sections about the solid $\angle$ F, let fall a $\perp$ IL upon the opposite plane GFH. P.11. B.11.
2. Draw LF, LG, LH, HI & GI in the planes which form the solid $\angle$. Pos.1. B. 1.
3. In the given straight line AB, take $AM = FG$. P. 3. B. 1.
4. At the point A, make a plane $\angle$ MAD = the plane $\angle$ GFH. P.23. B. 1.
5. Cut off $AD = FH$. P. 3. B. 1.
6. In the same plane MAD, make a plane $\angle$ MAE = to the plane $\angle$ GFL, P.23. B. 1.
7. Cut off $AE = FL$. P. 3. B. 1.
8. At the point E, in the plane MAD erect the $\perp$ EC. P.12. B.11.
9. Make $EC = LI$. P. 3. B. 1.
10. Draw AC. Pos.1. B. 1.

Preparation.

Draw ME, ED, CD & CM in the planes, MAD, CAD & MAC.

DEMONSTRATION.

BECAUSE in the $\triangle GFH$ & MAD , the sides FG & FH are $=$ to the sides AM & AD , each to each, (*Ref. 3. & 5.*) & $\angle GFH$ is $=$ to $\angle MAD$, (*Ref. 4.*)

1. GH will be $=$ to MD .

2. Likewise in the $\triangle GFL$ & AME , GL is $=$ to ME .
Therefore if GL be taken from GH & ME from MD .

} *P. 4. B. 1.*

3. LH will be $=$ to ED .

Ax. 3. B. 1.

And since in the $\triangle LHI$ & EDC , ED is $=$ to LH , LI is $=$ to EC & the $\angle DEC$ & $\angle HLI$, are $\angle$, (*Arg. 3. Ref. 9. & D. 3. B. 11.*)

4. IH will be $=$ to CD .

P. 4. B. 1.

Likewise in the $\triangle FLI$ & AEC , LI is $=$ to EC , & LF is $=$ to AE , besides $\angle FLI$ & $\angle AEC$, are $\angle$, (*Ref. 7. 9. & D. 3. B. 11.*)

5. Therefore FI is $=$ to AC .

P. 4. B. 1.

6. It may be demonstrated after the same manner that GI is $=$ to MC .
Since then the three sides HI , FI & FH of the $\triangle IFH$ are $=$ to the three sides DC , AC & AD , of the $\triangle CAD$ (*Arg. 4. & 5.*)

7. $\angle IFH$ will be $=$ to $\angle CAD$.

P. 8. B. 1.

8. Likewise $\triangle GFI$ is $=$ to the $\triangle MAC$ & $\angle GFI$ is $=$ to $\angle MAC$.
Therefore the plane $\angle GFH$ being $=$ to the plane $\angle MAD$, (*Ref. 4.*)

The plane $\angle IFH$ is $=$ to the plane $\angle CAD$ (*Arg. 7.*)

And the plane $\angle GFI$ is $=$ to the plane $\angle MAC$, (*Arg. 8.*)

Besides the plane $\angle GFH$, $\angle IFH$ & $\angle GFI$, form a solid $\angle F$.

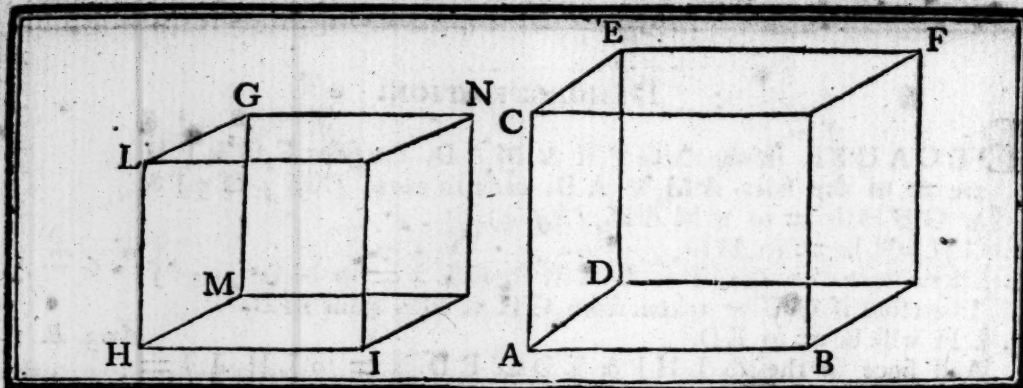
And the plane $\angle MAD$, $\angle CAD$ & $\angle MAC$, similarly situated as these already mentioned, form the solid $\angle A$.

9. It follows that the solid $\angle A$ is $=$ to the solid $\angle F$.

D. 9. B. 11.

Which was to be done.





PROPOSITION XXVII. PROBLEM V.

TO describe from a given straight line (A B), a parallelepiped similar, & similarly situated to one given (H N).

Given.

- I. A straight line A B.
- II. The $\square$ H N.

Sought.

From A B to describe a $\square$ A F, $\propto$ & similarly situated to a $\square$ H N.

Resolution.

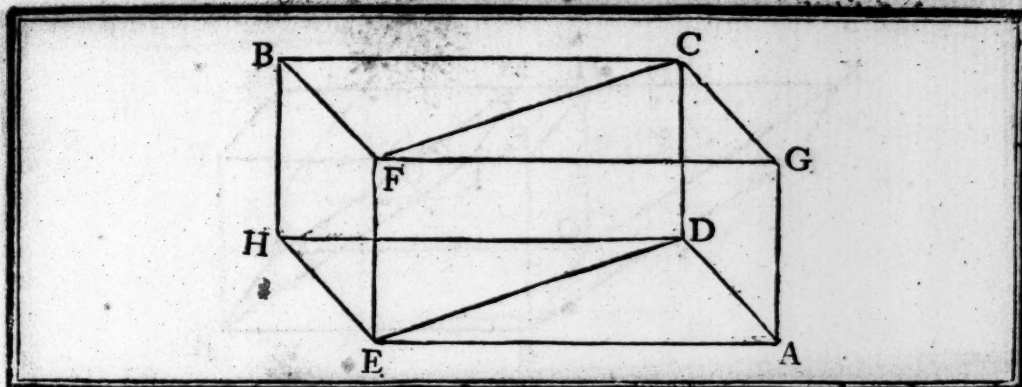
1. At the point A in the line A B make a solid $\forall$ C A D B, = to the solid $\forall$ H, or L H M I. P.26. B.11.
2. Cut A C so that $H I : H L = A B : A C$. P.12. B. 6.
3. Also A D so that $H L : H M = A C : A D$. P.31. B. 1.
4. Complete the pgrs. A E, B D & B C.
5. Complete the $\square$ A F.

DEMONSTRATION.

THE three pgrs. A E, B D & B C being $\propto$ & similarly situated with the three pgrs. H G, M I & L I of the $\square$ H N, each to each (Ref. 1. 2. 3. & 4. & D. 1. B. 6).
As also their opposite ones.

1. Consequently, the six planes or pgrs. which form the $\square$ A F, are $\propto$, & similarly situated to the six planes or pgrs. which form the given $\square$ H N. P.24. B.11.
2. Therefore the $\square$ A F described from A B, is similar & similarly situated to the given $\square$ H N. D. 9. B.11.

Which was to be done.



PROPOSITION XXVIII. THEOREM XXIII.

IF a parallelepiped (AB) be cut by a plane (FCDE) passing thro' the diagonals (FC & ED) of the opposite planes (BG & AH): it shall be cut into two equal parts.

Hypothesis.

The $\square$ AB is cut by a plane FD passing thro' the diagonals FC & ED of the opposite planes BG & AH.

Thesis.

The plane FD cuts the $\square$ AB into two equal parts.

DEMONSTRATION.

BECAUSE the plane FA is a pgr.

1. The sides EF & GA are = & pple. }

2. Likewise CD & GA are = & pple. }

3. Consequently, EF is = & pple. to CD.

4. Therefore ED is = & pple. to FC.

5. From whence it follows that FCDE is a pgr.

But the pgr. BCGF is = & pple. to the pgr. HDAE.

6. Consequently, the $\triangle$ BCF & FGC are = & $\propto$ to the $\triangle$ HDE & EDA.

Moreover, the pgrs. FEAG & GADC, are = & $\propto$ to the pgrs. BHDC & BHEF, each to each.

7. Therefore all the planes which form the prism BFD are = & $\propto$ to all the planes which form the prism DFG.

8. Therefore the prism BFD or BHEDCF is = & $\propto$ to the prism DFG or DEF CGA.

9. Consequently, the plane FCDE, cuts the $\square$ AB into two equal parts.

P.24. B.11.

P.33. B.11.

{ P. 9. B.11.

{ Ax.1. B. 1.

P.33. B. 1.

D.35. B. 1.

P.24. B.11.

{ P.34. B. 1.

{ P. 4. B. 1.

P.24. B.11.

P.24. B.11.

P.24. B.11.

P.24. B.11.

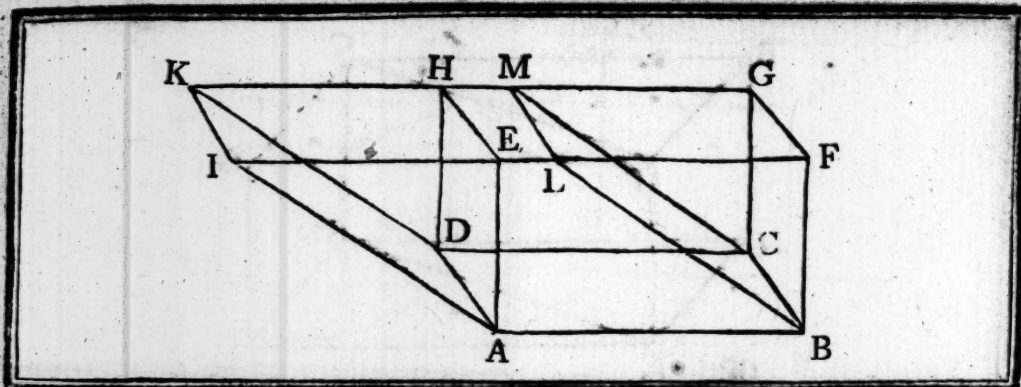
P.24. B.11.

P.24. B.11.

P.24. B.11.

Which was to be demonstrated.

N n



PROPOSITION XXIX. THEOREM XXIV.

PARALLELEPIPEDS (HB & KB) upon the same base (BD), and of the same altitude (AE), the insisting straight lines of which (AE, AI; BF, BL; DH, DK; CG, CM) are terminated in the same straight lines (IF, GK) in the plane opposite the base, are equal to one another.

Hypothesis.

Thesis.

- I. The $\square$ KB & HB have the same base BD.
- II. They have the same altitude AE.
- III. The insisting lines AE, AI, &c. of which, are terminated in the lines IF, GK.

$\square$ HB is $\equiv$ $\square$ KB.

DEMONSTRATION.

BECAUSE the pgrs. KC or KMCD, & HC or HGCD, have the same base DC, & their opposite sides KD, MC, & DH, CG, are terminated in KG which is pte. to DC (*Hyp. 3*).

1. The pgr. KC is $\equiv$ to the pgr. HC.

P. 35. B. 1.

Therefore if from those equal pgrs. be taken away the common trapezium HMCD.

2. The remainders, viz. the $\triangle$ KHD & MGC will be equal.

Ax. 3. B. 1.

3. Likewise $\triangle$ IEA is $\equiv$ to the $\triangle$ LFB.

4. The pgr. KE or KHEI, is also $\equiv$ to the pgr. MF or MGFL. Because they are each $\equiv$ to the pgr. DCBA, less the pgr. HMLE, (*D. 30. & P. 24. B. 11*).

But the plane GB or CF is $\equiv$ to the plane HA or DE, & the plane MB or LC is $\equiv$ to the plane KA or ID.

P. 24. B. 11.

5. Consequently, the prism HAKD is $\equiv$ to the prism GBMC. Therefore if to those equal prisms the part HMCBLEAD be added.

D. 10. B. 11.

6. The prism HAKD + part HMCBLEAD is $\equiv$ prism GBMC + part HMCBLEAD.

But prism HAKD + part HMCBLEAD = $\square$ KB.

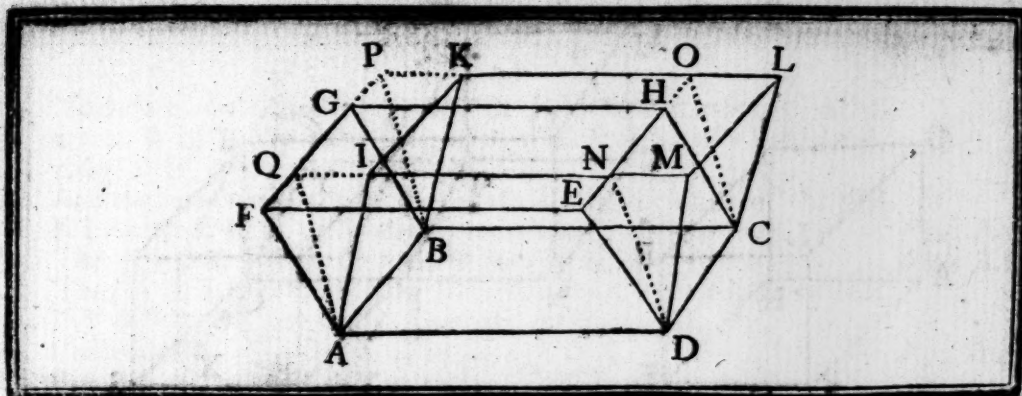
And prism GBMC + part HMCBLEAD = $\square$ HB.

Ax. 1. B. 2.

7. Therefore the $\square$ KB is $\equiv$ $\square$ HB.

Ax. 1. B. 1.

Which was to be demonstrated.



PROPOSITION XXX. THEOREM XXV.

PARALLELEPIPEDS (FGHEDCBA & IMLKBCA) upon the same base (ABCD) and of the same altitude, the insisting straight lines of which (AF, AI; DE, DM; BG, BK; CH, CL), are not terminated in the same straight lines in the plane opposite the base, are equal to one another.

Hypothesis.

- I. The $\square$ HA & LA are upon the same base AC.
- II. They have the same altitude.
- III. The insisting straight lines AF, AI, &c. are not terminated in the same straight lines.

Thesis.

$\square$ FHC is = $\square$ ILCA.

Preparation.

1. Produce LK & FG until they meet in P.
2. Produce IM until it meets FG in Q.
3. And EH to O.
4. Draw QA, PB, OC & ND.

} *Pos. 2. B. 1.*
Pos. 1. B. 1.

DEMONSTRATION.

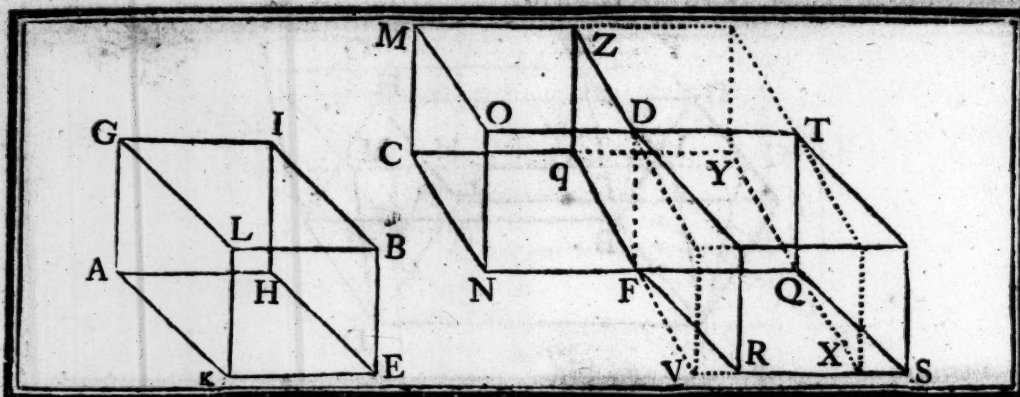
BECAUSE the $\square$ FHCA & QOCA have the same base ABCD, & their insisting straight lines AF, AQ; DE, DN; BG, BP; & CH, CO are terminated in the lines FP & EO.

1. The $\square$ FHCA is = to the $\square$ QOCA.
2. Likewise the $\square$ QOCA is = to $\square$ ILCA.
3. Therefore the $\square$ FHCA is = to the $\square$ ILCA.

P. 29. B. 11.

Ax. 1. B. 1.

Which was to be demonstrated.



PROPOSITION XXXI. THEOREM XXVI.

PARALLELEPIPEDS (KI & NZ) which are upon equal bases H & Nq, and of the same altitude, are equal to one another.

Hypothesis.

Thesis.

I. The $\square$ KI & NZ, have their bases KH & Nq equal.

The $\square$ KI is = to the $\square$ NZ.

II. They have the same altitude.

DEMONSTRATION.

CASE I.

If the insisting lines AG, &c. of the $\square$ KI; & the insisting lines CM, &c. of the $\square$ NZ, are $\perp$ to their bases; or if the inclinations of the insisting straight lines AG & MC are the same.

Preparation.

1. PRODUCE NF, & make FQ = AH. { Pof. 1. B. 1.
2. At the point F in FQ, make the plane ∇ QFR = plane ∇ HAK. { P. 3. B. 1.
3. Make FR = AK. P. 23. B. 1.
4. Complete the pgr. FQSR. P. 31. B. 1.
5. Complete likewise with the lines FQ & FD; FR & FD, the pgrs QTDF & DFR. P. 31. B. 1.
6. Complete the $\square$ DS. Pof. 2. B. 1.
7. Produce the straight lines Fq & RS until they meet in V. P. 31. B. 1.
8. Thro' the point Q, draw XQY, p^{le}. to Vq. { P. 36. B. 1.
9. Produce Cq, until it meets XY, in the point Y. { D. 1. B. 6.
10. Complete the $\square$ ZQ & VDTX.

BECAUSE the lines FQ & FR are = to AH & AK.

(Prep. 1. & 3).

And the ∇ QFR is = to the ∇ HAK (Prep. 2).

1. The pgr. FS is = & $\propto$ to the pgr. KH { P. 36. B. 1.
2. It may be demonstrated after the same manner that the pgrs FT & DR are = & $\propto$ to the pgrs. AI, & AL. { D. 1. B. 6.

Therefore, since the three pgrs. FS , FT , & DR , of the $\square DS$ are $=$ & $\propto$ to the three pgrs. AE , AI , & AL , of the $\square KI$, (*Arg. 1. & 2.*).

And the remaining pgrs. of the $\square DS$, likewise those of the $\square KI$ are $=$ & $\propto$ to those already mentioned; each to each.

3. The $\square DS$, will be $=$ & $\propto$ to the $\square KI$. P.24. B.11.
D.10. B.11.
- The $\square DX$ & DS , have the same base DQ , & their insitfing lines FV & FR , &c. are in the same plle. directions VS , &c.
4. Consequently, $\square DS$ is $=$ to the $\square DX$. P.29. B.11.
But the $\square DS$ is $=$ to the $\square KI$ (*Arg. 3.*)
5. Therefore the $\square DX$ is also $=$ to the $\square KI$. Ax.1. B. 1.
- The $\square MQ$ is cut by the plane FZ , plle. to the plane MN .
6. Consequently, the base Nq : base qQ $=$ $\square MF$: $\square ZQ$. P.25. B.11.
The $\square ZX$ is cut by the plane DQ , plle. to the plane ZY .
7. Consequently, the base FX : base qQ $=$ $\square DX$: $\square ZQ$. P.25. B.11.
P.35. B. 1.
But the pgr. FX is $=$ to the pgr. FS .
And the pgr. FS is $=$ to the pgr. HK . (*Arg. 1.*)
8. Consequently, the pgr. FX is $=$ to the pgr. HK . Ax.1. B. 1.
But the base HK is $=$ to the base qN (*Hyp. 1.*)
9. Hence the base qN $=$ to the base FX .
But the base qN : base qQ $=$ $\square MF$: $\square ZQ$ (*Arg. 6.*)
And the base qQ : base FX $=$ $\square ZQ$: $\square DX$. (*Conv. Arg. 7.*)
10. Hence the base qN : base FX $=$ $\square MF$: $\square DX$. P.22. B. 5.
But the base qN is $=$ to the base FX (*Arg. 9.*)
11. Consequently, the $\square MF$ is $=$ to the $\square DX$. P.14. B. 5.
But the $\square DX$ & KI are equal. (*Arg. 5.*)
12. Therefore, the $\square MF$ is $=$ to the $\square KI$. Ax.1. B. 1.

Which was to be demonstrated.

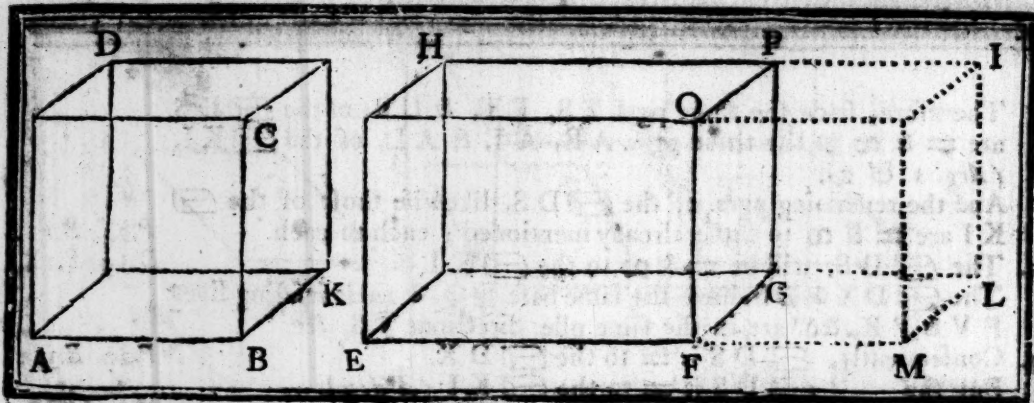
CASE II.

If the angles of inclination of the insitfing straight lines, AG &c. of $\square KI$ are not equal to the angles of inclination of the insitfing straight lines CM , &c. of the $\square MF$.

UPON the base KI , make a $\square$ having its insitfing straight lines, either $\perp$: or equally inclined with the insitfing straight lines of the $\square MF$, & in the same direction as those of $\square KI$.
And consequently, which will be equal to it (*P. 30. B. 11.*).
The remainder of the construction, & of the demonstration, are the same as in the foregoing case.

COROLLARY.

EQUAL parallelepipeds which have the same altitude, have equal bases.



PROPOSITION XXXII. THEOREM XXVII.
PARALLELEPIPEDS (BD & EP) which have equal altitudes (BC & FO), are to one another as their bases (AK & EG).

Hypothesis.
 The altitudes BC & FO, of the
 $\square$ BD & EP, are equal.

Thesis.
 $\square$ BD : $\square$ EP = base AK : base EG.

Preparation.

1. Produce EF to M.
2. Upon FG with FM, make the pgr. FL = pgr. KA, which will be in the same direction with the pgr. EG. So that the pgrs. EG & FL together, form the pgr. EL. P.44. B. 1.
3. Complete the $\square$ FI.

Pos. 2. B. 1.

DEMONSTRATION.

BECAUSE the base FL of the $\square$ FI, is = to the base AK of the $\square$ BD (*Prep. 2*).

1. The $\square$ FI is = to the $\square$ BD.
2. Consequently, $\square$ FI : $\square$ EP = $\square$ BD : $\square$ EP.
 But, $\square$ FI : $\square$ EP = base FL : base EG.
 And the base FL is = to the base AK (*Prep. 2*).
3. Therefore, $\square$ BD : $\square$ EP = base AK : base EG.

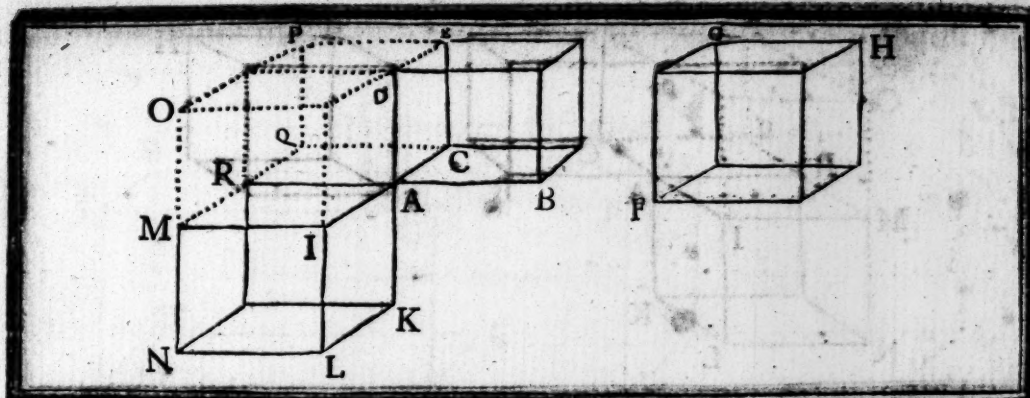
P.31. B.11.

P. 7. B. 5.

P.25. B.11.

{ *P.11. & 7.*
B. 5.

Which was to be demonstrated.



PROPOSITION XXXIII. THEOREM XXVIII.

SIMILAR parallelepipeds (EB & FH) are to one another in the triplicate ratio of their homologous sides (AB & GH).

Hypothesis.

The $\square$ EB & FH are $\square$, & the sides AB & GH are homologous.

Thesis.

The $\square$ EB is to the $\square$ FH in the triplicate ratio of AB to GH, or as $AB^3 : GH^3$.

Preparation.

1. Produce AB & make $AR = GH$.
2. From AR describe the $\square$ RL = & $\square$ to the $\square$ FH, so that the lines AC & AI; DA & AK be in the same straight line.
3. Complete the $\square$ AO, so as to form with $\square$ RL the $\square$ OK.
4. Complete likewise the $\square$ AP, so as to form with $\square$ OA, the $\square$ OC, & with the $\square$ EB the $\square$ PB.

{ Pof. 2. B. 1.
P. 3. B. 1.

P. 27. B. 11.

DEMONSTRATION.

BECAUSE the $\square$ EB & RL, are $\square$ (Prep. 2).

1. The pgr. AM is $\square$ to the pgr. CB.
2. Consequently, $AB : AC = AR : AI$.
3. And alternando $AB : AR = AC : AI$.
4. Likewise $AB : AD = AR : AK$.
5. And alternando $AB : AR = AD : AK$.
- And since AR is = to GH (Prep. 1).
6. The three ratios $AB : AR$, $AC : AI$, & $AD : AK$, are equal to one another & equal to the ratio of AB to GH. But the $\square$ PB is cut by the plane AE (Prep. 4).
7. Consequently, the base CB : base QA = $\square$ BE : $\square$ AP. And the base CB : base QA = $AB : AR$.
8. Therefore $AB : AR = \square$ BE : $\square$ AP.

D. 9. B. 11.

D. 1. B. 6.

P. 16. B. 5.

D. 1. B. 6.

P. 16. B. 5.

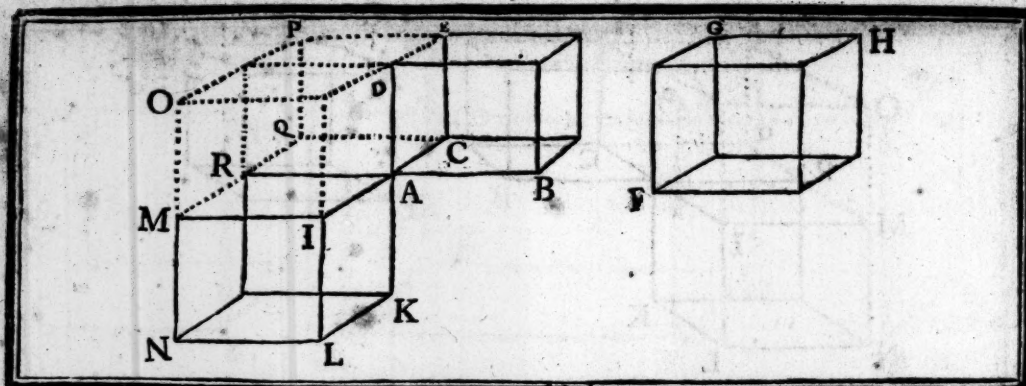
P. 25. B. 11.

P. 1. B. 6.

P. 11. B. 5.

Which was to be demonstrated.

* See Cor. 2. of this proposition.



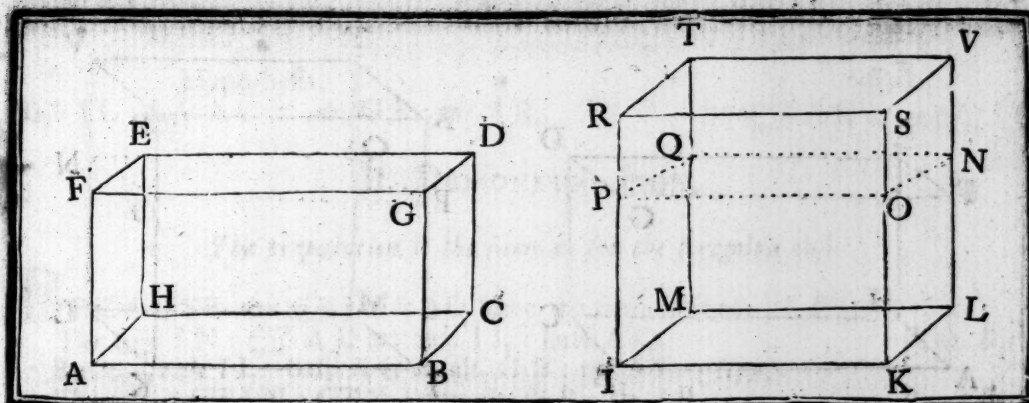
- The $\square$ O C is cut by the plane R D (*Prep. 4*).
9. Consequently, the base R C : base A M = $\square$ A P : $\square$ O A. P. 25. B. 11.
 And, the base R C : base A M = A C : A I. P. 1. B. 6.
10. Therefore, A C : A I = $\square$ A P : $\square$ O A. P. 11. B. 5.
- In fine, the $\square$ O K being cut by the plane A M (*Prep. 4*).
11. It may be demonstrated after the same manner.
 That A D : A K = $\square$ A O : $\square$ A N.
- But the three ratios A B : A R, A C : A I, & A D : A K are =
 to the ratio A B : G H (*Arg. 6*).
12. Consequently, the four $\square$ B E, A P, A O, & A N form a series
 of magnitudes in the same ratio (A B : G H). P. 11. B. 5.
13. Therefore, they are proportionals. D. 6. B. 5.
14. Consequently, the $\square$ B E is to the $\square$ A N in the triplicate ratio
 of A B to G H. D. 11. B. 5.
- But the $\square$ B E is to the $\square$ F H in the triplicate ratio of A B
 to G H, (or as A B³ to G H³). *

COROLLARY I.

FROM this it is manifest, that if four straight lines be continual proportionals, as the first is to the fourth, so is the parallelepiped described from the first to the similar & similarly described parallelepiped from the second; because the first straight line has to the fourth, the triplicate ratio of that which it has to the second.

* COROLLARY II.

ALL cubes being similar parallelepipeds (D. IX. & XXX. B. 11), similar parallelepipeds (A B & F H) are to one another as the cubes of their homologous sides (A B & G H) (expressed thus A B³ : G H³); because they are in the triplicate ratio of those same sides.



PROPOSITION XXXIV. THEOREM XXIX.

THE bases, (pgrs. AC & IL) and altitudes (GB & IR) of equal parallelepipeds, (AD & IV) are reciprocally proportional; and if the bases, (pgrs. AC & IL) and altitudes (GB & IR) be reciprocally proportional, the parallelepipeds are equal.

Hypothesis.

$\square AD$ is = to $\square IV$.

Thesis.

Base AC : base IL = alt. IR : alt. GB.

I. DEMONSTRATION.

The given parallelepipeds may be either.

- CASE 1. Of the same altitude }
 CASE 2. Of different altitudes } and equally inclined on their bases.
 CASE 3. Having different inclinations : as if one was $\perp$ to the
 base, and the other oblique.

CASE I.

When the $\square$ have the same altitude, that is, $IR = GB$.

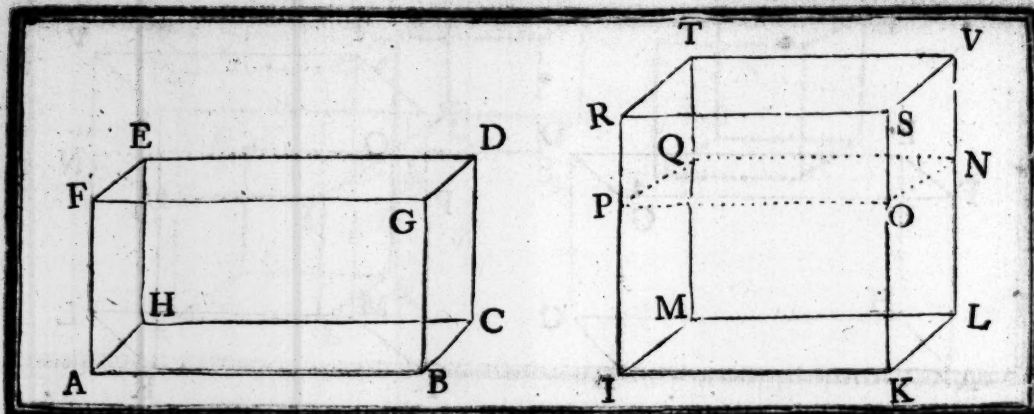
BECAUSE the given $\square$ are equal, & have the same altitude.

1. Their bases are equal (Cor. of P. 31. B. 11).
2. Therefore, the base AC : base IL = altitude IR : altitude GB. D. 6. B. 5.

CASE II.

When $IR > GB$.

○ ○



I. Preparation.

1. From the alt. RI, cut off the part PI = to the alt. BG.
2. Thro' the point P, pass the plane P O N Q, p^lle. to the base IL.

BECAUSE the parallelepipeds AD & IN have the same altitude (*I. Prep. 1*).

1. The $\square AD : \square IN = \text{base } AC : \text{base } IL$. P. 32. B. 11.
But the $\square AD$ is = to the $\square IV$ (*Hyp*).
2. Therefore, $\square AD : \square IN = \square IV : \square IN$ P. 7. B. 5.
3. Consequently, $\square IV : \square IN = \text{base } AC : \text{base } IL$. P. 11. B. 5.
The $\square IV$ is cut by the plane P O N Q (*I. Prep. 2*).
4. Therefore, $\square PV : \square IN = \text{base } PS : \text{base } KP$. P. 25. B. 11.
5. Therefore, componendo $\square IV : \square IN = \text{base } KR : \text{base } KP$. P. 18. B. 5.
But the base KR : base KP = RI : PI. P. 1. B. 6.
6. Wherefore, $\square IV : \square IN = RI : PI$. P. 11. B. 5.
But, $\square IV : \square IN = \text{base } AC : \text{base } IL$ (*Arg. 3*).
And PI = GB (*I. Prep. 1*).
7. Consequently, base AC : base IL = IR : BG. P. 11. B. 5.

CASE III.

When the $\square IV$ has a different inclination from the $\square AD$.

II. Preparation.

Describe a $\square$ of the same altitude with the $\square IV$, having the same inclination as the $\square AD$.

BECAUSE the described $\square$, has the same base & the same altitude with the $\square AD$ (*II. Prep*).

1. This $\square$ will be = to the given $\square IV$. P. 31. B. 11.
But this described $\square$ is in the reciprocal ratio of its base, & of its altitude with the $\square AD$ (*Case II*).
2. Therefore, the $\square IV$ will be also in reciprocal ratio with the $\square AD$. P. 7. B. 5.

Which was to be demonstrated.

Hypothesis.
Base IL : base AC = alt. GB : alt. IR .

Thesis.
 $\square AD$ is = $\square IV$.

II. DEMONSTRATION.

The preparation is the same as for the foregoing case.

- B**ECAUSE the $\square IN$ & AD have the same altitude (*I. Prep. 1*).
1. The $\square IN$: $\square AD$ = base IL : base AC . P. 32. B. 11.
But the base IL : base AC = alt. GB : alt. IR . (*Hyp*).
 2. Therefore $\square IN$: $\square AD$ = alt. GB : alt. IR . P. 11. B. 5.
And as PI is = BG . (*I. Prep. 1*).
 3. The $\square IN$: $\square AD$ = alt. PI : alt. IR . P. 7. B. 5.
But PI : IR = pgr. PK : pgr. KR . P. 1. B. 6.
And pgr. KP : pgr. KR = $\square IN$: $\square IV$. P. 32. B. 11.
 4. Therefore the $\square IN$: $\square AD$ = $\square IN$: $\square IV$. P. 11. B. 5.
But the $\square IN$ is the first & third terms of the proportion.
 5. Consequently, the $\square AD$ is = to the $\square IV$. P. 14. B. 5.

Which was to be demonstrated.

The demonstrations of the first and third cases in this hypothesis, are the same, for which reason we have omitted them.

R E M A R K I.

WHAT has been demonstrated in the propositions 25, 29, 30, 31, 33 & 34, concerning parallelepipeds, is also true with respect to triangular prisms; because such a prism is the half of its parallelepiped; (P. 28. B. 11.) from whence we may conclude.

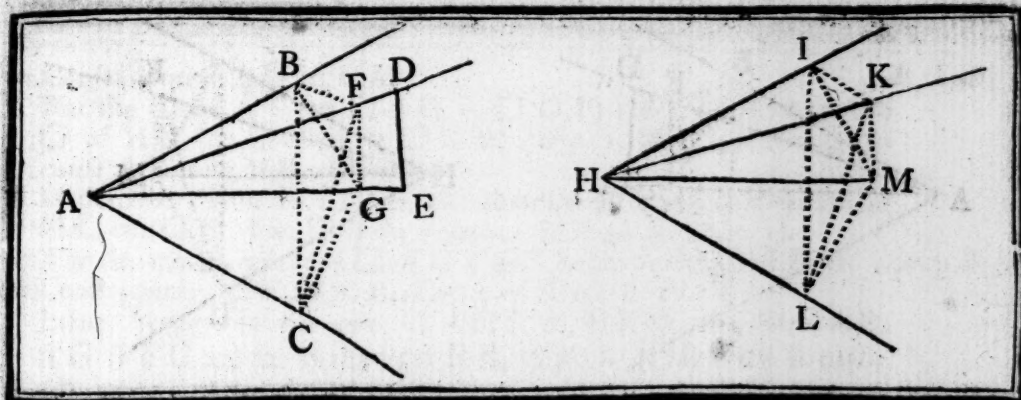
- I. If a triangular prism be cut by a plane p^{er}pe. to the opposite planes; the two prisms resulting from thence, will be to one another as the parts of the pgr., base of the whole prism.
- II. Triangular prisms which have the same, or equal bases, & have equal altitudes, are equal.
- III. Triangular prisms which have the same altitude, are to one another as their bases.
- IV. Similar triangular prisms, are to one another in the triplicate ratio, of their homologous sides.
- V. Equal triangular prisms, have their bases and altitudes reciprocally proportional, & triangular prisms whose bases and altitudes, are reciprocally proportional, are equal.

R E M A R K II.

*W*ITH the same properties prisms are endued, whose opposite planes are polygons. Since it has been demonstrated, (P. 20. B. 6.) that those opposite & similar polygons may be divided into the same number of similar triangles; therefore if thro' the homologous diagonals which form those triangles, planes, be passed: those planes will divide the polygon prisms, into as many triangular prisms as there are triangles in their opposite & ple. planes.

But what has been observed in the foregoing remark, is applicable to those triangular prisms. Consequently, we may conclude (P. 12. B. 5.) that polygon prisms are endued with the same properties.





PROPOSITION XXXV. THEOREM XXX.

IF from the vertices (A & H) of two equal plane angles (BAC & IHL), there be drawn two straight lines (AD & HK) above the planes in which the angles are, and containing equal angles ($\angle BAD = \angle IHK$ & $\angle DAC = \angle KHL$), with the respective sides of those angles, (viz. AD with AB & AC; HK with IH & HL), and from any two points (D & K) in those lines, (AD & HK), above the planes, there be let fall the perpendiculars (DE & KM), on the planes of the first named angles (BAC & IHL), and from the points (E & M), in which the perpendiculars meet those planes, the straight lines (AE & HM), be drawn to the vertices (A & H), of the angles first named: those straight lines (AE & HM), shall contain equal angles (DAE & KHM), with the straight lines (AD & HK) which are above the planes of the angles.

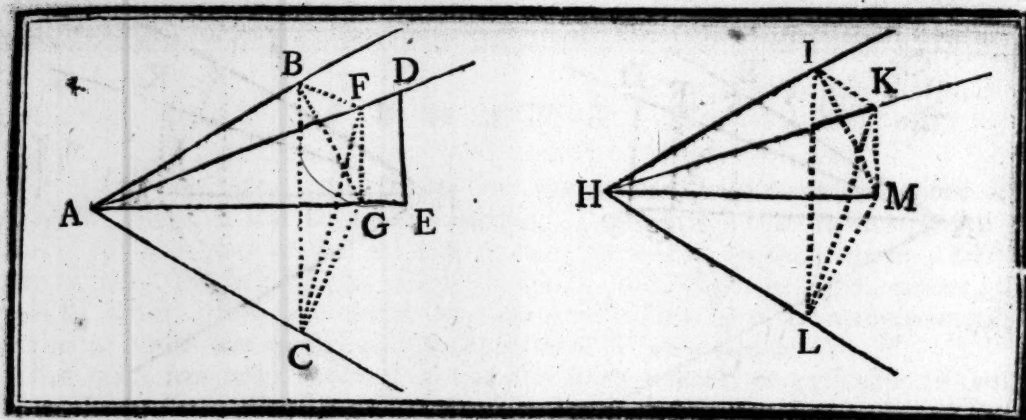
Hypothesis.

Thesis.

- I. Above the planes of the equal $\angle BAC$ & $\angle IHL$, & from their vertices A & H, there has been drawn AD & HK, containing $\angle BAD$ & $\angle DAC = \angle IHK$ & $\angle KHL$, each to each.
- II. From the two points D & K, in AD & HK, there has been let fall the $\perp$ DE & KM, on the planes BAC & IHL.
- III. From the points E & M, where the $\perp$ meet those planes, there has been drawn AE & HM, to the vertices A & H.

Preparation.

1. Make $AF = HK$. P. 3. B. 1.
2. Draw FG, p^lle. to DE, until it meets the plane BAC in G. P. 31. B. 1.
3. From the point G, in the plane BAC, draw CG, $\perp$ to AC; & GB, $\perp$ to AB. P. 12. B. 1.
4. From the point K in the plane IHL, draw IM, $\perp$ to HI; & ML, $\perp$ to HL. P. 12. B. 1.
5. Draw BF, BC & FC; IK, IL & LK. Pof. 1. B. 1.



DEMONSTRATION.

BECAUSE FG is p^lle. to DE which is $\perp$ to the plane BAC. (*Hyp.* 111).

1. The line GF is $\perp$ to the same plane BAC. P. 8. B. 11.
And the $\sphericalangle$ FGB, FGA & FGC are $\perp$. D. 3. B. 11.
2. Consequently, the $\square$ of AF is = to $\square$ of FG + $\square$ of GA. P. 47. B. 1.
But the $\square$ of AG is = to $\square$ of AB + $\square$ of BG. (*Prep.* 3.) & P. 47. B. 1.
3. Therefore, the $\square$ of AF is = to $\square$ FG + $\square$ AB + $\square$ BG. Ax. 1. B. 1.
But the $\square$ GB + $\square$ FG are = to the $\square$ BF (*Prep.* 3.) P. 47. B. 1.
4. Consequently, the $\square$ AF is also = to the $\square$ BF + $\square$ AB.
5. Therefore, $\sphericalangle$ ABF, is a $\perp$. P. 48. B. 1.
6. It may be demonstrated after the same manner that $\sphericalangle$ FCA, is a $\perp$.
7. That also the $\sphericalangle$ KIH & KLH, are $\perp$.
In the $\triangle$ FCA & KLH; the line HK is = to AF (*Prep.* 1.)
the $\sphericalangle$ ACF & KLH, are $\perp$ (*Arg.* 6. & 7.), & the $\sphericalangle$ FAC = $\sphericalangle$ KHL, (*Hyp.* 1). P. 26. B. 1.
8. Therefore the sides AC & CF are = to the sides HL & LK, each to each.
9. Likewise AB is = to HI & BF = IK.
10. Consequently, in the $\triangle$ BAC & IHL; the bases BC & IL are equal and the $\sphericalangle$ ACB & ABC = to the $\sphericalangle$ HLI & HIL, each to each. P. 4. B. 1.
Therefore if those equal $\sphericalangle$, be taken from the four $\perp$ ACG, ABG, HLM & HIM.
11. The remaining $\sphericalangle$ will be equal, viz. $\sphericalangle$ BCG = $\sphericalangle$ ILM & $\sphericalangle$ CBG = $\sphericalangle$ LIM. Ax. 5. B. 1.
Since then the $\triangle$ GBC & IML have their bases BC & IL equal (*Arg.* 10).
And the $\sphericalangle$ at those bases are equal, each to each, (*Arg.* 11).
12. The sides BG & CG will be = to the sides IM & ML. P. 26. B. 1.
In the $\triangle$ BAG & HIM, AB is = to HI (*Arg.* 9.) BG = IM, (*Arg.* 12.) & the $\sphericalangle$ ABG & HIM are $\perp$. (*Prep.* 3. & 4.)

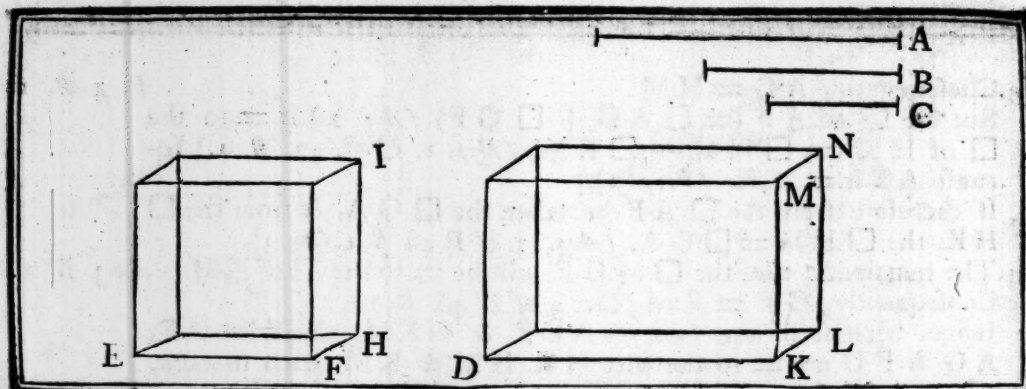
13. Consequently, $AG = HM$ P. 4. B. 1.
 But the $\square$ of AF ($= \square AG + \square GF$) (*Arg. 2.*) is $=$ to the
 $\square$ of HK ($= \square HM + \square KM$) (*Hyp. 1. & P. 47. B. 1.*) be-
 cause AF is $= HK$. (*Prep. 1.*)
 If therefore from the $\square AF$ be taken the $\square GA$, & from the $\square$
 HK , the $\square HM = \square GA$, (*Arg. 13. & P. 46. B. 1. Cor. 3.*)
 14. The remainder, *viz.* the $\square$ of GF will be $=$ to the $\square$ of KM . *Ax. 3. B. 1.*
 15. Consequently, $GF = KM$ (*Cor. 3. of P. 46. B. 1.*)
 Infine, because in the two $\triangle AGF$ & HKM , the sides AF ,
 AG & FG are $=$ to the sides HK , HM & KM , each to each,
 (*Prep. 1. & Arg. 13. & 15.*)
 16. The $\angle FAG$ or DAE is $=$ to the $\angle KHM$. P. 8. B. 1.

Which was to be demonstrated.

C O R O L L A R Y.

IF from the vertices A & H of two equal plane angles BAC & IHL , there be elevated two equal straight lines AF & HK ; containing with the respective sides, the $\angle BAF$ & FAC equal to the $\angle IHK$ & KHL ; each to each, & there be let fall from those points F & K (of those elevated straight lines) the perpendiculars FG & KM on the planes BAC & IHL : those $\perp FG$ & KM will be equal. (*Arg. 15.*)





PROPOSITION XXXVI. PROBLEM XXXI.

IF three straight lines (A, B, C) be proportionals, the parallelepiped (D N), described from these three lines as its sides, is equal to the equiangular parallelepiped (E I), described from the mean proportional (B).

Hypothesis.

Thesis.

- I. The straight lines A, B, & C are proportionals, that is, $A : B = B : C$. The $\square$ E I is = to the $\square$ D N.
- II. The $\square$ D N, is described from those three lines, that is, $DK = A$, $MK = B$, & $KL = C$.
- III. The equiangular $\square$ E I, is described from the mean proportional B, that is, $EF = FG = FH = B$.

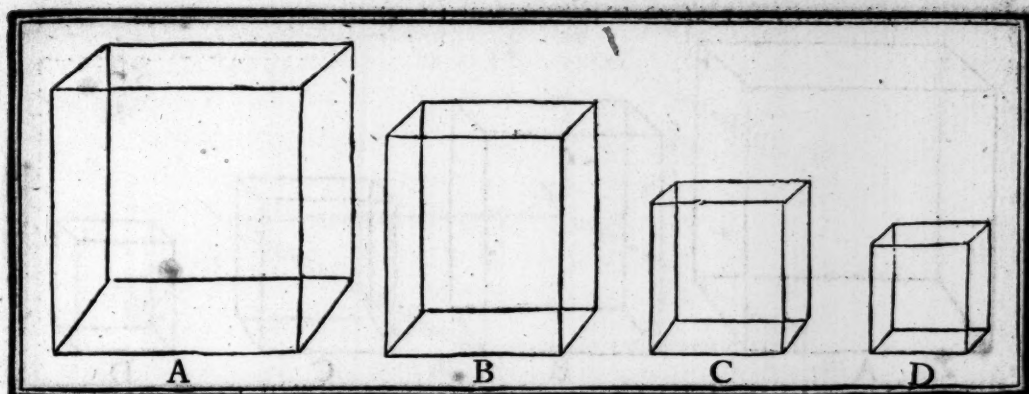
DEMONSTRATION.

BECAUSE $DK : EF = EF$ or $FH : KL$ (Hyp. 2).

And the plane ∇EFH is = to the plane ∇DKL (Hyp. 3).

1. The pgr. DL, base of $\square$ D N is = to the pgr. EH, base of $\square$ E I P. 14. B. 6. Moreover, the plane ∇GFE & ∇GFH contained by the elevated line FG, & the sides EF & FH, being = to the plane ∇MKD , & ∇MKL , contained by the elevated line KM, & DK, & KL, each to each, (Hyp. 3.), & $FG = KM$, (Hyp. 2. & 3.).
2. The $\perp$ let fall from the point G, on the base EH, will be = to the $\perp$ let fall from the point M on the base DL. (Cor. of P. 35. B. 11).
3. Consequently, $\square$ E I has the same altitude with the $\square$ D N. D. 4. B. 6. But the base EH of $\square$ E I is = to the base DL of $\square$ D N, (Arg. 1).
4. Therefore, $\square$ E I is = to the $\square$ D N. P. 31. B. 11.

Which was to be demonstrated.



PROPOSITION XXXVII. THEOREM XXXII.

IF four straight lines (A, B, C, & D) be proportionals, (*that is, if*, $A : B = C : D$) : the similar and similarly described parallelepipeds, from the two first (A & B), will be proportional to the similar and similarly described parallelepipeds, from the two last (C & D); and if the two similar and similarly described parallelepipeds, from the two lines (A & B); be proportional to the two other similar and similarly described parallelepipeds, from the two other straight lines (C & D); the homologous sides of the first (A & B), will be proportional to the homologous sides (C & D) of the last.

Hypothesis.

I. $A : B = C : D$.II. From A & B there has been described $\square$.

III. Also from C & D.

Thesis.

 $\square A : \square B = \square C : \square D$.

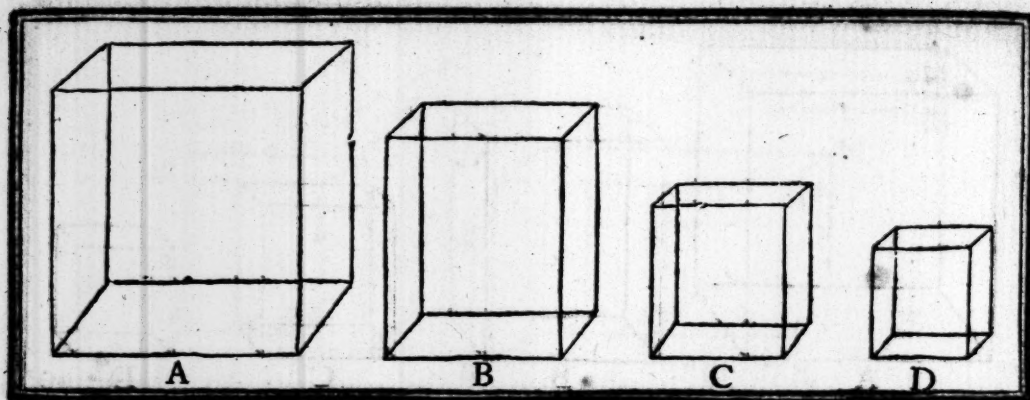
DEMONSTRATION.

BECAUSE the $\square A$ is $\propto$ to the $\square B$ (Hyp. 2).

1. The $\square A : \square B = A^3 : B^3$.2. Likewise, the $\square C : \square D = C^3 : D^3$.But the ratio of A to B being $=$ to the ratio of C to D (Hyp. 1).3. It follows, that three times the ratio of A to B is $=$ to three times the ratio of C to D, *that is*, $A^3 : B^3 = C^3 : D^3$.4. Consequently, the $\square A : \square B = \square C : \square D$.

{ P. 33. B. 11.
Cor.

Ax. 6. B. 1.
P. 11. B. 5.



Hypothesis.

- I. The $\square$ A is $\propto$ to the $\square$ B.
- II. Also the $\square$ C is $\propto$ to the $\square$ D.
- III. The $\square$ A : $\square$ B = $\square$ C : $\square$ D.

Thesis.

$$A : B = C : D.$$

II. DEMONSTRATION.

BECAUSE the $\square$ A is $\propto$ to the $\square$ B (Hyp. 1).

1. The $\square$ A : $\square$ B = A^2 : B^2 .

P. 33. B. 11.

Likewise the $\square$ C is $\propto$ to the $\square$ D (Hyp. 2).

2. The $\square$ C : $\square$ D = C^2 : D^2 .

P. 33. B. 11.

But the $\square$ A : $\square$ B = $\square$ C : $\square$ D (Hyp. 3).

3. Therefore, A^2 : B^2 = C^2 : D^2 .

P. 11. B. 5.

4. Consequently, $A : B = C : D$.

Ax. 7. B. 1.

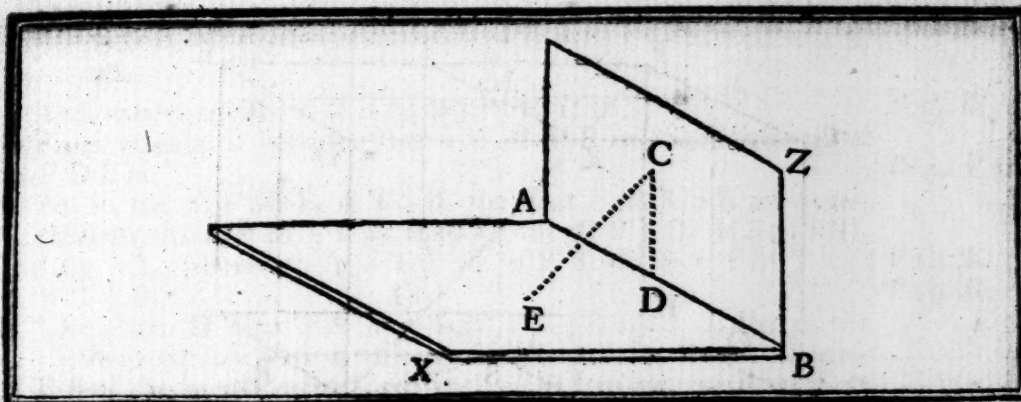
Which was to be demonstrated.

REMARK.

- I. **B**ECAUSE the triangular prism is the half of its parallelepiped (P. 28. B. 11.), it follows (Ax. 7. B. 1.), that the same truth is applicable to similar triangular prisms.

- II. It may be also applied to similar polygon prisms; because they may be divided by planes into triangular prisms. (Remark 2. of P. 34. B. 11).





PROPOSITION XXXVIII. THEOREM XXXIII.

IF two planes (A Z & A X) be perpendicular to one another ; and a straight line (C D) be drawn from the point (C) in one of the planes (A Z) perpendicular to the other (A X); this straight line shall fall on the common section (A B) of the planes.

Hypothesis.

The plane A Z, is $\perp$ to the plane A X.

Thesis.

The line C D drawn from the point C, situated in the plane A Z, $\perp$ to the plane A X, falls on the common section A B.

DEMONSTRATION.

If not,

There may be drawn a $\perp$ as C E, which will not fall on the common section A B.

Preparation.

From the point C, let fall on A B, in the plane A Z, a $\perp$ C D.

P.12. B. 1.

BECAUSE C D is $\perp$ to the common section A B (*Prep*).

1. C D will be $\perp$ to the plane A X.

But E C is $\perp$ to the same plane. (*Sup.*).

D. 4. B.11.

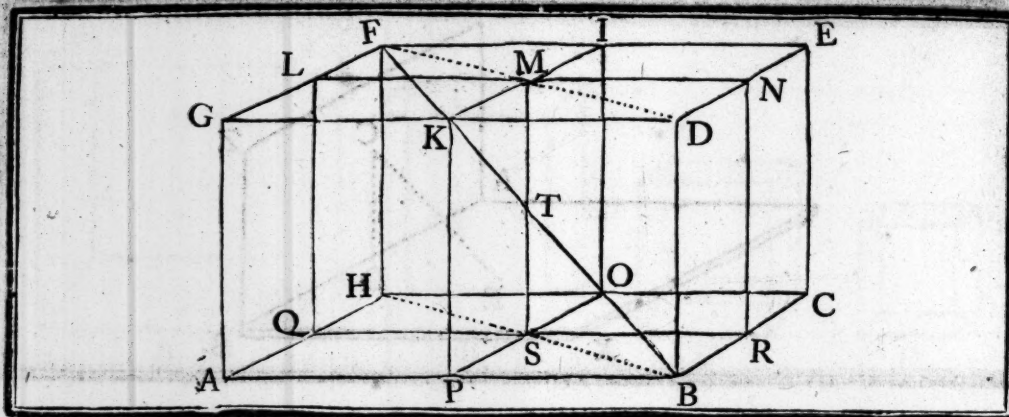
2. Therefore, from the same point C, there has been drawn to the plane A X, two $\perp$ E C & C D.

3. Which is impossible.

P.13. B.11.

4. Consequently, the $\perp$ C D let fall from the point C, of the plane A Z, to the plane A X (which is perpendicular to it) passes thro' their common section A B.

Which was to be demonstrated.



PROPOSITION XXXIX. THEOREM XXXIV.

IN a parallelepiped (A'E) if the sides (GD, AB; GF, AH; FE, HC; ED, & BC) of the opposite planes, (FA & EB; FC & GB) be divided each into two equal parts, the common section (MS) of the planes (IP & LR), passing thro' the points of section (K, P, O, I & L, Q, R, N) and the diameter (FB) of the parallelepiped (A'E) cut each other into two equal parts in the point (T).

Hypothesis.

Thesis.

- I. In the $\square$ AE, having for diam FB; the sides DG, AB, &c. are bisected in the points K, P, &c. The common section MS of those planes, & the diam. FB, cut each other into two equal parts in the point T.
- II. The planes KO & LR, have been passed thro' the points, K, P, O, I, & L, Q, R, N.

Preparation.

Draw SB, SH, FM, & MD.

Pos. 1. B. 1.

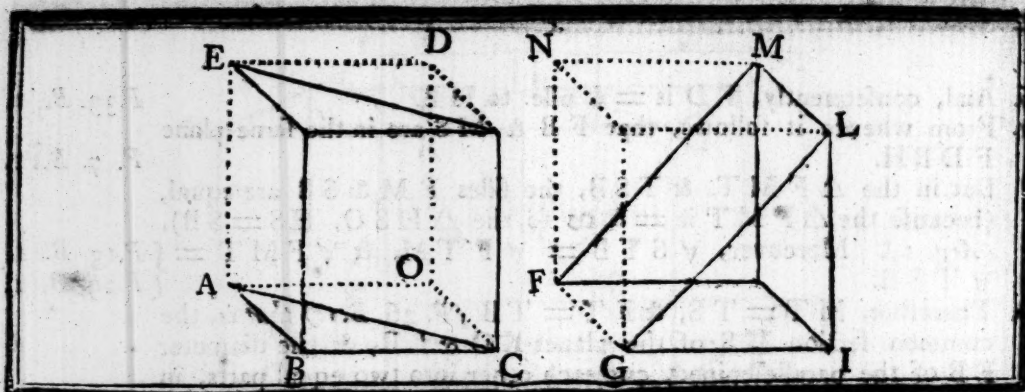
DEMONSTRATION.

- THE sides HQ & SQ being = to the sides BR & SR (Hyp. 1). P. 34. B. 1.
And the $\angle$ HQS = $\angle$ SRB. P. 29. B. 1.
1. The base HS of the $\triangle$ HSQ will be = to the base SB of the $\triangle$ BSR, & $\angle$ HSQ = $\angle$ RSB. P. 4. B. 1.
 2. Consequently, $\angle$ RSH + $\angle$ RSB = 2 $\angle$. P. 13. B. 1.
 3. Wherefore, $\angle$ HSB is a straight line. Ax. 1. B. 1.
 4. It may be demonstrated after the same manner, that FD is a straight line. P. 14. B. 1.
 5. Moreover, BD being = & pple. to AG & AG = & pple. to FH. P. 34. B. 1.
- The line BD will be = & pple. to FH. { P. 9. B. 11.
Ax. 1. B. 1.

6. And, consequently, FD is $\parallel$ & plle. to HB . P.33. B. 1.
 7. From whence it follows, that FB & MS are in the same plane $FDBH$. P. 7. B.11.
 But in the $\triangle FMT$, & TSB , the sides FM & SB are equal,
 (because the $\triangle FMT$ is $\parallel$ & $\propto$ to the $\triangle HSO$, $HS=SB$),
 (Arg. 1). Moreover, $\angle STB = \angle FTM$, & $\angle FMT = \angle TSB$. { P.15. B. 1.
P.29. B. 1.
 8. Therefore, $MT=TS$, & $FT=TB$ (P. 26. B.1.) *that is*, the
 common section MS of the planes KO & LR , & the diameter
 FB of the parallelepiped, cut each other into two equal parts, in
 the point T .

Which was to be demonstrated.





PROPOSITION XL. THEOREM XXXV.

IF two triangular prisms (FL & EC) have the same altitude (LI & AE), and the base of one (as CL) is a parallelogram (FI), and the base of the other (EC) a triangle (ABC): if the parallelogram be double of the triangle, the first prism (LF) will be equal to the second (EC).

Hypothesis.

Thesis.

- I. In the prisms FL & EC; the alt. LI is = to the alt. AE. The prism FL is = to the prism EC.
- II. The base of the prism LF is a pgr. FI, & the base of the prism EC a $\triangle ABC$.
- III. The pgr. FI is double of the $\triangle ABC$.

Preparation.

Complete the $\square$ NI & BD.

DEMONSTRATION.

BECAUSE the pgr. FI, base of the prism FL, is double of the $\triangle ABC$, base of the prism EC (*Hyp. 2. & 3*).
And the pgr. BO is also double of the $\triangle ABC$.

1. The pgr. FI is = to the pgr. BO.
Moreover, the altitude LI being = to the altitude AF (*Hyp. 1*),
2. The $\square$ BD is = to the $\square$ NI.
The given prism LF is the half of the $\square$ ND. }
And the prism EC is the half of the $\square$ BD. }
3. Consequently, the prism FL is = to the prism EC.

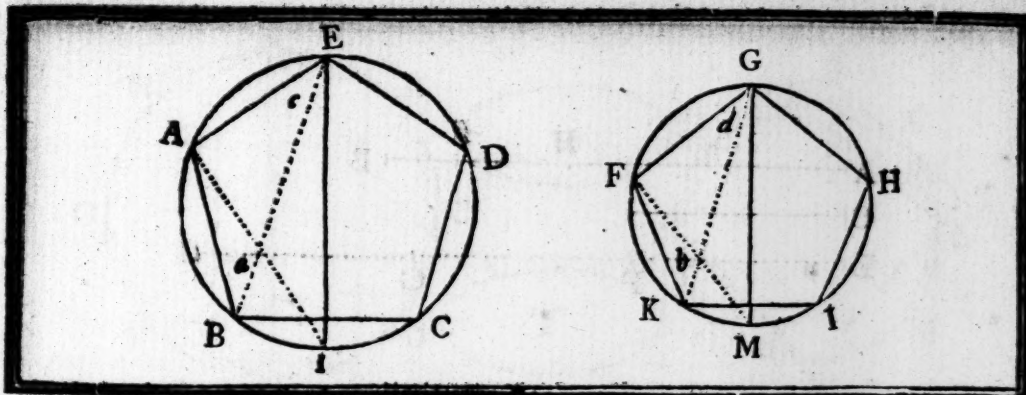
P.41. B. 1.

P.31. B.11.

P.28. B.11.

Ax.7. B. 1.

Which was to be demonstrated.



PROPOSITION I. THEOREM I.

SIMILAR polygons (ABCDE & FGHIK), inscribed in circles are to one another as the squares of their diameters (EL & GM).

Hypothesis.

I. The polygons ABCDE & FGHIK are $\propto$.

II. They are inscribed in circles.

Thesis.

Polyg. : ACE : polyg. FIH = the $\square$ of the diam. EL : $\square$ of the diam. GM, or as diam. EL² : diam. GM².

Preparation.

1. In the $\odot$ ACD, draw AL, & BE, also diam. EL.
2. In the $\odot$ FMH, draw the homologous lines FM & GK, also the diameter GM.

} Pof. 1. B. 1.

DEMONSTRATION.

BECAUSE the polygons ABCDE & GFKIH are $\propto$ (Hyp. 1). And the $\sphericalangle$ A or EAB is = to $\sphericalangle$ GFK, & AE : AB = FG : FK (D. 1. B. 6).

1. The $\triangle$ ABE is equiangular with the $\triangle$ FGK. P. 6. B. 6.
2. Wherefore, $\triangle$ ABE is $\propto$ to $\triangle$ GFK, & $\sphericalangle a = \sphericalangle b$, also $\sphericalangle c = \sphericalangle d$.

But $\sphericalangle$ ELA is = $\sphericalangle$ EBA, or a , & $\sphericalangle$ GMF = $\sphericalangle$ GKF or b . P. 21. B. 6.

3. Consequently, $\sphericalangle$ ELA is = to $\sphericalangle$ GMF. Ax. 1. B. 1.

4. Likewise, $\sphericalangle$ EAL = $\sphericalangle$ GFM. P. 31. B. 3.

And, because, in the two $\triangle$ ALE & GFM, the two $\sphericalangle$ ELA & EAL of the first are = to the two $\sphericalangle$ GMF & GFM of the second (Arg. 3. & 4).

5. The third $\sphericalangle$ AEL of the $\triangle$ EAL will be = to the third $\sphericalangle$ FGM of the $\triangle$ FMG.

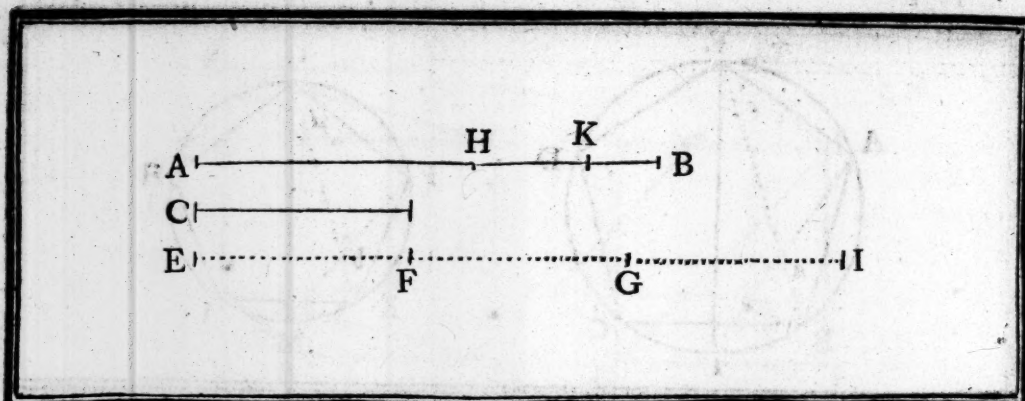
6. Therefore, EL : AE = GM : GF. P. 32. B. 1.

7. And alternando EL : GM = AE : GF. P. 4. B. 6.

But AE & GF are homologous sides of the polygons ABD & FHK. Besides, EL & GM are the diameters of the $\odot$ in which those polygons are inscribed.

8. Wherefore, polyg. ABCDE : polyg. FKHG = EL² : GM². P. 22. B. 1.

Which was to be demonstrated.



L E M M A.

IF from the greater (A B), of two unequal magnitudes (A B & C), there be taken more than its half (viz. A H), and from the remainder (H B) more than its half (viz. H K), and so on: there shall at length remain a magnitude (K B), less than the least (C), of the proposed magnitudes.

Preparation.

1. Take a multiple E I of the least C, which may surpass A B, & be $> 2 C$. *Pos. 1. B. 5.*
2. From A B, take a part H A $>$ the half of A B. *Pos. 2. B. 5.*
3. From the remainder H B, take H K $>$ the half of H B.
4. Continue to take more than the half from those successive remainders, until the number of times, be equal to the number of times, that C is contained in its multiple E I. *Pos. 2. B. 5.*

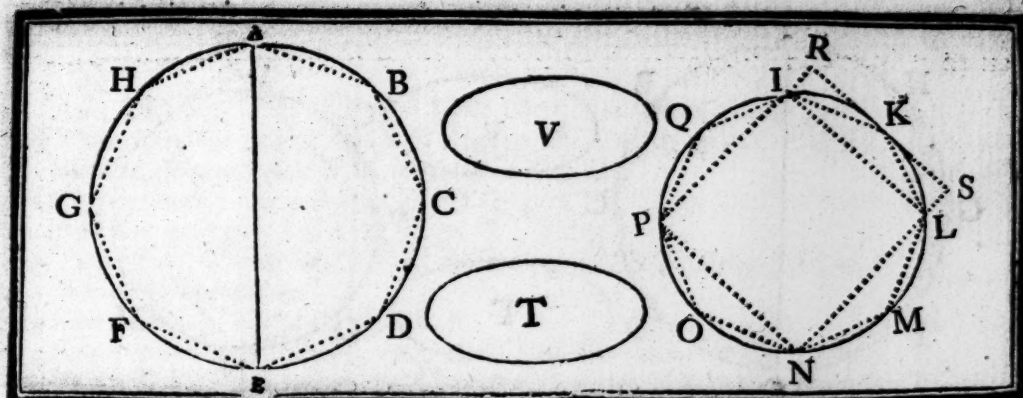
DEMONSTRATION.

BECAUSE the magnitude E I is a multiple greater than twice the least magnitude C (*Prep. 1*).

If there be taken from it a magnitude G I $=$ C.

1. The remainder E G will be $>$ the half of E I. *P. 19. B. 5.*
But E I is $>$ A B (*Prep. 1*).
2. Consequently, the half of E I is $>$ the half of A B.
3. Therefore, G E will be much $>$ the half of A B.
But H B is $<$ the half of A B (*Prep. 2*).
4. Much more then G E is $>$ H B.
5. Therefore, E F, the half of E G, is $>$ the half of H B.
And K B is $<$ the half of H B (*Prep. 3*).
6. Consequently, E F is $>$ K B.
And as the same reasoning may be continued until a part (E F) of the multiple of the magnitude C be attained, which will be equal to C (*Prep. 4*).
7. It follows, that the magnitude C will be $>$ the remaining part (K B) of the greater A B.

Which was to be demonstrated.



PROPOSITION II. THEOREM II.

CIRCLES (AFD & ILP), are to one another as the squares of their diameters (AE & IN).

Hypothesis.

In the circles AFD & ILP there has been drawn the diameters AE & IN.

Thesis.

$$\odot AFD : \odot ILP = AE^2 : IN^2.$$

DEMONSTRATION.

If not,

AE^2 is to IN^2 as the $\odot AFD$ is to a space T (which is $<$ or $>$ the $\odot ILP$).

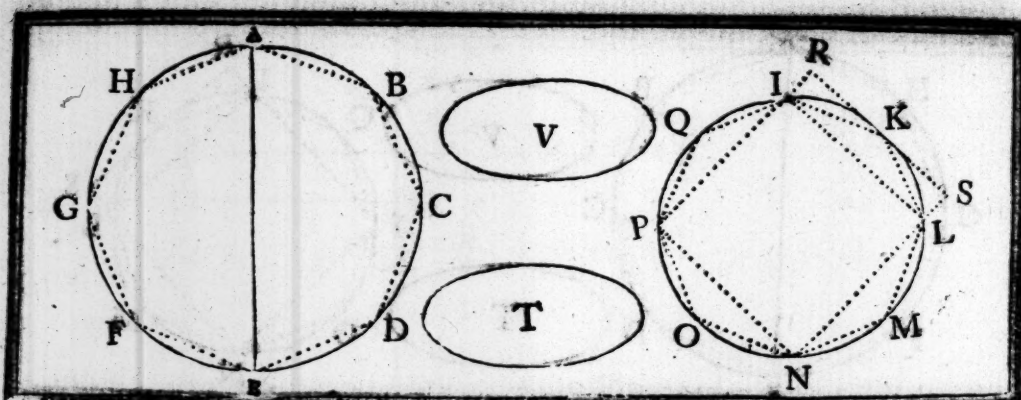
I. Supposition.

Let T be $<$ $\odot ILP$ by the space V. that is, $T + V = \odot ILP$.

I. Preparation.

1. In the $\odot ILP$ describe the $\square ILNP$. P. 6. B. 4.
2. Divide the arches IL, LN, NP, & PI into two equal parts in the points K, M, O, & Q. P. 30. B. 3.
3. Draw the lines IK, KL, LM, MN, NO, OP, PQ & QI. P. 1. B. 1.
4. Thro' the point K, draw SR pple. to LI. P. 31. B. 1.
5. Produce NL & PI to R & S; which will form the rgle. SRIL.
6. Inscribe in the $\odot ADF$ a polygon $\propto$ to the polygon of the $\odot ILP$.

Qq



BECAUSE the $\square$ described about the $\odot$ ILP is $>$ the $\odot$ itself.

1. The half of this $\square$ will be $>$ the half of the $\odot$ ILP.
But the inscribed $\square$ ILNP is = to half of the circumscribed $\square$ (the side of the circumscribed $\square$ being = to the diameter, & the $\square$ of the diameter = $\square$ LI + $\square$ LN = $2 \square$ LI).

Ax. 8. B. 1.

P. 19. B. 5.

2. Therefore, the $\square$ LIPN is $>$ the half of the $\odot$ ILP.

P. 47. B. 1.

Ax. 1. B. 1.

The rgle. SI is $>$ the segment LKI (*Prep. 5. & Ax. 8. B. 1.*)

3. Consequently, the half of the rgle. SI is $>$ the half of the segment LKI.

P. 19. B. 5.

The $\triangle$ LKI is = to half of the rgle. SI.

P. 41. B. 1.

4. Therefore, the $\triangle$ LKI is $>$ the half of the segment LKI.

P. 19. B. 5.

5. It may be proved after the same manner, that all the $\triangle$ LMN, NOP, &c. are each $>$ the half of the segment in which it is placed.

6. Wherefore, the sum of all those triangles will be $>$ the sum of the half of all those segments.

Continuing to divide the segments KI, IL, &c. as also the segments arising from those divisions.

It will be proved after the same manner.

7. That the triangles formed by the straight lines drawn in those segments, are together $>$ the half of the segments in which those triangles are placed.

Therefore, if from the $\odot$ ILP be taken more than its half, viz. the $\square$ ILNP, & from the remaining segments (LKI, IQP, &c.) be taken more than the half, & so on.

8. There will at length remain segments which together, will be $<$ V.

Lem. B. 12.

But the $\odot$ ILP is = $T + V$ (*1. Sup.*)

Therefore, taking those segments LKI, &c. from the $\odot$ ILP.

And the space V, from $T + V$ (which is $>$ those segments).

9. The remainder, viz. the polygon IKLMNOPQ will be $>$ T. *Ax. 5. B. 1.*

But the polyg. ADFK : polyg. ILQ = $\square$ of AE : $\square$ of IN. *P. 1. B. 12.*

And the $\square$ of $AE : \square$ of $IN = \odot ACEG : T$. (*Sup.*)

10. Therefore, the polyg. $ADFH : \text{polyg. } ILOQ = \odot ACEG : T$. *P. 11. B. 5.*

But the polygon $ADFH$ is $< \odot ACEG$. *Ax. 8. B. 1.*

11. Consequently, the polygon $ILOQ$ is $< T$. *P. 14. B. 5.*

But the polygon $ILOQ$ is $> T$. (*Arg. 9.*)

12. Therefore, T will be $> \& <$ the polyg. $ILOQ$ (*Arg. 9. & 11.*)

13. Which is impossible.

14. Therefore, T is not $< \odot ILP$.

15. From whence it follows, that the $\square$ of the diameter (AE) of a $\odot (ACEG)$, is not to the $\square$ of the diameter (IN) of another $\odot (ILP)$, as the first $\odot (ACEG)$ to a space $<$ the second $\odot (ILP)$.

II. Supposition.

Let the space T be $>$ the circle ILP .

II. Preparation.

Take a space V , such that

$T : \odot ACEG = \odot ILP : V$.

BECAUSE the $\square$ of $AE : \square$ of $IN = \odot ACEG : T$.

16. Invertendo $T : \odot ACEG = \square$ of $IN : \square$ of AE .

{ *P. 4. B. 5.*
Cor.

But $T : \odot ACEG = \odot ILP : V$. (*II. Prep.*)

Moreover, T is $> \odot ILP$. (*II. Sup.*)

17. Consequently, the $\odot ACEG$ is also $> V$. *P. 14. B. 5.*

Besides $T : \odot ACEG = \square$ of $IN : \square$ of AE (*Arg. 16.*)

And $T : \odot ACEG = \odot ILP : V$. (*II. Prep.*)

18. Therefore, the $\square$ of $IN : \square$ of $AE = \odot ILP : V$. *P. 11. B. 5.*

But $V < \odot ACEG$. (*Arg. 17.*)

And it has been demonstrated (*Arg. 15.*), that the $\square$ of the diameter (IN) of a $\odot (ILP)$, is not to the $\square$ of the diameter of another $\odot (ACEG)$; as the first $\odot (ILP)$ to a space $<$ the second $\odot (ACEG)$.

19. Consequently, V is not $<$ the $\odot ILP$.

20. Therefore, T is not $>$ the $\odot ILP$.

Therefore, the space T being neither $<$ nor $>$ the $\odot ILP$,

(*Arg. 14. & 19.*)

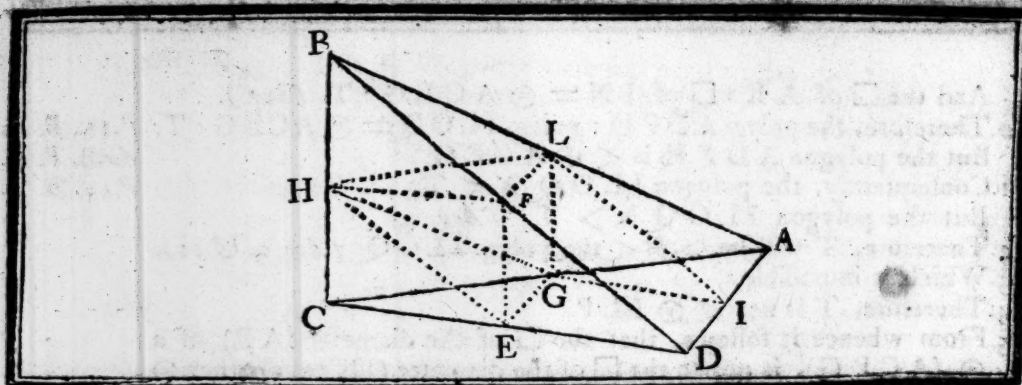
21. T will be $=$ to this $\odot ILP$.

22. Consequently, the $\odot ACEG : \odot ILP = \square$ of $AE : \square$ of IN . *P. 7. B. 1.*

Which was to be demonstrated.

C O R O L L A R Y.

CIRCLES are to one another as the polygons inscribed in them (*P. 1. B. 12. & P. 11. B. 5.*)



PROPOSITION III. THEOREM III.

EVERY pyramid ($ABCD$) having a triangular base (ACD), may be divided into two equal and similar prisms, ($IDEFLG$ & $GLFHC$), and into two equal and similar pyramids, ($LGIA$ & $LFHB$), which are similar to the whole pyramid; and the two prisms together are greater than half of the whole pyramid ($ABCD$).

Hypothesis.

$ABCD$ is a pyramid whose base ADC is a Δ .

Thesis.

- I. The part $IDEFLG$ is a prism $=$ & $\propto$ to the part $GLFHC$.
- II. The part $ALGI$ is a pyramid $=$ & $\propto$ to the part $BLFH$.
- III. Those pyramids $ALGI$ & $BLFH$ are $\propto$ to the pyramid $ABCD$.
- IV. The prisms $IDEFLG$ & $GLFHC$ are together $>$ than the half of the pyr. $ABCD$.

I. Preparation.

1. Cut all the sides of the pyramid $ABCD$ into two equal parts, in the points $L, F, H, E, G,$ & I .
2. Draw the lines LF, FH, FE, GE, GI & IL , also $LG,$ & LH .

P.10. B. 1.

Pos.1. B. 1.

DEMONSTRATION.

BECAUSE in the ΔBCD the sides BD & BC are divided into two equal parts in the points F & H (Prep. 1).

1. $BH : HC = BF : DF$.
3. Consequently, FH is pple. to DC .
3. Likewise, FE is pple. to BC .
4. Therefore, $FECH$ is a pgr.
5. It may be proved after the same manner, that $LFEG$ & $LGCH$ are pgrs.
- And since FH & HL are pple. to EC & GC . (Arg. 2. & 5).
6. The planes passing thro' LFH & ECG will be pple.
7. Therefore, $LGECHE$ will be a prism.
8. Likewise, $LFEDIG$ will be also a prism.

P.19. B. 5.

P. 2. B. 6.

D.35. B. 1.

P.15. B.11.

D.13.B. 11.

But those two prisms have the same altitude LG, & the pgr. GIDE which is the base of the prism LD is double of the $\triangle CEG$, base of the prism LC.

9. Therefore, the prism LD is = to the prism LC.

Which was to be demonstrated. 1.

BECAUSE the side BD is cut into two equal parts in F, that FE & DE are pple. to BC & FH, each to each. (*Prep. 1. & Arg. 2. & 3.*)

10. The $\triangle FDE$ is = & $\propto$ to $\triangle BFH$.

11. The $\triangle FED$ & ILG are also equal.

12. Therefore, $\triangle BFH = \triangle LIG$.

And since the other sides of the pyramid ABCD are divided into two equal parts.

It may be easily proved that,

13. $\triangle BLF$ is = to the $\triangle LAI$, $\triangle BLH = \triangle AGL$, & $\triangle LFH = \triangle AGI$.

14. From whence it follows, that those parts BLHF & ALGI are equal & $\propto$ pyramids.

Which was to be demonstrated. 11.

THE line FH, is pple. to DC. (*Arg. 2.*)

15. Therefore, $\triangle BFH$ is $\propto \triangle BDC$.

Likewise, all the triangles which form the pyramids BLHF & ALGI are $\propto$ to all the triangles of the whole pyramid ABCD.

16. Therefore, the pyramids BLHF & ALGI, are $\propto$ to the pyramid ABCD.

Which was to be demonstrated. 111.

II. Preparation.

Draw GH & EH.

THE line BH being = to HC (*I. Prep. 1.*) $FH = EC$ (*Arg. 4.*) & $\angle ECH = \angle FHB$ (*P. 29. B. 1.*)

17. Consequently, the $\triangle ECH$ is = to the $\triangle BFH$.

18. Also the $\triangle HGC$ & $\triangle GEC$ are = & $\propto$ to the $\triangle BLH$ & $\triangle LHF$.

19. Therefore, the pyramid LFHB is = to the pyramid HGEC.

But the pyramid ECHG is only a part of the prism ECHFLG.

20. Therefore, the prism ECHFLG is > the pyramid ECHG.

21. Consequently, this prism ECHFLG is also > the pyramid LFHB.

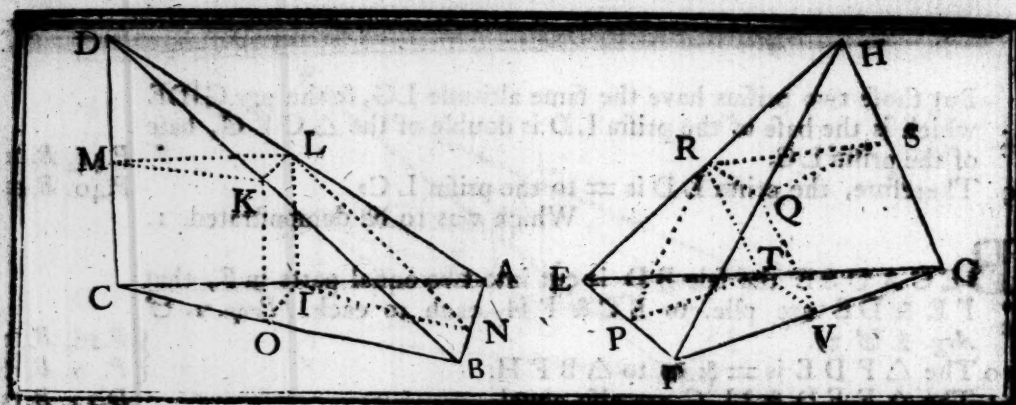
The prism LG ECHF is = to the prism EFLGID, & the pyramid LFHB = to the pyramid AIGL (*Arg. 9. & 14.*)

22. Therefore, the prism EFLGID is also > the pyramid AIGL.

23. Therefore, the two prisms ECHFLG & EFLGID together, will be > the two pyramids BLFH & LAIG together.

24. From whence it follows, that the two prisms ECHFLG & EFLGID together, are > the half of the given pyr. ABCD.

Which was to be demonstrated. 1v.



PROPOSITION IV. THEOREM IV.

IF there be two pyramids ($ABCD$ & $EFGH$) of the same altitude, upon triangular bases (ABC & EFG), and each of them be divided into two equal pyramids similar to the whole pyramid, (viz. the pyramid $ABCD$ into the pyramids $DLKM$ & $ANIL$, and the pyramid $EFGH$ into the pyramids $HRQS$ & $REPT$); and also into two equal prisms, (viz. the pyramid $ABCD$ into the prisms LB & LC , and the pyramid $EFGH$ into the prisms RF & RG); and if each of these pyramids ($DLKM$, $ANIL$, $HRQS$, & $REPT$) be divided in the same manner as the first two, and so on. The base (ABC), of one of the first two pyramids ($ABCD$), is to the base (EFG) of the other pyramid ($EFGH$), as all the prisms contained in the first pyramid ($ABCD$), is to all the prisms contained in the second ($EFGH$), that are produced by the same number of divisions.

Hypothesis.

Thesis.

- I. The triangular pyramids $ABCD$ & $EFGH$, have the same altitude.
- II. Each of them are cut into two equal prisms LB & LC ; also RF & RG , & into two equal pyramids similar to the whole pyramid.
- III. Each of those pyramids $DLKM$, $ANIL$, $RTPE$, & $RQSH$, are supposed to be divided in the same manner as the first two, & so on.

The sum of all the prisms contained in the pyramid $ABCD$ is to the sum of those contained in the pyramid $EFGH$, being equal in number; as the base ABC , of the pyramid $ABCD$ is to the base EFG , of the pyramid $EFGH$.

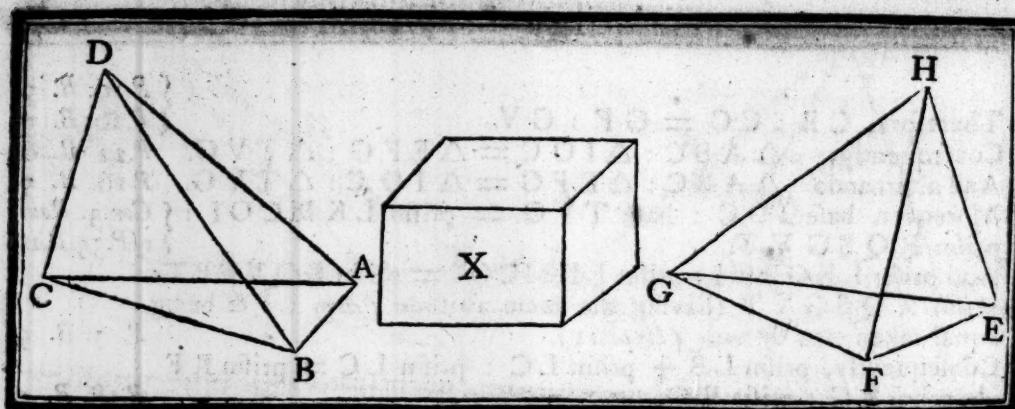
DEMONSTRATION.

BECAUSE the pyramids $ABCD$ & $EFGH$ have equal altitudes, & the prisms LB , LC , RF & RG have each the half of this altitude, (Hyp. 1. & P. 3. B. 12).

1. Those prisms LB , LC , RF & RG have the same altitude. The lines BC & FG are cut into two equal parts in the points O & V .

Ax. 7. B. 1.

P. 3. B. 12.



PROPOSITION V. THEOREM V.

PYRAMIDS (ABCD & EFGH) of the same altitude, which have triangular bases (ABC & EFG): are to one another as their bases, (ABC & EFG).

Hypothesis.

Thesis.

- I. The pyramids ABCD & EFGH have for bases the $\triangle ABC$ & EFG. Pyram. ABCD : pyram. EFGH = base ABC : base EFG.
 II. They have the same altitude.

DEMONSTRATION,

If not,

Pyramid ABCD : pyramid EFGH > base ABC : base EFG.

Preparation.

1. Take a solid X which may be > the pyramid ABCD, so that $X : \text{pyram. EFGH} = \text{base ABC} : \text{base EFG}$.
2. Divide the pyramids ABCD & EFGH as directed in P. 3. B. 12.

BECAUSE the two prisms resulting from the first division, are > the half of the pyramid ABCD; & the four following, resulting from the second division, are > than the halves of the pyramids resulting from the first division, & so on.

1. It is evident, that the sum of all the prisms contained in the pyramid ABCD, will be > the solid X, which was supposed to be < the pyramid ABCD.

P. 3. B. 12.

Lem. B. 12.

But all the prisms contained in the pyramid $ABCD$, are to all the prisms contained in the pyramid $EFGH$, as the base ABC is to the base EFG .

P. 4. B. 12.

And the solid X : pyramid $EFGH =$ base ABC : base EFG (*Prep. 1*).

2. Consequently, all the prisms contained in the pyramid $ABCD$ are to all the prisms contained in the pyramid $EFGH$, as the solid X is to the pyramid $EFGH$.

P. 11. B. 5.

But all the prisms contained in the pyramid $ABCD$, are $>$ the solid X . (*Arg. 1*).

3. Therefore, all the prisms contained in the pyramid $EFGH$, are $>$ the pyramid $EFGH$ itself.

P. 14. B. 5.

4. Which is impossible.

Ax. 8. B. 1.

5. Consequently, a solid (as X) which is $<$ the pyramid $ABCD$, cannot have the same ratio to the pyramid $EFGH$, which the base ABC , has to the base EFG .

And as the same demonstration holds for any other solid greater than the pyramid $ABCD$.

6. It follows, that the pyramid $ABCD$: pyramid $EFGH =$ base ABC : base EFG .

COROLLARY I.

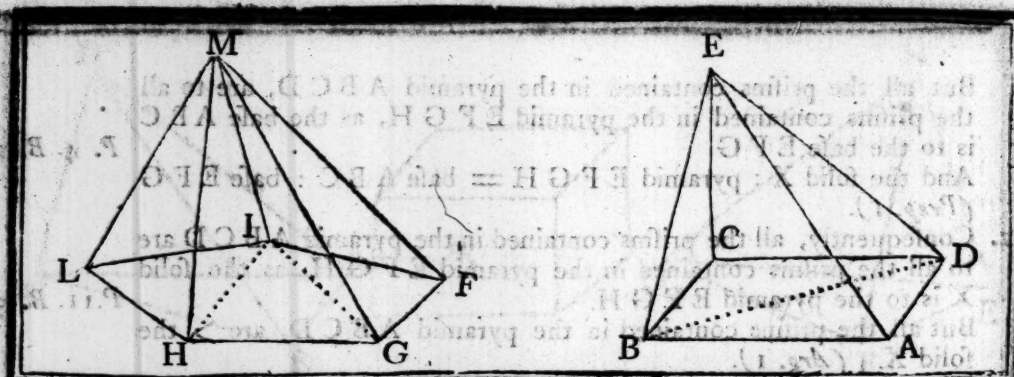
PYRAMIDS of the same altitude, & which have equal triangles for their bases : are equal. (*P. 14. & 16. B. 5.*)

COROLLARY II.

EQUAL pyramids which have equal triangles for their bases : have the same altitude.



R-1



PROPOSITION VI. THEOREM VI.

PYRAMIDS (FGLIM & ABCDE) of the same altitude, which have polygons (FGHLI, & ABCD) for their bases: are to one another as their bases.

Hypothesis.

Thesis.

- I. The pyramids FGHIL & ABCD, *Pyram. MFGHLI : pyram. ABCDE*
have polygons for their bases. $= \text{base FILHG} : \text{base ABCD}$.
- II. They have the same altitude.

Preparation.

1. Divide the bases FILHG & ABCD into triangles, by drawing the lines GI, FH, & DB.
2. Let planes be passed thro' those lines & the vertices of the pyramids, which will divide each of those pyramids into as many pyramids as each base contains triangles.

DEMONSTRATION.

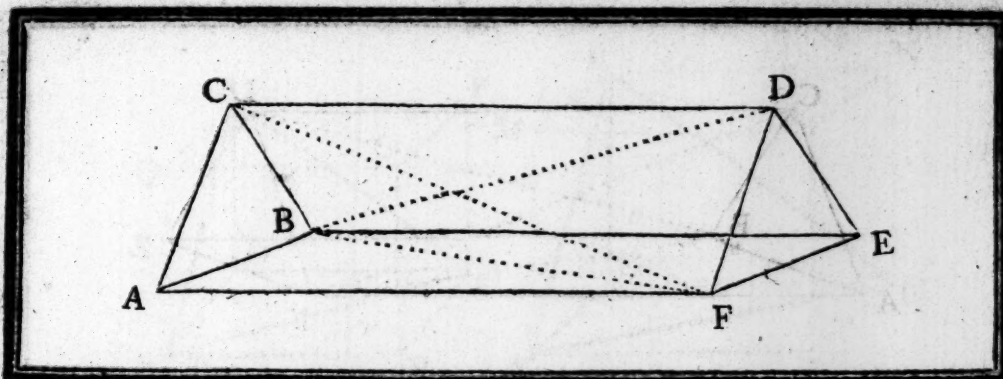
BECAUSE the triangular pyramids ILMH & ABDE have the same altitude. (*Hyp. 11. & Prep. 2*).

1. The pyramid ILMH : pyr. ABDE = base IHL : base ABD. } *P. 5. B. 12.*
2. Likewise, pyr. GIHM : pyr. ABDE = base HIG : base ABD. }
3. Consequently, pyr. ILMH + pyr. GIHM : pyr. ABDE = base IHL + base HIG : base ABD. *P. 24. B. 5.*
4. Moreover, pyr. FIGM : pyr. ABDE = base FIG : base ABD. *P. 5. B. 12.*
5. Therefore, pyr. ILMH + pyr. GIHM + pyr. FIGM : pyr. ABDE = base IHL + base HIG + base FIG : base ABD. *P. 24. B. 5.*
- But pyr. ILMH + pyr. GIHM + pyr. FIGM are = to the pyr. MFGHLI, & the base IHL + base HIG + base FIG = base FILHG. *Ax. 1. B. 2.*
6. Consequently, pyr. MFGHLI : pyr. ABDE = base FILHG : base ABD. *P. 7. B. 5.*

It may be proved after the same manner, that

7. Pyr. MFGHLI : pyr. BDCB = base FILHG : base BDC.
8. Therefore, pyr. MFGHLI : pyr. ABCDE = base FILHG : base ADCB. *P. 25. B. 5.*

Which was to be demonstrated.



PROPOSITION VII. THEOREM VII.

EVERY triangular prism (ADE) : may be divided (by planes passing through the $\triangle BCF$ & BDF) into three pyramids (ACBF, BDEF & DCBF) that have triangular bases, and are equal to one another.

Hypothesis.
The given prism ADE has a triangular base.

Thesis.
The prism ADE may be divided into three equal triangular pyramids, ACBF, BDEF, DCBF.

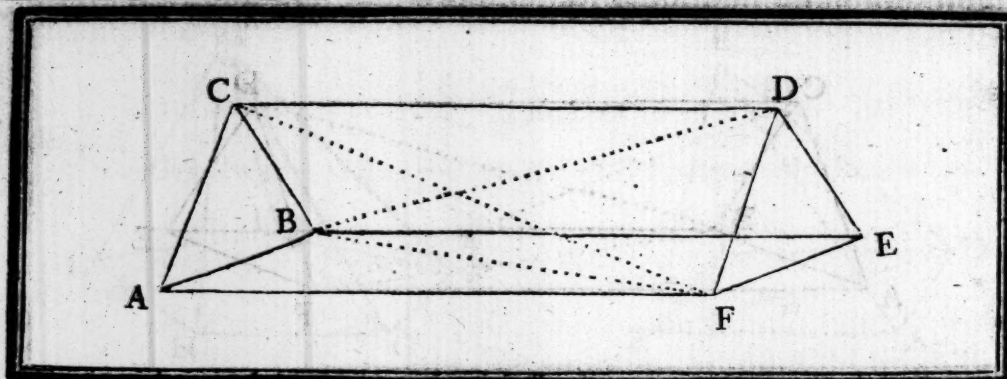
Preparation.

1. In the pgr. DA draw any diagonal CF.
2. From the point F in the pgr. AE, draw the diag. BF.
3. From the point B in the pgr. CE, draw the diag. BD.
4. Let a plane be passed thro' CF & BF, also thro' BF & BD.

DEMONSTRATION.

BECAUSE AD is a pgr. cut by the diagonal CF. (*Prep. 1*).

1. The $\triangle ACF$ base of the pyramid ABCF is = to the $\triangle CFD$, base of the pyramid BCFD. P.34. B. 1.
But those pyramids ABCF & BCFD, have their vertices at the point B.
2. Therefore, the pyramid ABCF is = to the pyramid BCFD. { P. 5. B.12. Cor. 1.
Likewise, the pgr. EC is cut by its diagonal BD. (*Prep. 3*).
3. Therefore, the $\triangle CBD$, base of the pyramid BCFD is = to the $\triangle BDE$, base of the pyramid BDEF. P.34. B. 1.
And those pyramids BCFD, &c. have their vertices at the point F.
4. Consequently, the pyramid BCFD is = to the pyramid BDEF. { P. 5. B.12. Cor. 1.
But the pyramid ABCF is also = to the pyramid BCFD. (*Arg. 2*).
5. Therefore, the pyramids ABCF, BCFD, & BDEF are equal. *Ax.1. B. 1.*



6. Consequently, the triangular prism (ADE) may be divided into three triangular pyramids.

Which was to be demonstrated.

COROLLARY I.

FROM this it is manifest, that every pyramid which has a triangular base, is the third part of a prism which has the same base, & is of an equal altitude with it.

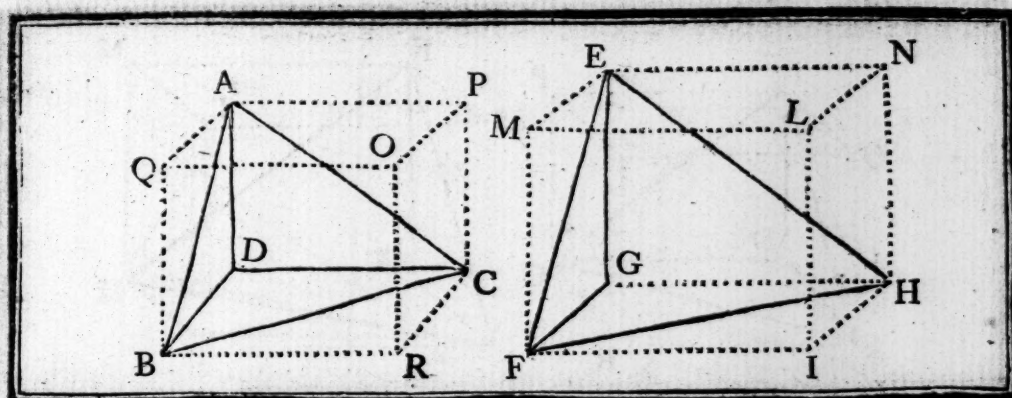
COROLLARY II.

EVERY pyramid which has a polygon for base, is the third part of a prism which has the same base, & is of an equal altitude with it; since it may be divided into prisms having triangular bases.

COROLLARY III.

PRISMS of equal altitudes are to one another as their bases, because pyramids upon the same bases, & of the same altitude, are to one another as their bases. (P. 6. B. 12).





PROPOSITION VIII. THEOREM VIII.

SIMILAR pyramids (ABCD & EFGH) having triangular bases (BDC & FGH) : are to one another in the triplicate ratio of that of their homologous sides.

Hypothesis.

The $\propto$ pyramids ABCD & EFGH have triangular bases BDC & FGH, whose homologous sides are BD & FG, &c.

Thesis,

The pyramid ABCD is to the pyramid EFGH; in the triplicate ratio of BD to FG, that is, as $DB^3 : FG^3$.

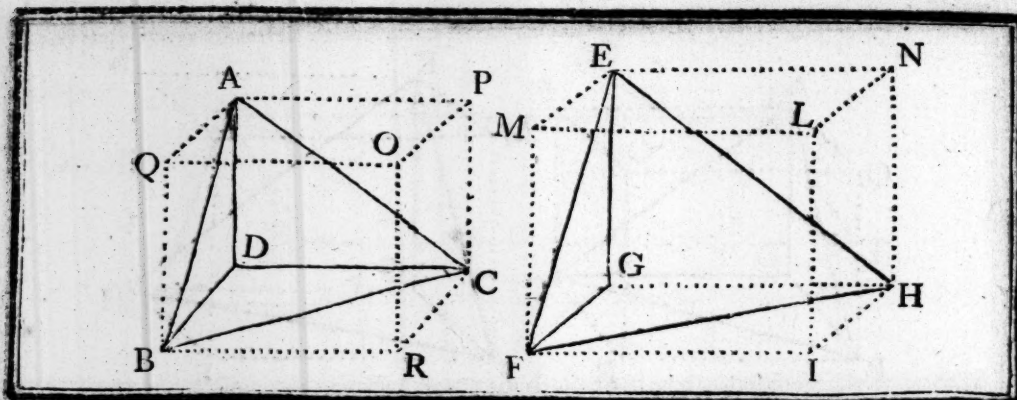
Preparation.

1. Produce the planes of the $\triangle BDC$, $\triangle ABD$ & $\triangle ADC$; complete the pgrs. DR, DQ & DP. P.31. B. 1.
2. Draw PO & OQ pple. to AQ & AP, & produce them to O. P.31. B. 1.
3. Join the points O & R; & QC will be a $\square$ which will have the same altitude with the pyramid ABCD.
4. After the same manner describe the $\square$ MH.
5. Infine, Join the points Q & P, also M & N, homologous to the points B & C; also F & H.

DEMONSTRATION.

BECAUSE the pyramids ABCD & EFGH are $\propto$ (Hyp.).

1. All the triangular planes which form the pyramid ABCD are $\propto$ to all the triangular planes which form the pyramid EFGH, each to each. D. 9. B.11.
2. Consequently, $AD : BD = EG : GF$, &c. D. 1. B. 6.
3. And the plane $\vee ADB$ is = to the plane $\veve EGF$. P. 5. B. 6.
4. Therefore the pgr. DQ is $\propto$ to the pgr. MG. D. 1. B. 6.
5. Likewise, the pgr. DR & GI; DP, & GN are $\propto$; as also their opposite ones AO, EL; QR, MI. P.24. B.11.



6. Consequently, $AR \& EI$ are $\propto$ $\square$.
 7. Therefore, $\square AR : \square EI = DB^3 : FG^3$.
 And since the lines $QP \& BC$; $MN \& FH$, are diagonals similarly drawn in the equal & pple. pgrs. $OA \& RD$; $EL \& IG$.
 (Prep. 5).
 8. The parts $BQAPCD \& FMENHG$ will be $\propto$ prisms : & each equal to the half of its $\square$.
 9. Consequently, the prism $BQPC$: prism $FMNH = BD^3 : FG^3$.
 But the pyramid $ABDC$ is the third part of the prism $BQPC$,
 & the pyramid $EFGH$ is the third part of the prism $FMNH$.
 10. Therefore, the pyramid $ABCD$: pyramid $EFGH = BD^3 : FG^3$.
 Which was to be demonstrated.

D. 9. B. 11.
 P. 33. B. 11.

D. 9. B. 11.
 P. 28. B. 11.
 P. 15. B. 5.
 P. 34. B. 11.
 Rem. 1.

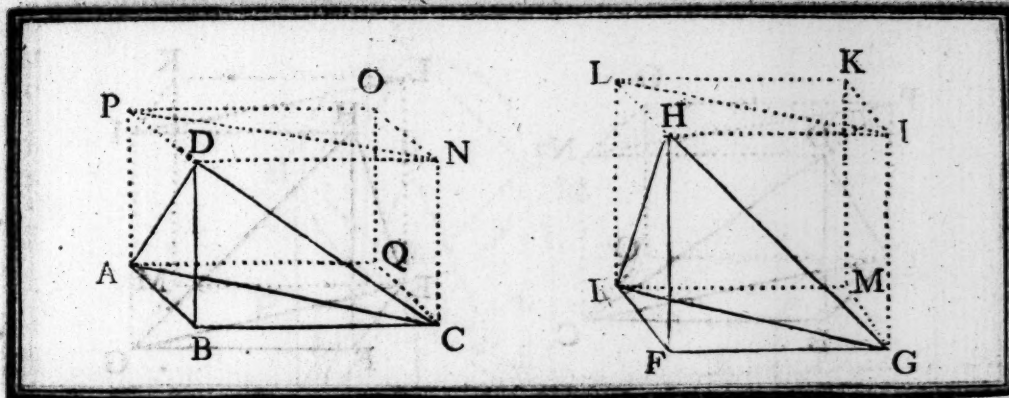
P. 7. B. 12.
 Cor. 1.

P. 15. B. 5.

C O R O L L A R Y.

FROM this it is evident, that similar pyramids which have polygons for their bases, are to one another in the triplicate ratio of their homologous sides, (because they may be divided into triangular pyramids; which are similar, taken two by two.





PROPOSITION IX. THEOREM IX.

THE bases (ABC & EFG), and altitudes (BD & FH), of equal pyramids, ($ABCD$ & $EFGH$), having triangular bases, are reciprocally proportional, (*that is*, the base ABC : base EFG = altitude FH : altitude BD), and triangular pyramids ($ABCD$ & $EFGH$), of which the bases (ABC & EFG), and altitudes (BD & FH), are reciprocally proportional: are equal to one another.

Hypothesis:

Thesis:

- I. The pyram. $ABCD$ & $EFGH$ are triangular. Base ABC : base EFG = altitude FH : altitude BD .
 II. The pyram. $ABCD$ is = to the pyram. $EFGH$. FH : altitude BD .

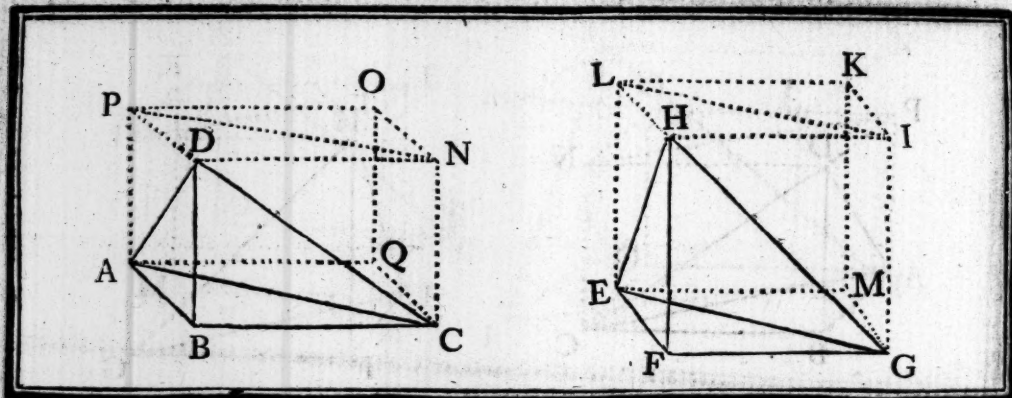
Preparation.

Complete the $\square BO$ & FK having the same altitude with the pyramids $ABCD$ & $EFGH$; as also the prisms $BAPNC$ & $FELIG$.

I. DEMONSTRATION.

BECAUSE the prisms PNB & LIF , have the same base & altitude with the given pyramids $ABCD$ & $EFGH$. (*Prep.*)

1. Each prism will be triple of its pyramid, (*that is*, the prism PNB triple of the pyramid $ABCD$, & the prism LIF triple of the pyramid $EFGH$). { P. 7. B. 12.
Cor. 1.
2. Consequently, the prism PNB is = to the prism LIF . Ax. 6. B. 1.
 But the $\square BO$ is double of the prism PNB , & the $\square FK$ double of the prism LIF . P. 28. B. 11.
3. Therefore, the $\square BO$ is = to the $\square FK$. Ax. 6. B. 1.
 But the equal $\square$ (BO & FK) have their bases and altitudes reciprocally proportional (*that is*, base BQ : base FM = altitude FH : altitude BD).
 And those $\square$ are each sextuple of their pyramids, (*that is*, the $\square BO$ is = six pyramids $ABCD$, & the $\square FK$ = six pyramids $EFGH$. Arg. 1. & 3).



Moreover, the base of the pyramid $ABCD$ is the half of the base of the $\square BQ$.

And the base of the pyramid $EFGH$ is the half of the base of the $\square FK$.

4. Consequently, base ABC : base EFG = alt. FH : alt. BD .

$P.41. B. 1.$

$P.15. B. 5.$

$P.11. B. 5.$

Which was to be demonstrated.

Hypothesis.

Thesis.

- I. The pyramids $ABCD$ & $EFGH$ are triangular. The triangular pyramid $ABCD$ is =
 II. Base ABC : base EFG = alt. FH : alt. BD . to the triangular pyramid $EFGH$.

II. DEMONSTRATION.

BECAUSE the $\triangle ABC$: $\triangle EFG$ = FH : BD . (*Hyp. 2*).

And the pgr. BQ is double of the $\triangle ABC$, the pgr. FM double of the $\triangle EFG$.

$P.41. B. 1.$

1. It follows, that the pgr. BQ : pgr. FM = FH : BD .

$P.15. B. 5.$

But $\square BQ$ has for base the pgr. BQ , & for alt. BD .
 And $\square FK$ has for base the pgr. FM , & for alt. FH . } (*Prep.*)

2. Consequently, the $\square BQ$ is = to the $\square FK$.

$P.34. B.11.$

But the $\square BQ$ & $\square FK$ are each double of the prisms PNB & LIF .

$P.18. B.11.$

And those prisms PNB & LIF are each triple of their pyramids $ABCD$ & $EFGH$.

$P. 7. B.12.$

$\{ \text{Cor. 1.}$

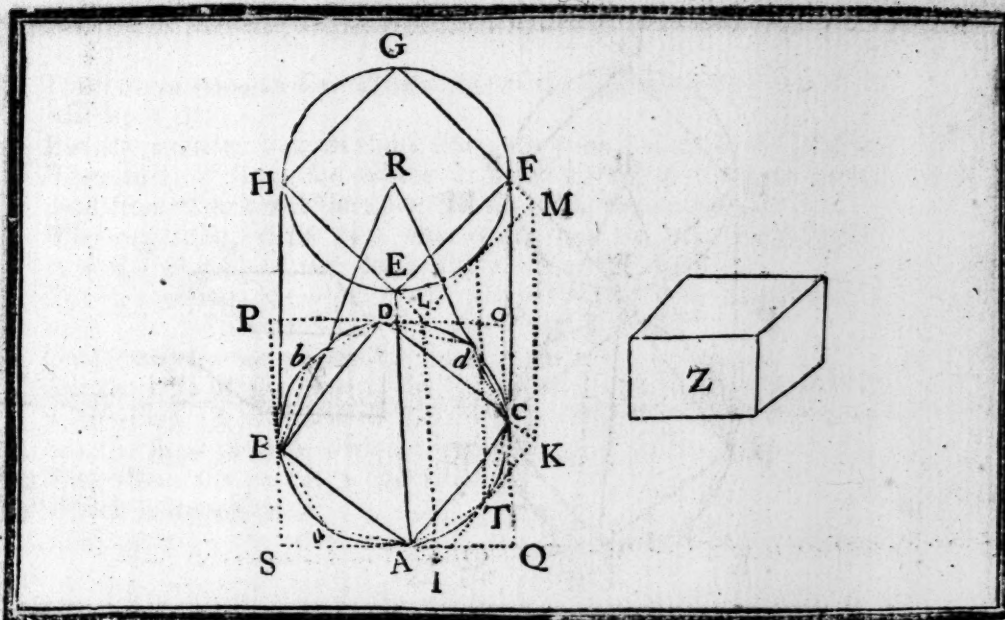
3. Therefore, the triangular pyramid $ABCD$ is = to the triangular pyramid $EFGH$.

$Ax.7. B. 1.$

Which was to be demonstrated.

C O R O L L A R Y.

EQUAL polygon pyramids have their bases and altitudes reciprocally proportional; & polygon pyramids whose bases & altitudes are reciprocally proportional are equal.



PROPOSITION X. THEOREM X.

EVERY cone (BRC) is the third part of the cylinder (HGF EABD C) which has the same base, (BDCA) and the same altitude (BH) with it.

Hypothesis.

The cone BRC, & the cylinder HFADC, have the same base BDCA, & the same altitude BH.

Thesis.

The cone BRC is equal to the third part of the cylinder HFCABD.

DEMONSTRATION.

If not,

The cone will be $<$ or $>$ the third part of the cylinder, by a part = Z.

I. Supposition.

Let the third part of the cylinder HC be = cone BRC + Z.

I. Preparation.

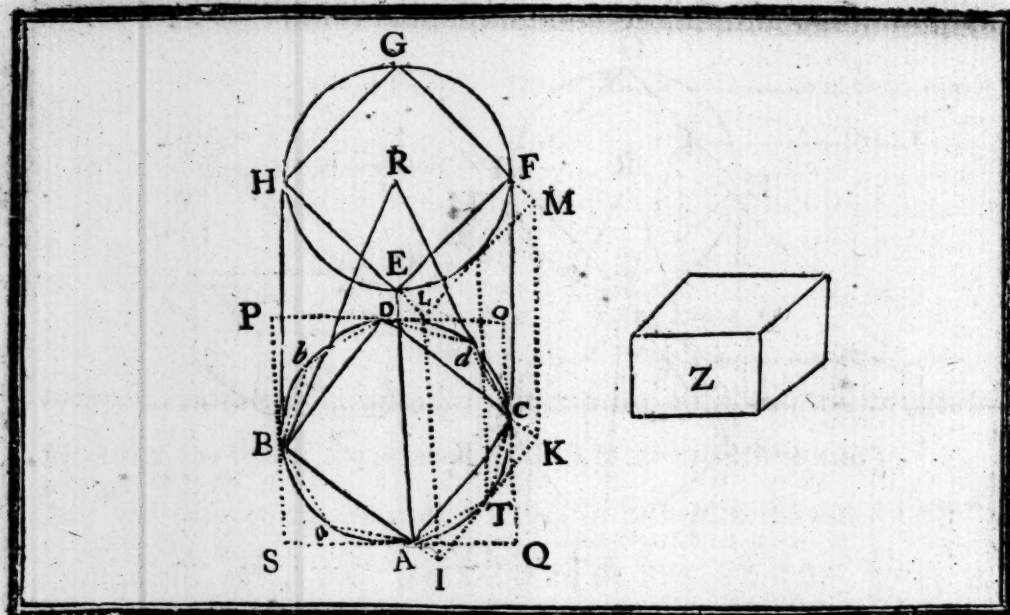
1. **I**N the base ABDC of the cone & cylinder, describe the $\square$ ABDC.
2. About the same base describe the $\square$ POQS.
3. Upon those squares erect two $\square$ s, the first $\square$ FHBC, upon the inscribed $\square$, & the second, on the circumscribed $\square$, which will touch the superior base with its p^lle. planes, in the points H, G, F, & E, * having the same altitude with the cylinder, & the cone.

P. 6. B. 4.

P. 7. B. 4.

S f

* We have omitted a part of the preparation in the figure to avoid confusion.



4. Bisect the arches $A T C$, $C d D$, $D b B$, & $B a A$, in T, d, b , & a . P.30. B. 3.
5. Draw $A T$, & $T C$, &c. Pos.1. B. 1.
6. Thro' the point T , draw the tangent $I T K$, which will cut $B A$ & $D C$ produced, in the points I & K & complete the pgr. $A K$. P.17. B. 3.
7. Upon the pgr. $A K$, erect the $\square$ $ALFK$, & upon the $\triangle AIT$, TAC , & TCK the prisms ETI , ETF , & TFK , having all the same altitude with the cylinder & cone.
8. Do the same with respect to the other segments $A a B$, $B b B$, &c.

BECAUSE the $\square$ $POQS$ is described about, & the $\square$ $BDCA$ described in the $\odot$. (*Prop. 2*).

1. The $\square$ $POQS$ is double of the $\square$ $BDCA$. P.47. B. 1.
And the $\square$ described upon those squares having the same altitude, (*Prop. 3*).
2. Therefore, the $\square$ upon $POQS$ is double of the $\square$ upon $BDCA$. P.32. B.11.
But the $\square$ upon $POQS$ is $>$ the given cylinder. Ax.8. B. 1.
3. Therefore, the $\square$ upon $BDCA$ is $>$ the half of the same cylinder. P.19. B. 5.
And since the $\triangle TAC$ is the half of the pgr. AK . P.41. B. 1.
4. The prism ETF , described upon this $\triangle TAC$, will be the half of the $\square$ upon the pgr. AK . P.28. B.11.
The $\square$ described upon the pgr. AK is $>$ the element of the cylinder, which has for base the segment ATC . P.34. B.11.
Rem.1. Cor.3.
Ax.8. B. 1.
5. Consequently, prism ETF described upon $\triangle TAC$ is $>$ half of the element of the cylinder which has for base segment ATC . P.19. B. 5.
6. Likewise, all the other prisms described after the same manner, will be $>$ the half of the corresponding parts or elements of the cylinder. Therefore, there may be taken from the whole cylinder more than the half, (viz. the $\square$ upon the $\square$ $BDCA$), & from those remaining elements (viz. $C F E A T$, &c.) more than the half; (viz. the prisms ETF , &c.), & so on.

7. Until there remains several elements of the cylinder which together will be $< Z$. *Lem. B. 12.*
 But the cylinder is $=$ to three times the cone $BRC + Z$. (*Sup.*).
 Therefore, if from the whole cylinder be taken those elements *(Arg. 7.)*
 And from three times the cone $BRC + Z$, the magnitude Z .
 8. The remaining prism (viz. that which has for base the polygon $AaBbDdCT$) will be $>$ the triple of the cone. *Ax. 4. B. 1.*
 But this prism is the triple of the pyramid of the same base & altitude (viz. of the pyramid $TAaBbDdCTR$). *P. 7. B. 12.*
Cor. 2.
 9. Consequently, the pyramid $ABDCR$ is $>$ the given cone. *Ax. 7. B. 1.*
 But the base of the cone is the $\odot$ in which this polygon $ABDC$ is inscribed, (& which is consequently $>$ this polygon), & this cone has the same altitude with the pyramid.
 10. Therefore, the part is $>$ the whole.
 11. Which is impossible. *Ax. 8. B. 1.*
 12. Consequently, the cone is not $<$ the third part of the cylinder,

II. Supposition.

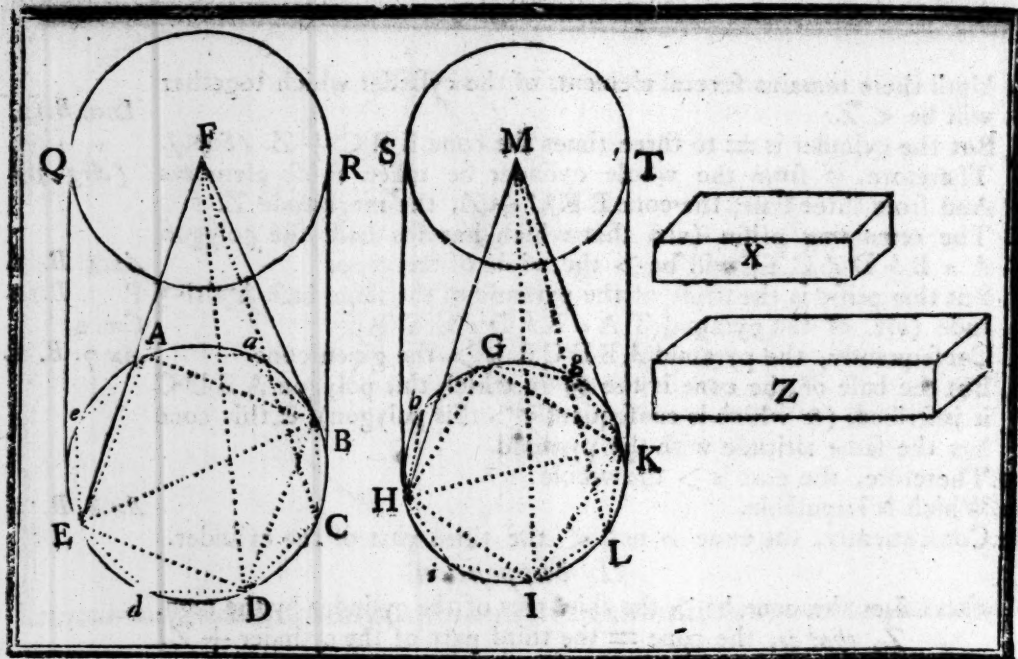
Let the cone be $>$ the third part of the cylinder by the mgn. Z , that is, the cone $=$ the third part of the cylinder $+ Z$.

II. Preparation.

Divide the given cone into pyramids, in the same manner that the cylinder was divided in the first supposition.

- I**F from the given cone be taken the pyramid which has for base the $\square$ $ABDC$, (which is greater than the half of the whole base of the given cone, being the half of the circumscribed $\square$, *Arg. 1.* & this $\square$ being $>$ the base of the cone, *Ax. 8. B. 1.*), & from the remaining segments, the pyramids corresponding to those segments, (*as has been done in the cylinder Arg. 7.*).
13. There will remain several elements of the cone which together will be $< Z$. *Lem. B. 12.*
 Therefore, if from the cone those elements be taken which are $< Z$, & from the cylinder $+ Z$, the magnitude Z .
 14. The remainder, viz. the pyramid $AaBbDdCTR$ is $=$ to the third part of the cylinder. *Ax. 5. B. 1.*
 But the pyr. $AaBbDdCTR$ is $=$ to the third part of the prism, *P. 7. B. 12.*
 which has for base the same polyg. $AaBbDdCT$, & the same alt. *Cor. 2.*
 15. Therefore, the given cylinder, is $=$ to this prism. *Ax. 6. B. 1.*
 But the base of the given cylinder is $>$ the base of the prism since this second is inscribed in the first. (*I. Prep. 4. & 5.*)
 16. Therefore, the part is $=$ to the whole.
 17. Which is impossible. *Ax. 8. B. 1.*
 18. Therefore, the third part of the cylinder is not $<$ the cone.
 And it has been demonstrated (*Arg. 12.*), that the third part of the cylinder is not $>$ the cone.
 19. Therefore, the cone is the third part of the cylinder of the same base & altitude,

Which was to be demonstrated.



PROPOSITION XI. THEOREM XI.
CONES (EABDF & HGKIM), and cylinders (QRBE & STKH) of the same altitude, are to one another as their bases.

Hypothesis,

The cones EABDF & HGKIM, as likewise the cylinders QRBE & STKH have the same altitude.

Thesis.

- I. Cone EFB : cone HMK = base EABD : base HGKI.
 II. Cylinder QRBE : cylinder STKH = base EABD : base HGKI.

DEMONSTRATION.

If not, The cone EFB : Z (which is $<$ or $>$ the cone HMK) = base EABD : base HGKI.

I. Supposition.

Let Z be $<$ the cone HMK by a magnitude X, that is, let the cone HMK = Z + X.

I. Preparation.

1. IN $\odot$ GHKI base of cone HMK; describe $\square$ GHKI. P. 6. B. 4.
2. Divide the cone into pyramids (as in II. Sup. of P. 10.).
3. In the bases of the cones EFB & HMK, draw diam. EB & HK.
4. In the $\odot$ EABD base of the cone EFB, describe a polyg. α to the polyg. HbGgKL iH, & divide it as the cone HMK.

BECAUSE the cone HMK has been divided into pyramids. (Prep. 2.).
 If those pyramids be taken from the cone (as was done in the foregoing proposition. Arg. 13.).

1. The sum of the remaining elements will be $<$ X.
 Therefore, if those elements be taken from the cone HMK, & the magnitude X from Z + X.

Lem. B. 12.

2. The remaining pyramid $HbGgKLIiM$ will be $> Z$.
 But those polygons inscribed in the $\odot EABD$ & $HGKI$ are $\odot$. (*Prep. 4.*)
3. Therefore, $\odot AEDB : \odot GHKI = \text{polyg. } Cdea : \text{polyg. } ibgL$. { *P. 2. B. 12. Cor.*
- But, $\odot AEDB : \odot GHKI = \text{cone } EFB : Z$. (*Sup.*)
 And the pyramid $DdEeAaBCF$: pyramid $HbGgKLIiM$
 $= \text{polygon } Cdea : \text{polygon } ibgL$. *P. 6. B. 11.*
4. Consequently, pyramid $DdEeAaBCF$: pyramid $HbGgKLIiM$
 $= \text{cone } EFB : Z$. *P. 11. B. 5.*
 But the pyramid $DdEeAaBCF$ is $<$ cone EFB . *Ax. 8. B. 1.*
5. Therefore, the pyramid $HbGgKLIiM$ is $< Z$. *P. 14. B. 5.*
6. But this pyramid is $> Z$. (*Arg. 2.*)
7. Therefore, it will be $> \& < Z$. (*Arg. 2. & 6.*)
8. Which is impossible.
9. Therefore, the supposition of $Z <$ the cone HMK is false.
10. Wherefore, the base of the cone EFB is not to the base of the cone HMK (the cones having the same altitude) as the cone EFB to a magnitude $Z <$ the cone HMK .

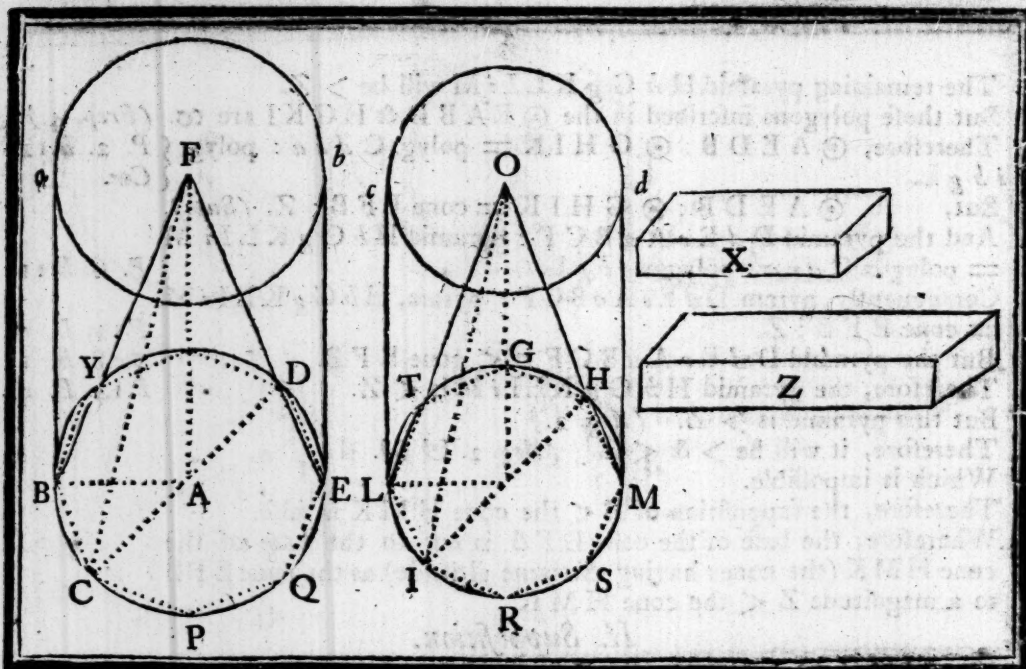
II. Supposition.

Let Z be $>$ the cone HMK .

II. Preparation.

Take a magnitude X such that $Z : \text{cone } EFB = \text{cone } HMK : X$.

- B**ECAUSE Z is $>$ the cone HMK . (*I. Sup.*)
11. The cone EFB is $> X$. *P. 14. B. 5.*
 But the cone $EFB : Z = \text{base } EABD : \text{base } HGKI$. (*Sup.*) { *P. 4. B. 5. Cor.*
12. Therefore, base $HGKI : \text{base } EABD = Z : \text{cone } EFB$. { *P. 11. B. 5.*
13. Consequently, base $HGKI : \text{base } EABD = \text{cone } HMK : X$.
 But it has been demonstrated (*Arg. 10.*), that the base of a cone is not to the base of another cone, having the same altitude, as the first cone is to a magnitude $<$ the second.
14. Therefore, X is not $<$ the cone EFB .
 But X is $<$ the cone EFB . (*Arg. 10.*)
15. Consequently, X will be $< \& \text{ not } <$ this cone EFB . (*Arg. 11. & 14.*)
16. Which is impossible.
17. From whence it follows, that the supposition of $Z >$ the cone HMK is false.
 Therefore, the magnitude Z being neither $<$ nor $>$ the cone HMK . (*Arg. 9. & 17.*)
18. It will be $=$ to the cone HMK .
19. Hence cone $EFB : \text{cone } HMK = \text{base } EABD : \text{base } HGKI$. *P. 7. B. 5.*
 Which was to be demonstrated. 1.
- B**ECAUSE the cone EFB is the third part of the cylin. $QRBE$ } *P. 10. B. 12.*
 And the cone HMK is the third part of the cylin. $HSTK$. }
20. The cylin. $QRBE : \text{cyl. } HSTK = \text{base } EABD : \text{base } HGKI$. *P. 15. B. 5.*
 Which was to be demonstrated. 11.



PROPOSITION XII. THEOREM XII.

SIMILAR cones (BFE & LOM), and cylinders (B a b E & L c d M) have to one another the triplicate ratio of that which the diameters (CD & IH) of their bases (BYDEP & LTHMR), have.

Hypothesis.

The cones BFE & LOM, likewise the cylinders B a b E & L c d M, are $\propto$.

Thesis.

- I. The cone BFE is to the cone LOM in the triplicate ratio of CD to IH; or as $CD^3 : IH^3$.
- II. The cyl. B a b E is to the cyl. L c d M, in the triplicate ratio of CD to IH; or as $CD^3 : IH^3$.

DEMONSTRATION.

If not,

The cone BFE is to a magnitude Z (which is $<$ or $>$ the cone LOM) as $CD^3 : IH^3$.

I. Supposition.

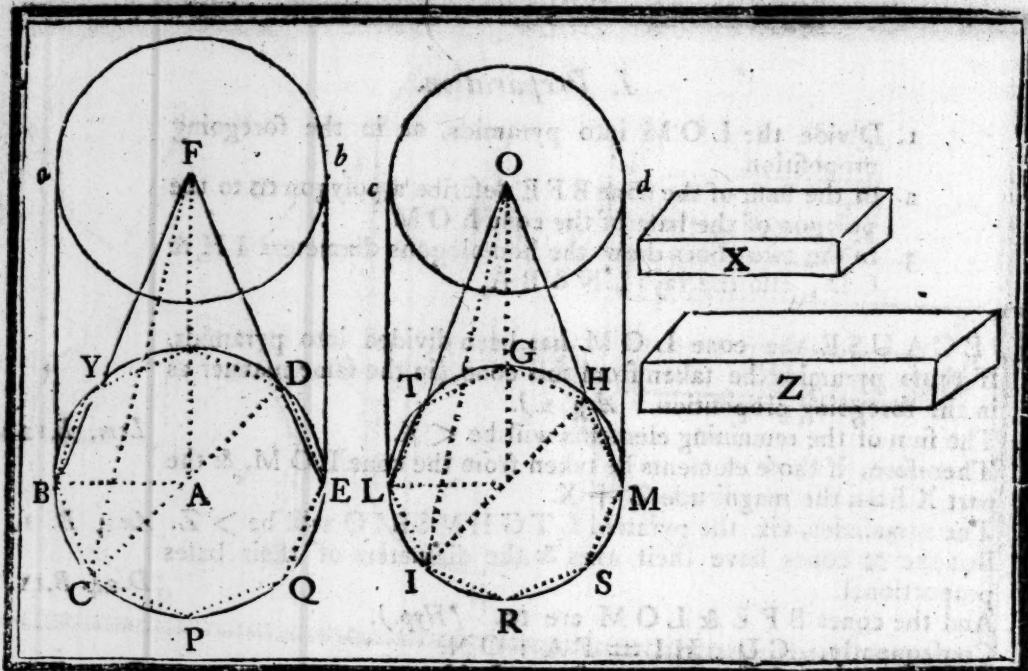
Let Z be $<$ the cone LOM by the magnitude X, that is, the cone LOM $= Z + X$.

I. Preparation.

1. Divide the L O M into pyramids, as in the foregoing proposition.
2. In the base of the cone B F E describe a polygon $\propto$ to the polygon of the base of the cone L O M.
3. In the two cones draw the homologous diameters I H & C D ; also the rays L N & B A.

BECAUSE the cone L O M has been divided into pyramids. If those pyramids be taken from this cone (in the same manner as in the foregoing proposition. *Arg. 1.*).

1. The sum of the remaining elements will be $< X$. *Lem. B. 12.*
Therefore, if those elements be taken from the cone L O M, & the part X from the magnitude $Z + X$.
2. The remainder, viz. the pyramid L T G H M S R I O will be $> Z$. *Ax. 4. B. 1.*
But the $\propto$ cones have their axes & the diameters of their bases proportional. *D. 24. B. 11.*
And the cones B F E & L O M are $\propto$. (*Hyp.*).
3. Consequently, $CD : HI = FA : ON$.
But, $CD : HI = CA : IN$. *P. 15. B. 5.*
4. Therefore, $CA : IN = FA : ON$. *P. 11. B. 5.*
5. And alternando $CA : FA = IN : ON$. *P. 16. B. 5.*
The $\triangle FAC$ & $\triangle ION$ have the $\angle CAF = \angle ION$. (*Prep. 3.*)
And the sides $CA, AF ; IN, ON$ about those equal angles proportional. (*Arg. 5.*).
6. Wherefore, the $\triangle FAC$ is $\propto$ to the $\triangle ION$. *D. 1. B. 6.*
7. Consequently, $CF : CA = IO : IN$. *P. 4. B. 6.*
8. Likewise, the $\triangle BCA$ is $\propto$ to the $\triangle LIN$. ($\angle BAC$ being $= \angle LNI$). (*Prep. 3.*).
9. Therefore, $CA : BC = IN : IL$. *P. 4. B. 6.*
But, $CF : CA = IO : IN$. (*Arg. 7.*).
10. Consequently, $CF : BC = IO : IL$. *P. 22. B. 5.*
In the $\triangle CAF$ & $\triangle BAF$, the side CA is $=$ to BA (*D. 15. B. 1.*)
 AF is common, & $\angle CAF = \angle BAF$. (*Prep. 3.*).
11. Therefore, the base BF is $=$ to the base CF . *P. 4. B. 1.*
12. In like manner, LO is $=$ to OI .
But, $CF : BC = OI : IL$. (*Arg. 10.*).
13. Therefore, $BF : BC = LO : IL$. *P. 7. B. 5.*
14. And invertendo, $BC : BF = IL : OL$. *P. 4. B. 5.*
15. Consequently, the three sides of the $\triangle BFC$ are proportional to the three sides of the $\triangle LOI$. *Cor.*
16. From whence it follows, that those $\triangle BFC$ & $\triangle LOI$ are $\propto$. *P. 5. B. 6.*
17. It may be demonstrated after the same manner, that all the triangles which form the pyramid $BDQF$ are $\propto$ to all the triangles which form the pyramid $LHSO$, each to each.



And as the bases of those pyramids are $\propto$ polygons. (*Prep. 2.*)

18. The pyramid B D Q F is $\propto$ to the pyramid L H S O. D. 9. B. 11.

But those pyramids being $\propto$.

19. The pyramid B D Q F : pyramid L H S O = $CB^3 : IL^3$. { P. 8. B. 12.

But, $CA : BC = IN : IL$. (*Arg. 9.*) { Cor.

20. Therefore invert. $BC : CA = IL : IN$. { P. 4. B. 5.

21. And alternando, $BC : LI = CA : IN$. { Cor.

22. Consequently, $BC : LI = CD : IH$. P. 16. B. 5.

23. Therefore, three times the ratio of B C to L I is = to three times { P. 15. B. 5.

the ratio of C D to I H, that is, $BC^3 : LI^3 = CD^3 : IH^3$. { P. 11. B. 5.

But $CB^3 : IL^3 =$ pyramid B D Q F : pyramid L H S O. (*Arg. 19.*)

24. Consequently, pyramid B D Q F : pyramid L H S O = $CD^3 : IH^3$. P. 11. B. 5.

But the cone B F E : Z = $CD^3 : IH^3$. (*Sup.*)

25. Therefore, the pyram. B D Q F : pyram. L H S O = cone B F E : Z. P. 11. B. 5.

But the pyramid B D Q F being < cone B F E. Ax. 8. B. 1.

26. The pyramid L H S O will be also < Z. P. 14. B. 1.

But the pyramid L H S O is > Z. (*Arg. 2.*)

27. Consequently, the pyram. L H S O will be < & > Z. (*Arg. 2. & 26.*)

28. Which is impossible.

29. Therefore, the supposition of Z < the cone L O M or L T G {

H M S R I O is false.

30. From whence it follows, that the cone B F E is not to a magnitude less than the cone L O M, in the triplicate ratio of the diameter C D to the diameter I H.

H. Supposition.

Let Z be $>$ the cone L O M.

II. Preparation.

Take a magnitude X, such that $Z : \text{cone B F E} = \text{cone L O M} : X$.

BECAUSE Z is $>$ than the cone L O M. (*II. Sup.*)

31. The cone B F E will be $>$ X.

P. 14. B. 3.

But $CD^3 : IH^3 = \text{cone B F E} : Z$. (*Sup.*)

32. Therefore, invert. $IH^3 : CD^3 = Z : \text{cone B F E}$.

P. 4. B. 5.
Cor.

But $Z : \text{cone B F E} = \text{cone L O M} : X$. (*II. Prep.*)

33. Consequently, $IH^3 : CD^3 = \text{cone L O M} : X$.

P. 11. B. 5.

And it has been demonstrated (*Arg. 30.*), that a cone is not to a magnitude less than another cone in the triplicate ratio of the diameters of their bases.

34. Therefore, X is not $<$ the cone B F E.

But X is $<$ the same cone. (*Arg. 31.*)

35. From whence it follows, that X will be $<$ the cone, & will not be $<$ at the same time.

36. Which is impossible.

37. Therefore, the supposition of Z being $>$ the cone L O M, is false.

Therefore, the magnitude Z being neither $<$ nor $>$ the cone L O M. (*Arg. 29. & 37.*)

38. It will be equal to it.

39. Consequently, the cone B F E : cone L O M = $CD^3 : IH^3$.

P. 7. B. 5.

Which was to be demonstrated. 1.

The cylinder B a b E, being triple of the cone B F E.

And the cylinder L c d M, the triple of the cone L O M.

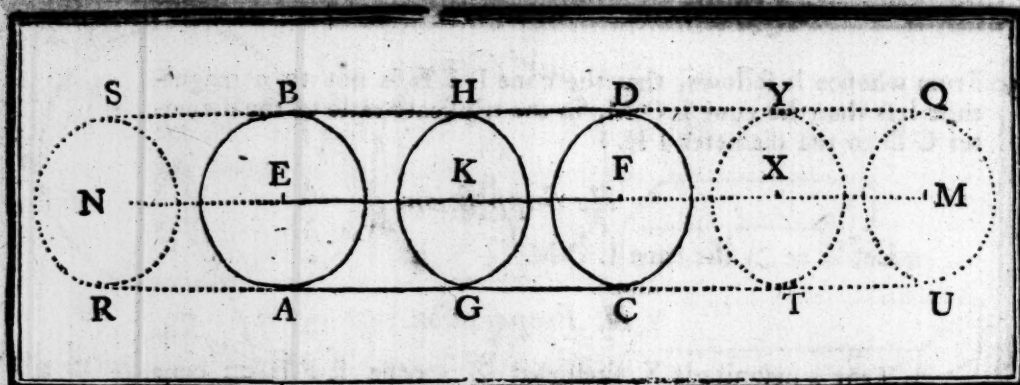
P. 10. B. 12.

40. The cylinder B a b E : cylinder L c d M = $CD^3 : IH^3$.

P. 15. B. 5.

Which was to be demonstrated. 11.

T t



PROPOSITION XIII. THEOREM XIII.

IF a cylinder (ABDC) be cut by a plane (HG) parallel to its opposite planes (BA & DC) : It divides the cylinder into two cylinders (ABHG & GHDC), which are to one another as their axes, (EK & KF) (*that is*, the cylinder ABHG : cylinder GHDC = axis EK : axis KF).

Hypothesis.

The cylin. AD is cut by a plane HG,
p^{lle}. to the opposite planes AB & DC.

Thesis.

Cylin. AH : cylin. HC = axis EK :
axis FK.

Preparation.

1. Produce the axis EF of the cylinder ABDC both ways towards N & M. Pos. 2. B. 1.
2. In the axis NM produced, take several parts = to EK & FK; as EN = EK, & FX, &c. each = FK. P. 3. B. 2.
3. Thro' those points N, X & M pass the planes SR, TY & VQ, p^{lle}. to the opposite planes BA & DC.
4. From the points N, X & M, describe on those planes the $\odot$ SR, TY & VQ each = to the opposite $\odot$ BA & DC. Pos. 3. B. 2.
5. Complete the cylinders SA, CY & TQ.

DEMONSTRATION.

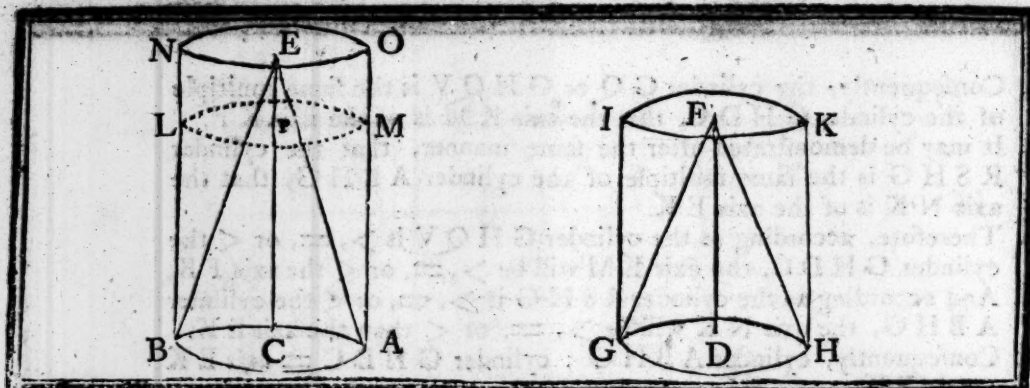
BECAUSE the axes FX & XM of the cylinders DT & TQ are equal to the axis FK, of the cylinder GD. (*Prep. 2*).

1. Those cylinders DT, TQ & GD will be to one another as their bases. P. 11. B. 12.
But those bases are equal. (*Prep. 4*).
2. Therefore, those cylinders TD, TQ & GD are also equal. P. 14. B. 5.
But there are as many equal cylinders CY, TQ &c. which together are equal to the cylinder GQ, as there are parts FX, XM, &c. each equal to the axis KF, which together are equal to MK.

3. Consequently, the cylinder GQ or $GHQV$ is the same multiple of the cylinder $GHDC$, that the axis KM is of the axis KF .
4. It may be demonstrated after the same manner, that the cylinder $RSHG$ is the same multiple of the cylinder $ABHG$, that the axis NK is of the axis EK .
5. Therefore, according as the cylinder $GHQV$ is $>$, $=$, or $<$ the cylinder $GHDC$, the axis KM will be $>$, $=$, or $<$ the axis KF . And according as the cylinder $RSHG$ is $>$, $=$, or $<$ the cylinder $ABHG$, the axis NK will be $>$, $=$, or $<$ than the axis EK .
6. Consequently, cylinder $ABHG$: cylinder $GHDC$ = axis EK : axis KF .

D. 5. B. 5.





PROPOSITION XIV. THEOREM XIV.

CYLINDERS (NOAB & IKGH), and cones (BEA & GFH) upon equal bases (BA & GH) : are to one another as their altitudes (CE & DF).

Hypothesis.

The cylinders NOAB & GIKH, as also the cones BEA & GFH, have equal bases.

Thesis.

- I. Cylinder NOAB : cylinder IKGH = alt. CE : alt. DF.
 II. Cone BEA : cone GFH = alt. CE : alt. DF.

Preparation.

1. In the axis of the greater cylinder AONB, take a part PC = to the altitude of the cylinder GIKH.
2. Thro' the point P, pass a plane LM, pple. to the base BA, which will divide the cylinder AONB into two cylinders, viz. BAML & LMON.

DEMONSTRATION.

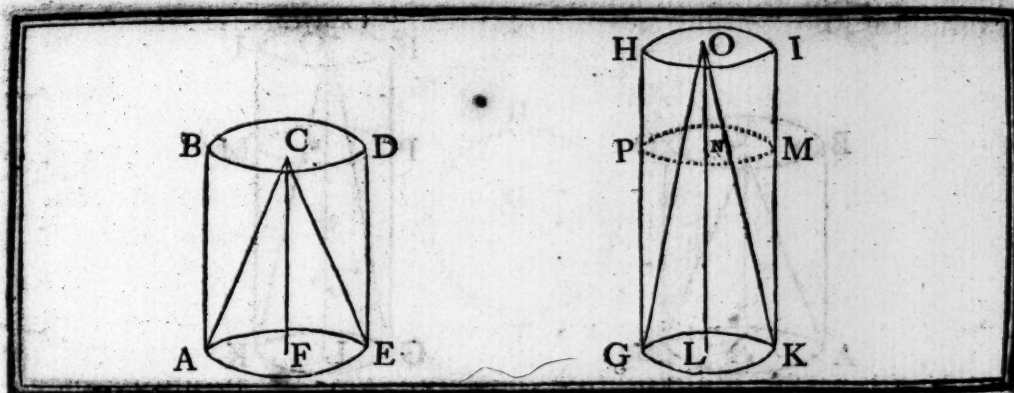
BECAUSE the cylinder BNOA is cut by a plane pple. to its base, (*Prep. 2.*).

1. The cylinder NOML : cylinder LMAB = PE : PC. P.13. B.12.
2. Consequently, cylinder NOML + LMAB : cylinder LMAB = PE + PC : PC. P.18. B. 5.
 But the cylinder NOML + LMAB is = to the cylip. BNOA, PE + PC = EC. Ax.1. B. 1.
 Moreover, the cylinder LMAB is = to cylinder IGHK, & PC = DF. (*Prep. 1.*).
3. Therefore, the cylinder BNOA : cylinder IGHK = alt. EC : alt. DF. P. 7. B. 1.

Which was to be demonstrated. 1.

The cone BEA is the third part of the cylinder BNOA. }
 And the cone GFH the third part of the cylinder GIKH. } P.10. B.12.

4. Consequently, the cone BEA : cone GFH = alt. EC : alt. DF. P.15. B. 5.
 Which was to be demonstrated. 11.



PROPOSITION XV. THEOREM XV.

THE bases (AE & GK), and altitudes (CF & OL), of the equal cylinders (ABDE & GHIK), and cones (ACE & GOK) : are reciprocally proportional, (*that is*, the base AE : base GK = alt. LO : alt. CF). And the cylinders and cones whose bases and altitudes are reciprocally proportional : are equal to one another.

Hypothesis.

Thesis.

- I. The cylinders ABDE & GHIK are equal. Base AE : base GK = alt. LO : alt. CF.
 II. The cones AEC & GOK are equal.

Preparation.

1. From the greater LO, cut off the altitude LN = the altitude CF.
2. Thro' the point N, pass a plane PM p^{le}. to the opposite planes of the cylinder HIKG.

P. 3. B. 1.

I. DEMONSTRATION.

BECAUSE the cylinder GHIK & PMKG have the same base.

1. The cylinder GHIK : cylinder PMKG = alt. LO : alt. LN. P. 14. B. 12.
 But the cylinders ABDE & GHIK are equal. (Hyp. 1.).

2. Consequently, the cylinder ABDE : cylinder PMKG = alt. LO : alt. LN.

P. 7. B. 5.

Moreover, the cylinders ABDE & PMKG have the same altitude. (Prep. 1.).

3. Therefore, the cylinder ABDE : cylinder PMKG = base AE : base GK.

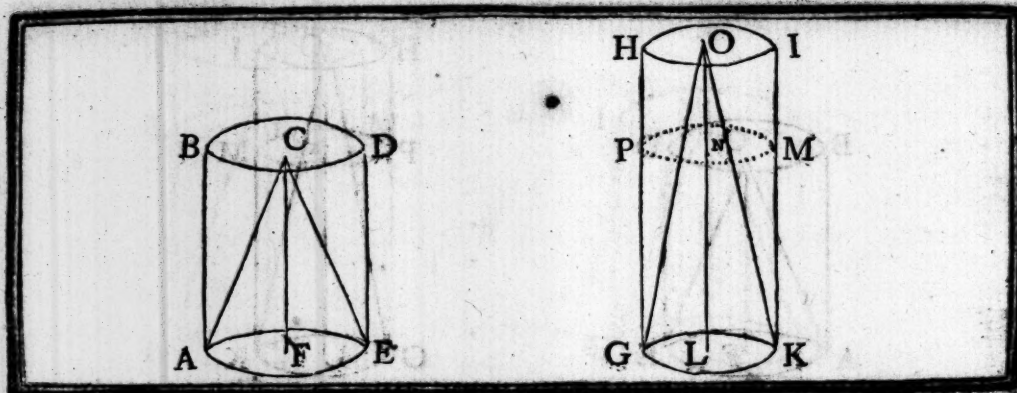
P. 11. B. 12.

But the cylinder ABDE : cylinder PMKG = alt. LO : alt. LN. (Arg. 2.).

And the alt. LN is = to the alt. GF. (Prep. 1.).

4. From whence it follows, that base AE : base GK = alt. LO : alt. CF. { P. 11. B. 5.
 { P. 7. B. 5.

Which was to be demonstrated.



Hypothesis.

Base GK : base AE = alt. CF : alt. LO.

Thesis.

I. Cyl. ABDE is = to cyl. GHIK.
II. The cone ACE is = to the cone GOK.

II. DEMONSTRATION.

BECAUSE the cylinders GPMK & ABDE, have the same altitude, (*Prop. 2*).

1. The cylinder GPMK : cylinder ABDE = base GK : base AE. *P. 11. B. 12.*
But the base GK : base AE = alt. CF : alt. LO, (*Hyp.*)
2. Consequently, the cyl. GPMK : cyl. ABDE = alt. CF : alt. LO. *P. 11. B. 5.*
Moreover, the cylinders GPMK & HIKG have the same base.
3. Therefore, the cyl. GPMK : cyl. HIKG = alt. LN : alt. LO. *P. 14. B. 12.*
But the altitude LN is = to the altitude CF, (*Prop. 1*).
4. From whence it follows that the cylinder GPMK : cylinder GHIK = altitude CF : altitude LO. *P. 7. B. 5.*
But the cylinder GPMK : cylinder ABDE = alt. CF : alt. LO. (*Arg. 2*).
5. Therefore the cylinder GPMK : cylinder ABDE = cylinder GPMK : cylinder GHIK. *P. 11. B. 5.*
6. Consequently, the cylinder ABDE is = to the cylinder GHIK. *P. 14. B. 5.*

Which was to be demonstrated I.

The cones ACE & GOK being each the third part of the cylinders ABDE & GHIK.

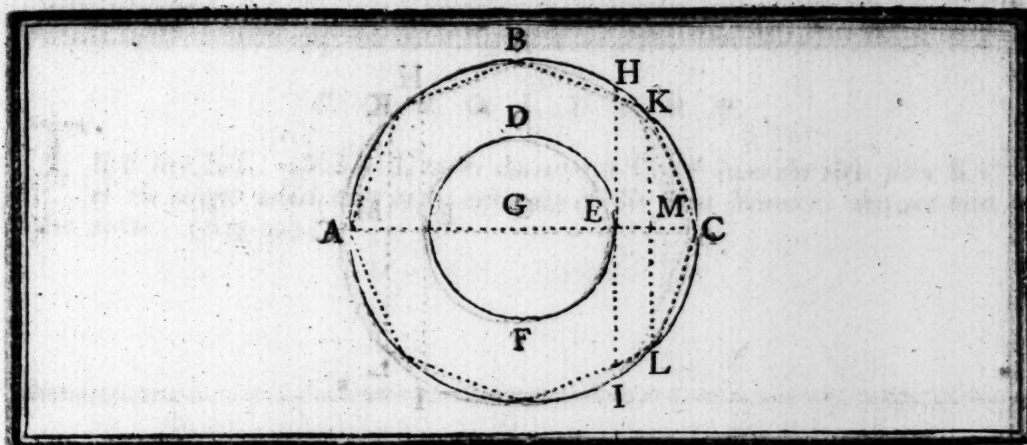
P. 10. B. 12.

And those cylinders being equal (*Arg. 6*).

1. The cone ACE is = to the cone GOK.

As. 7. B. 1.

Which was to be demonstrated. II.



PROPOSITION XVI. PROBLEM I.

TWO unequal circles (ABCI & DEF) being given, having the same center (G): to describe in the greater (ABCI) a polygon of an even number of equal sides, that shall not meet the lesser circle (DEF).

Given.

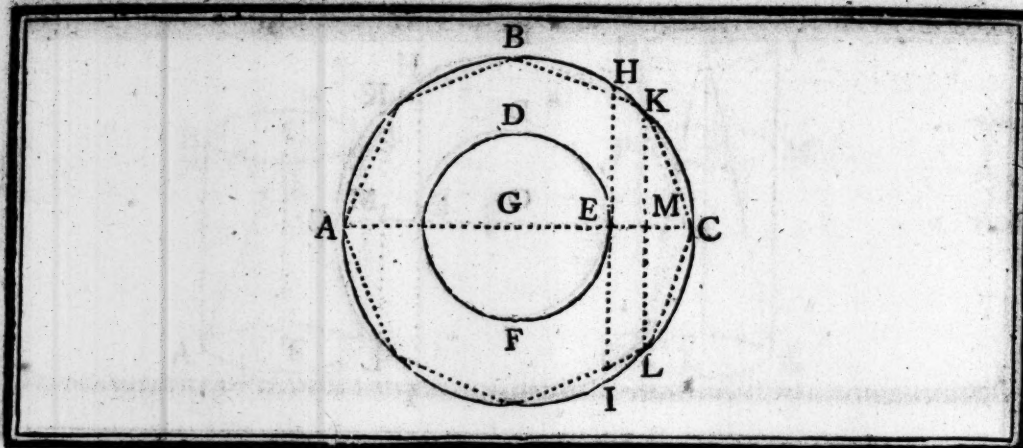
Two unequal $\odot$ ABCI & DEF having the same center G.

Sought.

To describe in the greater $\odot$ ABCI, a polygon of an even number of equal sides, that shall not be lesser $\odot$ DEF.

Resolution.

1. Draw the diameter AC in the greater $\odot$ ABCI which will cut the $\odot$ of the $\odot$ DEF in the point E.
2. Thro' the point E, draw the tangent HEI to the $\odot$ DEF & produce it until it meets the $\odot$ of the $\odot$ ABCI in the points H & I. P. 16. B. 3.
Pos. 2. B. 1.
3. Cut the semi $\odot$ ABC into two equal parts in the point B. P. 30. B. 3.
4. Divide the semi arch BC into two equal parts, & so on until the arch KC be $\angle$ the arch HC. Lem. B. 12.
5. Draw the chord KC & apply it around in the $\odot$ of the $\odot$ ABCI. P. 1. B. 4.
Pos. 1. B. 1.



Preparation.

From the point K, let fall the $\perp$ K M upon the diameter { P.11. B. 1.
A C, & produce it until it meets the $\odot$ in L. { P.1. B. 1.

DEMONSTRATION.

BECAUSE the semi $\odot$ ABC, is divided into two equal parts at the point B. (Ref. 3.).

And the divisions have been continued until the arch K C has been attained. (Ref. 4.).

1. It follows, that this arch K C will measure the $\odot$, an even number of times without a remainder, (because it measures the semi $\odot$. Ref. 3. & 4.).

2. Consequently, the line K C (chord of the arch K C) will be the side of a polygon, having an even number of equal sides inscribed in the $\odot$.

Moreover, the two $\angle$ HEM & KME being two $\angle$. (Ref. 2. & Prep.).

3. The line K M or K L is p^{er}pendic^{ular} to H E or H I.

P.28. B. 1.

But the line H I is a tangent of the $\odot$ D E F in E. (Ref. 2.).

4. Consequently, K L does not meet the $\odot$ D E F.

D.35. B. 1.

But K C is $<$ K L (P. 15. B. 3.) because K C is remoter from the center than K L. (Prep.).

5. Much more then K C will not meet the $\odot$ D E F.

P.15. B. 1.

And since the other sides of the polygon inscribed in the $\odot$ A B C I are each $=$ to K C. (Ref. 5.).

6. It may be demonstrated after the same manner, that they do not meet the $\odot$ D E F.

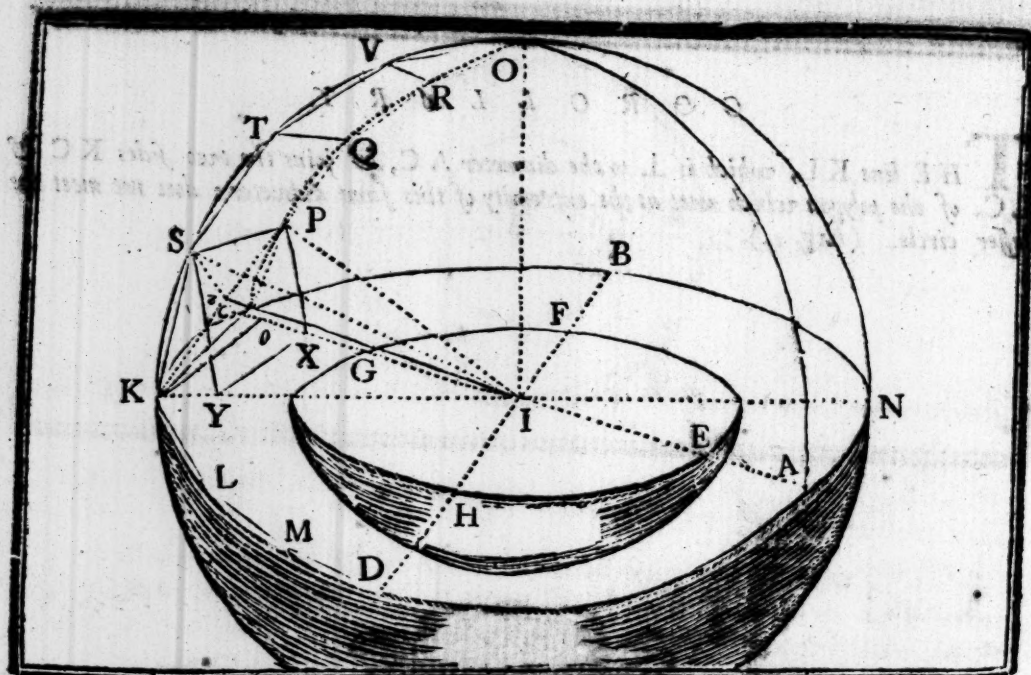
7. Consequently, there has been described in the $\odot$ A B C I, a polygon having an even number of equal sides, which does not meet the $\odot$ D E F.

Which was to be done.

C O R O L L A R Y.

THE line KL, which is $\perp$ to the diameter AC, & joins the two sides KC & LC, of the polygon which meet at the extremity of this same diameter: does not meet the lesser circle. (Arg. 4.).





PROPOSITION XVII. PROBLEM II.

TWO spheres (K O N & G F E H) having the same center (I) being given : to describe in the greater (K O N) a polyhedron (K C S P T Q V R O &c.), the superficies of which shall not meet the lesser sphere.

Given.

Sought.

Two concentric spheres K O N & G F E H. I. A polyhedron K P T R V O &c. described in the greater sphere K O N.
II. The superficies of which polyhedron must not touch the lesser sphere G F E H.

Resolution.

1. Cut the spheres by a plane K B N D passing thro' their center.
2. In the $\odot$ ABCD, draw the diameters AC & BD, intersecting each other at right angles. { Pos. 1. B. 1.
P. 12. B. 1.
3. In this greater $\odot$ ABCD, describe the polygon CKLMD &c. so as not to meet the lesser $\odot$ G F E H. P. 16. B. 12.
4. Draw the diameter K I N.
5. From the center I, erect on the plane of $\odot$ ABCD, the $\perp$ IO, & produce it to the surface of the greater sphere in O. { P. 12. B. 12.
Pos. 2. B. 1.
6. Thro' I O, & the diameters AC, BD, & K N, pass the planes A O C, B O D, & K O N.
7. Divide the arches A O C & K O N into an even number of parts in the points P, Q, R, S, T, & V, &c. so that each of those parts be equal to C K.
8. Draw the straight lines S P, T Q, V R.

I. Preparation.

1. From the points P & S, let fall the $\perp$ PX & SY upon the plane of the $\odot$ ABCD.
2. Draw YX.

P.12. B.12.

DEMONSTRATION.

BECAUSE the planes KON & COA pass thro' IO. (Ref.6).
And that IO is $\perp$ to the plane of the $\odot$ ABCD. (Ref. 5.).

1. Those planes KON & COA, are $\perp$ to the plane of this $\odot$.
But the points P & S are in those planes COA & KON.
And from those points have been let fall the $\perp$ PX & SY. (I. Prep.).
2. Consequently, the points Y & X are in the lines KN & CA.
In the $\triangle CXP$ & KYS ; $\angle PXC = \angle SYK$. (I. Prep. 1).
Moreover, $\angle PCX = \angle SKY$. (P.27.B.3), & $CP = KS$, (Ref.7).
3. Therefore, the sides PX & XC are $\equiv$ to the sides SY & YK.
But the rays KI & CI are equal.
Therefore, if the equals XC & YK be taken from them.
4. The remainders, viz. IX & YI will be equal.
5. Consequently, $IX : XC = IY : YK$.
6. From whence it follows, that XY is pple. to KC.
But PX which is $\equiv$ to SY (Arg. 3.) is also $\perp$ on the same plane with SY. (I. Prep. 1.).
7. Therefore, PX is also pple. to YS.
8. Likewise, SP is $\equiv$ & pple. to XY.
But XY is pple. to KC. (Arg. 6.).
9. Therefore, SP is also pple. to KC.
10. Consequently, the sides of the quadrilateral figure KSPC are in the same plane.
11. It may be demonstrated after the same manner, that the sides of the quadrilateral figures TQPS, VRQF, & of the $\triangle ROV$, are each in the same plane.
12. And as it may be demonstrated in this manner, that the whole sphere is incompassed with such like quadrilateral figures and triangles.
13. Consequently, there has been described in the greater sphere a polyhedron RPCKTVO, &c.

P.18. B.11.

P.38. B.11.

P.26. B. 1.

D.15. B. 1.

Ax.3. B. 1.

P. 7. B. 5.

P. 2. B. 6.

P. 6. B.11.

P.33. B. 1.

P. 9. B.11.

P. 7. B.11.

Which was to be demonstrated 1.

II. Preparation.

1. From the center I, let fall on the plane KSPC, the $\perp$ IZ.
2. Join the points ZP, ZC, ZS, & ZK; SI & PI.
3. From the point K, & in the plane ABCD, let fall the $\perp$ K ϕ on the diameter CA.

P.11. B.11.

Pos.1. B. 1.

P.12. B. 1.

BECAUSE in the $\triangle KCI$, the line YX is pple. to KC. (Arg.6).

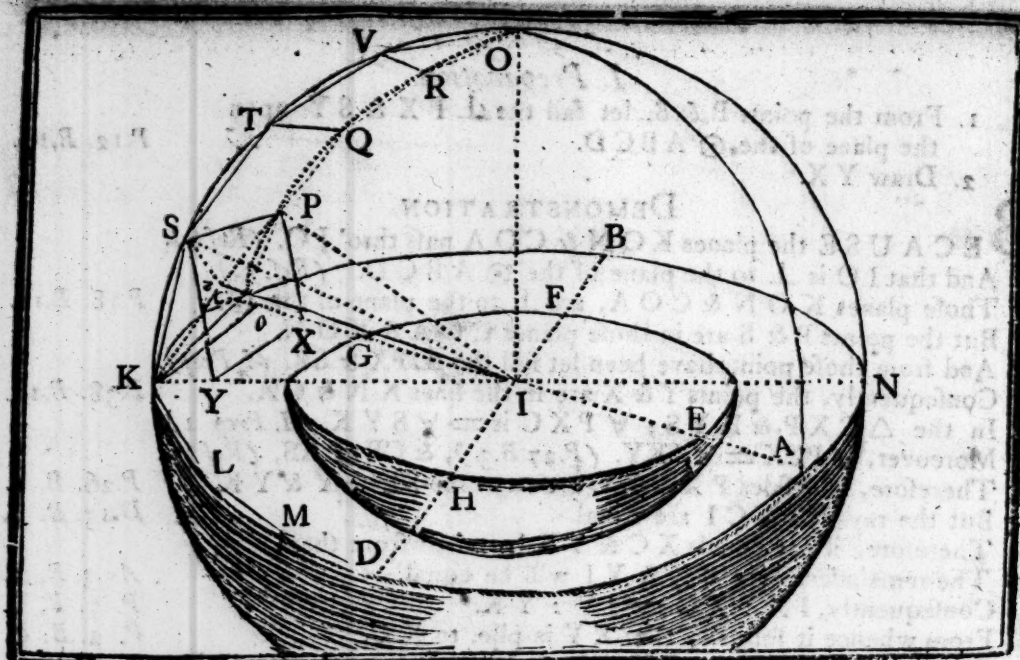
14. $IC : CK = IX : XY$.
But IC is $>$ IX.
15. Therefore, $CK > XY$.
But PS is $\equiv$ to XY. (Arg. 8.).
16. From whence it follows, that CK is also $>$ PS.
17. It may be demonstrated after the same manner, that SP is $>$ TQ, & $TQ >$ VR.

P. 2. B. 6.

Ax.8. B. 1.

P.14. B. 5.

P. 7. B. 5.



The $\angle$ IZP, IZC, IZK, & IZS are $\angle$ (II. Prep. 1. D. 3. B. 11). { D. 16. B. 1.
& IC is $\equiv$ IP $\equiv$ IS $\equiv$ IK. { D. 15. B. 1.

Moreover, IZ is common to the $\triangle$ IZP, IZC, IZK, & IZS. { P. 47. B. 1.
18. Therefore, ZP $\equiv$ ZC $\equiv$ ZK $\equiv$ ZS. { P. 46. B. 1.

19. Consequently, the $\odot$ described from the center Z, at the distance { Cor. 3.
ZP, will pass thro' the points K, S & C, & the quadrilateral
figure RSPC will be described in a $\odot$. { D. 3. B. 4.

But the four sides of the quadrilateral figure were equal; the arches
which subtend them will be so also, & will be each a quadrant of
the $\odot$. (P. 28. B. 3.).

But KS, CK & CP, are equal (Ref. 7.) & CK is $>$ SP. (Arg. 16.).

20. From whence it is manifest, that the three sides KS, CK, & CP,
subtend more than the three quadrants of the $\odot$; &, consequently,
CK (which is $\equiv$ to KS & CP) subtends more than a quadrant. { P. 33. B. 6.

21. Consequently, the $\angle$ CZK at the center is $>$ $\angle$.

22. Hence it follows, that the $\square$ of KC is $>$ $\square$ of ZC + $\square$ of ZK. { P. 12. B. 2

But the $\square$ of ZC is $\equiv$ to the $\square$ of ZK, (P. 46. B. 3. Cor. 3.).

Because, ZC is $\equiv$ to ZK. (Arg. 18.).

23. Therefore, the $\square$ of KC is $>$ the double of the $\square$ of ZC.

The $\angle$ AIK is $>$ $\angle$ (being $\equiv$ $\angle$ AID + $\angle$ DIK, & $\angle$ DIA
being a $\angle$. Ref. 2.).

Moreover, $\angle$ AIK is $\equiv$ $\angle$ ICK + $\angle$ IKC.

24. Consequently, $\angle$ ICK + $\angle$ IKC are $>$ $\angle$. { P. 32. B. 1.

But $\angle$ ICK is $\equiv$ to $\angle$ CKI (P. 5. B. 1.) because KI is $\equiv$ to CI. { D. 15. B. 1.

25. Therefore, 2 $\angle$ ICK are $>$ a $\angle$, & $\angle$ ICK $>$ half of a $\angle$. { Ax. 7. B. 5.

26. Wherefore, in $\triangle$ CKI, the $\angle$ CKI is $<$ half a $\angle$.

But $\angle ICK$ is $>$ half a $\angle$. (*Arg.* 25.)

27. From whence it follows, that in the $\triangle C\theta K$, the side $K\theta$, opposite to the $\angle K C \theta$ or $K C I$ is $>$ the side $C\theta$, opposite to the $\angle C K \theta$. *P. 18. B. 1.*

28. Consequently, the $\square$ of $K C$ (which is $=$ to the $\square$ of $K\theta +$ the $\square$ of $C\theta$. *P. 47. B. 1.*) is $<$ $2 \square$ of $K\theta$.

And it has been demonstrated (*Arg.* 23.) that the $\square$ of $K C$ is $>$ the double of the $\square$ $Z C$.

29. Wherefore, $2 \square$ of $K\theta$ will be $>$ $2 \square$ of $Z C$.

30. Hence, the $\square$ of $K\theta$ is $>$ the $\square$ of $Z C$.

But the $\square$ of $I C$ is $=$ to the $\square$ of $I Z +$ the $\square$ of $Z C$.

And the $\square$ of $I K$ ($=$ to the $\square$ of $I C$. *D. 15. B. 1. & P. 47. B. 1. Cor. 3.*) is $=$ to the $\square$ of $I\theta +$ the $\square$ of $K\theta$.

31. Therefore, $\square$ of $I Z + \square$ of $Z C$ are $=$ to $\square$ of $I\theta + \square$ of $K\theta$. *Ax. 1. B. 1.*

Therefore, if from one side be taken the $\square$ of $Z C$, & from the other the $\square$ $K\theta$, (which are unequal, *Arg.* 30).

32. The remainder, viz. the $\square$ of $I Z$ will be $>$ the $\square$ of $I\theta$. *Ax. 5. B. 1.*

33. Consequently, $I Z$ is $>$ $I\theta$.

But the line $K\theta$, (which is $\perp$ to the diameter $A C$. *II. Prep. 3.* is without the sphere $E F G H$, & cannot meet it. *P. 16. B. 12. Cor.*), that is, $I\theta$ is $>$ $I G$.

And $I\theta$ is $<$ $I Z$. (*Arg.* 33.)

34. Much more then $I Z$, (which is much $>$ $I G$) does not meet the surface of the sphere $E F G H$.

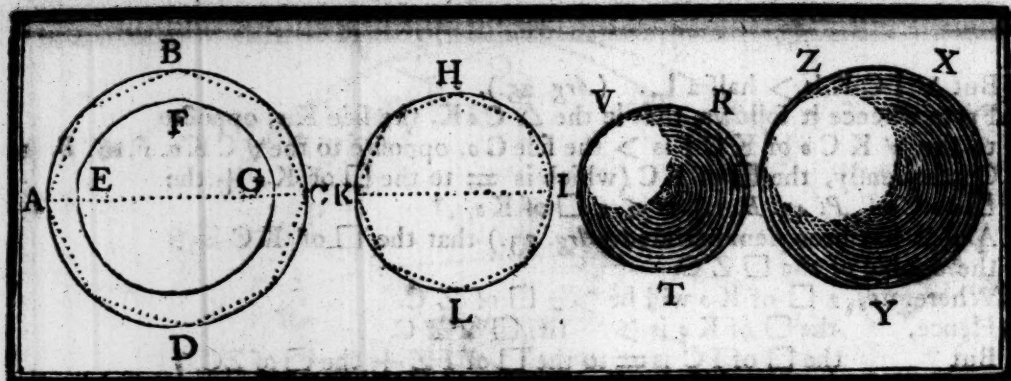
35. Wherefore the plane $K S P C$, in which Z is the point nearest the center I , does not touch this sphere $E F G H$.

36. It may be demonstrated after the same manner, that all the other planes which form the polyhedron do not meet the sphere $E F G H$.

37. Consequently, there has been described in the greater sphere $K O N$ a polyhedron $K P T R V O$, &c. whose planes do not meet the lesser sphere. Which was to be demonstrated. 11.

C O R O L L A R Y.

IF in two spheres there be described two similar polyhedrons; those polyhedrons will be to one another in the triplicate ratio of the diameters of the spheres in which they are described: For those polyhedrons being similar, are bounded by the same number of planes similar each to each, (*D. 9. B. 11.*); consequently each polyhedron may be divided into pyramids, having all their vertices at the center of the sphere, & for bases the planes of the polyhedron, besides all the pyramids contained in the first polyhedron are similar to all the pyramids contained in the second polyhedron, each to each; consequently, they are to one another, (viz. the pyramids of the first polyhedron to the pyramids of the second) in the triplicate ratio of their homologous sides; that is, of the semi diameters of their spheres. (*Cor. P. 8. B. 12.*) From whence it follows, (*P. 12. B. 5.*) that all the pyramids composing the first polyhedron, are to all the pyramids composing the second polyhedron in the triplicate ratio of the semi diameters of their spheres; & (*P. 11. & 15. B. 5.*) that the first polyhedron is to the second in the triplicate ratio of the diameters of their spheres.



PROPOSITION XVIII. THEOREM XVI.

SPHERES (ABCD & HILK) have to one another the triplicate ratio of that which their diameters (AC & KI) have.

Hypothesis.

AC is the diameter of the sphere ABCD,
& KI the diameter of the sphere HILK.

Thesis.

Sphere ABCD : sphere HILK =
 $AC^3 : KI^3$.

DEMONSTRATION.

If not,

A Sphere $<$ or $>$ the sphere ABCD will be to a sphere
 $HILK = AC^3 : KI^3$.

I. Supposition.

Let the sphere VRT be $<$ the sphere ABCD, so that
the sphere VRT : sphere HILK = $AC^3 : KI^3$.

I. Preparation.

1. Place the sphere VRT so as to have the same center with the sphere ABCD, as EFG (which is $=$ to the sphere VRT).
2. In the greater sphere ABCD describe a polyhedron the superficies of which does not meet the lesser sphere EFG. P.17. B.12.
3. In the sphere HILK describe a polyhedron ω to that in the sphere ABCD.

BECAUSE the polyhedrons ABCD & KHIL are ω .
(I. Prep. 1. & 2.).

1. The polyhedron ABCD : polyhedron KHIL = $AC^3 : KI^3$. { P.17. B.12.
Cor.

- And since the sphere VRT : sphere $HILK = AC^3 : KI^3$. (*I. Sup.*)
 Moreover, the sphere VRT is $\equiv$ to the sphere EFG . (*Prop.*)
2. It follows, (invertendo) that the sphere $HILK$: sphere EFG $\left\{ \begin{array}{l} P. 4. B. 3 \\ Cor. \\ P. 7. B. 11. \end{array} \right.$
 $= KI^3 : AC^3$.
 3. From whence it follows, that the sphere $HILK$: sphere EFG
 $= \text{polyg. } KHIL : \text{polyg. } ABCD$. P. 11. B. 3.
 But the sphere $HILK$ is $>$ the polyhedron $KHIL$. Ax. 8. B. 1.
 4. Therefore, the sphere EFG (or its equal VRT) is also $>$ the polyhedron $ABCD$. P. 14. B. 5.
 But the sphere EFG is contained in the polyhedron $ABCD$ (*Prop. 2*).
 5. Consequently, the part will be $>$ the whole.
 6. Which is impossible.
 7. Consequently, the cube of the diameter (AC) of a sphere ($ABCD$) is not to the cube of the diameter (KI) of another sphere ($HILK$) as a sphere VRT , less than the first sphere ($ABCD$), is to this second sphere $HILK$.

II. Supposition.

Let the sphere ZXY be $>$ the sphere $ABCD$, so that the sphere ZXY : sphere $HILK = AC^3 : KI^3$.

II. Preparation.

Take a sphere VRT , such that the sphere $ABCD$: sphere $VRT = AC^3 : KI^3$.

BECAUSE the sphere XZY : sphere $HILK = AC^3 : KI^3$. (*II. Sup.*)

And the sphere $ABCD$: sphere $VRT = AC^3 : KI^3$. (*II. Prep.*)

8. The sphere XZY : sphere $HILK =$ sphere $ABCD$: sphere VRT .

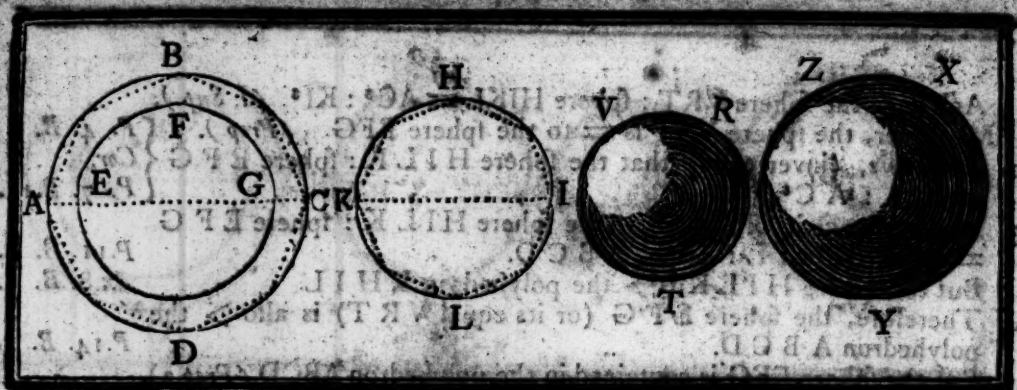
But the sphere XZY is $>$ the sphere $ABCD$. (*II. Sup.*)

9. Consequently, the sphere $HILK$ is also $>$ the sphere VRT .

But it has been demonstrated (*Arg. 7.*), that the cube of the diameter (AC) of a sphere ($ABCD$) is not to the cube of the diameter (KI) of another sphere ($HILK$), as a sphere $ABCD$ is to a sphere less than $HILK$.

10. Therefore, the sphere VRT is not $<$ the sphere $HILK$ (as has been proved, *Arg. 9.*)

11. Consequently, the sphere XZY is not $>$ the sphere $ABCD$, (as has been supposed).



Therefore, as the supposed sphere cannot be either $<$ or $>$ the sphere $ABCD$.
 12. It will be equal to it.
 13. From whence it follows, that the sphere $ABCD$: sphere $HILK$
 $= AC^2 : KI^2$. P. 7. B. 5.

C O R O L L A R Y.

S P H E R E S are to one another as the similar polyhedrons described in them. (Cor.)
 P. 17. B. 12. & P. 11. B. 5.

Take a sphere VRT . Such that the sphere $ABCD$: sphere
 $VRT = AC^2 : KI^2$ **S I N I F**

B E C A U S E the sphere XYZ : sphere $HILK = AC^2 : KI^2$.
 (II. 24.)

And the sphere $ABCD$: sphere $VRT = AC^2 : KI^2$. (II. 24.)
 8. The sphere XYZ : sphere VRT : sphere $ABCD$: sphere

VRT .
 But the sphere XYZ : sphere $HILK = AC^2 : KI^2$. (II. 24.)
 9. Consequently, the sphere VRT : sphere $HILK$: sphere $ABCD$: sphere

VRT .
 But it has been shewn, that the sphere VRT : sphere $HILK$: sphere $ABCD$: sphere

VRT .
 Therefore, the sphere $ABCD$: sphere $HILK$: sphere VRT : sphere XYZ : sphere $ABCD$: sphere

VRT .
 Therefore, the sphere $ABCD$: sphere $HILK$: sphere VRT : sphere XYZ : sphere $ABCD$: sphere

